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Two-Factors Each Component of Which Contains a Specified Vertex

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Abstract

It is shown that if G is a graph of order n with minimum degree $\delta(G)$, then for any set of k specified vertices $\{v_1, v_2, \cdots, v_k\} \subset V(G)$, there is a 2-factor of G with precisely k cycles $\{C_1, C_2, \cdots, C_k\}$ such that $v_i \in V(C_i)$ for $(1 \le i \le k)$ if $n = 3k, \delta(G) \ge \frac{7k-2}{3}$ or $3k+1 \le n \le 4k, \delta(G) \ge \frac{2n+k-3}{3}$ or $4k \le n \le 6k-3, \delta(G) \ge 3k-1$ or $n \ge 6k-3, \delta(G) \ge \frac{n}{2}$. Examples are described that indicate this result is sharp.

1 INTRODUCTION

A 2-factor of a graph G is a collection of vertex disjoint cycles $\{C_1, C_2, \cdots, C_r\}$ that are subgraphs of G that span G (i.e. $\cup_{i=1}^{r} V(C_i) = V(G)$). If $r = 1$, then the cycle C_1 spans all of the vertices of G, and so is a hamiltonian cycle and the graph G is hamiltonian.

A graph G of order n with $\delta(G) \geq n/2$ is hamiltonian, which is a classical result of Dirac [3]. Moreover, it was shown in [1] that the same degree condition implies that there is a 2-factor with precisely H cycles for any m, $(1 \leq m \leq n/4)$. The second author posed the following question at the Fifth Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications in Prague in July of 1998.

Question Given any set of k vertices $\{v_1, v_2, \cdots, v_k\}$ in a graph G of order $n \geq 6k - 3$ and with $\delta(G) \geq n/2$, is there a 2-factor of G with precisely k cycles $\{C_1, C_2, \cdots, C_k\}$ such that $v_i \in V(C_i)$ for $(1 \leq i \leq k)$?

We will prove the following result, which gives a positive answer to the above Question.

Main Theorem

Let G be a graph of order n with minimum degree $\delta(G)$. If for a positive integer k,

a) $n = 3k, \delta(G) \geq \frac{7k-2}{3}$ or

b) $3k + 1 \leq n \leq 4k, \delta(G) \geq \frac{2n+k-3}{3}$ or

c) $4k \leq n \leq 6k - 3, \delta(G) \geq 3k - 1$ or

d) $n \geq 6k - 3, \delta(G) \geq \frac{n}{2}$,

then for any set of k specified vertices $\{v_1, v_2, \cdots, v_k\}$ there is a 2-factor of G with k cycles C_i such that $v_i \in V(C_i)$ for $1 \leq i \leq k$.

The assumption on the minimum degree is sharp in all cases. Let $G = (A + K_{a-1,a+1} + K_{2k-2a}) + B$, where $a = \lceil 2k/3 \rceil$, $A \cong K_a$ and $B \cong K_{k-a}$. Then G cannot be partitioned into triangles $C_1, \cdots, C_k$ such that $|V(C_i) \cap (V(A) \cup V(B))| = 1$, while $\delta(G) = 3k - a - 1 = \lfloor (7k-3)/3 \rfloor$. Let $G = (A + K_{2a-1} + K_{n-k-2a-1}) + B$, where $a = \lceil (n-k+1)/3 \rceil$, and $A \cong K_a$ and $B \cong K_{k-a}$. Then $\delta(G) = n - a - 1 = \lfloor (2n+k-4)/3 \rfloor$, though G does not contain vertex disjoint cycles $C_1, \cdots, C_k$ such that $|V(C_i) \cap (V(A) \cup V(B))| = 1$. Suppose $n \geq 4k$, and let $G = A + K_{2k-1} + K_{n-3k+1}$ where $A \cong K_k$. Then $\delta(G) = 3k - 2$, though G does not contain vertex disjoint cycles $C_1, \cdots, C_k$ such that $|V(C_i) \cap V(A)| = 1$. Finally, let $G = K_{\lfloor (n-1)/2 \rfloor, \lceil (n+1)/2 \rceil}$. Then $\delta(G) = \lfloor (n-1)/2 \rfloor$, though G does not contain a 2-factor.

A key step in the proof of the Main Theorem is to show the existence of vertex disjoint cycles that contain the specified vertices. This type of **packing** result is of interest in its own right. Thus, the following Theorem 1 will be the first step of the proof presented in the next section.

Theorem 1 *Let G be a graph of order n. If for a positive integer k,*
a) $n = 3k, \delta(G) \geq \frac{7k-2}{3}$ or
b) $3k+1 \leq n \leq 4k, \delta(G) \geq \frac{2n+k-3}{3}$ or
c) $4k \leq n \leq 6k-3, \delta(G) \geq 3k-1$ or
d) $n \geq 6k-3, \delta(G) \geq \frac{n}{2}$,
then for any set of k specified vertices $\{v_1, v_2, \cdots, v_k\}$, there is a collection of k vertex disjoint cycles C_i such that $|V(C_i)| \leq 5$ and $v_i \in V(C_i)$ for $1 \leq i \leq k$.

The next step is to show that this collection of cycles can be transformed into a collection of cycles that are a 2-factor of G. This type of **partition** result is also of interest in its own right. Thus, the following Theorem 2 will be the second step of the proof presented in the third section.

Theorem 2 *Let G be a graph of order n. Suppose that for a given set of k specified vertices $\{v_1, v_2, \ldots, v_k\}$, there is a collection of k vertex disjoint cycles C_i, such that $v_i \in V(C_i)$ for $1 \leq i \leq k$. If $\sigma_2(G) \geq n = |V(G)|$, $\delta(G) \geq k+1$, $\sigma_2(G) + \delta(G) \geq n + 3k - 2$, then there exist disjoint cycles $H_1, \ldots, H_k$ satisfying $v_i \in V(H_i), 1 \leq i \leq k$, and $V(G) = \cup_{i=1}^{k} V(H_i)$.*

The condition $\delta(G) \geq k+1$ is necessary, since there are graphs G of order n and $\delta(G) \leq k$ which have no desired 2-factor if all neighbors of a vertex v with $d(v) = \delta(G) = k$ belong to the set of k specified vertices.

Note that in all four cases a) - d) of the Main Theorem and of Theorem 1 the assumptions of Theorem 2 ($\delta(G) \geq k+1, \sigma_2(G) + \delta(G) \geq 3\delta(G) \geq n + 3k - 2$) are satisfied.

Notation used will be standard and will follow [2]. The vertex set and edge set of a graph G will be denoted by $V(G)$ and $E(G)$ respectively. If $v \in V(G)$, then $N(v) = \{u \in V(G) : vu \in E(G)\}$, and will be called the neighborhood of v, and if $U \subset V(G)$, then $N(v) \cap U$ will be denoted by just $N_U(v)$, the neighborhood of v restricted to U. The degree of a vertex v, which is $|N(v)|$, will be denoted by $d(v)$, and the degree restricted to a subgraph U will be denoted by $d_U(v)$.

2 Packing of cycles

In this section we will prove Theorem 1. In this proof, a short cycle means a cycle of length less than or equal to 5.

Proof: Assume that Theorem 1 is not true, and let G be a maximal counterexample. Since $n \geq 3k$, G is not complete. Let x and y be nonadjacent vertices in G. By the maximality of G, $G + xy$ contains vertex disjoint short cycles $\{C_1, C_2, \cdots, C_k\}$ such that $v_i \in V(C_i)$ for $(1 \leq i \leq k)$. We may assume that $xy \in E(C_k)$. Then $\{C_1, C_2, \cdots, C_{k-1}\}$ are vertex disjoint short cycles in G such that $v_i \in V(C_i)$ for $(1 \leq i \leq k-1)$, $v_k \notin \cup_{i=1}^{k-1} V(C_i)$, and $\sum_{i=1}^{k-1} |V(C_i)| \leq n - 3$. Among all possible choices of a set of vertex disjoint short cycles $\{C_1, C_2, \cdots, C_{k-1}\}$ such that $v_i \in V(C_i)$ for $(1 \leq i \leq k-1)$ and $v_k \notin \cup_{i=1}^{k-1} V(C_i)$, select one collection such that

(1) $\sum_{i=1}^{k-1} |V(C_i)|$ is as small as possible,

and

(2) subject to (1), $\sum_{i=1}^{k-1} d_{C_i}(v_k)$ is as small as possible.

We also assume that in this selection any permutation of the vertices $\{v_1, v_2, \cdots, v_k\}$ can be used.

Let $C_i = (v_i, v_i^+, \cdots, v_i^-, v_i)$ for $(1 \leq i \leq k-1)$, let $L = G[\cup_{i=1}^{k-1} V(C_i)]$, and let $H = G - L$.

Claim 1 *$d_{C_i}(h) \leq 3$ for $h \in V(H)$ and for $1 \leq i \leq k-1$.*

Proof: If $d_{C_i}(h) \geq 4$ for $h \in V(H) - \{v_k\}$, then it is straightforward to check that the cycle C_i can be replaced by a shorter cycle containing v_i and h. In fact, the same can be said for v_k, except in this case the cycle C_i is replaced by a shorter cycle containing v_k and not v_i.

Claim 2 *Suppose u_1 and u_2 are distinct vertices in $N_H(v_k)$. Then the number of edges between $\{v_k, u_1, u_2\}$ and $V(C_i)$ is at most 7 for $1 \leq i \leq k-1$.*

Proof: This is clearly true if $d_{C_i}(u_1), d_{C_i}(u_2)$ or $d_{C_i}(v_k) \leq 1$, so assume that this is not true. In the case when $|V(C_i)| = 5$, observe that if $d_{C_i}(u_1) = 3$, then u_1 must be adjacent to three consecutive vertices of C_i other than v_i. Then, any adjacency of v_k other than v_i will result in a cycle of length less than C_i containing v_k. Hence $d_{C_i}(u_1), d_{C_i}(u_2) \leq 2$, which verifies the claim in this case. If $|V(C_i)| = 4$, and if $d_{C_i}(u_j) \geq 3$ for $j = 1$ or 2, then it is straightforward to check that the cycle C_i can be replaced by a cycle of length 3 containing v_k. Hence, $d_{C_i}(u_1) + d_{C_i}(u_2) + d_{C_i}(v_k) \leq 2 + 2 + 3 = 7$, which verifies the claim. This leaves the case when $C_i = K_3$. If the claim is not true, then all of the 9 edges, except for possibly 1, are between $\{v_k, u_1, u_2\}$ and $V(C_i)$, and it is easy to find two disjoint triangles in these 6 vertices with v_i and v_k in different cycles. This completes the verification of the claim.

First, we deal with the cases (a) and (b), and assume that $\delta(G) \geq \frac{2n+k-3}{3}$.

Claim 3 $N_H(v_k) = V(H) - \{v_k\}$.

Proof: Suppose v_k and $x \in V(H) - \{v_k\}$ are nonadjacent. Then $d_H(v_k) + d_H(x) \leq |V(H)| - 1$, since $|N_H(v_k) \cap N_H(x)| \leq 1$. Note also that $d_H(x) \leq |V(H)| - 2$. Hence

$$d_L(v_k) + 2d_L(x) \geq 3\delta(G) - (d_H(v_k) + d_H(x)) - d_H(x)$$

$$\geq 2n + k - 3 - (|V(H)| - 1) - (|V(H)| - 2)$$

$$= 2|V(L)| + k > \sum_{i=1}^{k-1}(2|V(C_i)| + 1).$$

This implies that $d_{C_i}(v_k) + 2d_{C_i}(x) \geq 2|V(C_i)| + 2$ for some i, $1 \leq i \leq k-1$. By Claim 1, this is possible only if $|V(C_i)| = 3$, $d_{C_i}(x) = 3$, and $d_{C_i}(v_k) \geq 2$. We may assume that v_k and v_i^+ are adjacent. Then, by replacing C_i with (v_i, v_i^-, x, v_i), we get a new cycle system, which contradicts the choice rule (2).

Let u_1 and $u_2 \in N_H(v_k)$. Since $N_H(u_1) = N_H(u_2) = \{v_k\}$, $d_H(u_1) + d_H(u_2) + d_H(v_k) = |V(H)| + 1$. Hence

$$d_L(u_1) + d_L(u_2) + d_L(v_k) \geq 3\delta(G) - (|V(H)| + 1)$$

$$\geq 2n + k - 3 - |V(H)| - 1 = 2|V(L)| + k - 4 + |V(H)|.$$

On the other hand, $d_L(u_1) + d_L(u_2) + d_L(v_k) \leq 7(k-1)$ by Claim 2. This is possible only if $|V(L)| = 3(k-1)$ and $|V(H)| = 3$, that is, $n = 3k$. This is not possible either, since we have assumed $\delta(G) \geq \frac{7k-2}{3} > \frac{2n+k-3}{3}$ in this case.

This settles the cases (a) and (b).

Next, we deal with the cases (c) and (d). Note that $\delta(G) \geq \max\{3k-1, n/2\}$ in these cases. By Claim 1, $d_H(h) \geq \delta(G) - 3(k-1) \geq 2$ for $h \in V(H)$. Let u_1 and $u_2 \in N_H(v_k)$. For $(0 \leq j \leq 3)$, let r_j denote the number of cycles C_i for $(1 \leq i \leq k-1)$ such that $d_{C_i}(v_k) = j$. Thus, $k - 1 = r_0 + r_1 + r_2 + r_3$. This implies that

$$d_L(v_k) = 3r_3 + 2r_2 + r_1 \leq k + 2r_3 + r_2 - 1,$$

and by Claim 2

$$d_L(u_1) + d_L(u_2) \leq 4r_3 + 5r_2 + 6(r_1 + r_0) = 6k - 2r_3 - r_2 - 6.$$

This implies on the average,

$$d_L(u) \leq 3k - r_3 - r_2/2 - 3$$

for vertices $u \in N_H(v_k)$. Let $r = r_3 + r_2/2$. The previous inequalities give lower bounds for degrees in H, in particular it implies

$$d_H(v_k) \geq \delta(G) - k - 2r + 1,$$

and on the average

$$d_H(u) \geq \delta(G) - 3k + r + 3$$

for $u \in N_H(v_k)$. Since there is no short cycle in H that contains v_k, $N_H(v_k)$ is an independent set and the neighborhoods in H of the vertices in $N_H(v_k)$ are disjoint except for v_k. Thus, $|V(H)| \geq d_H(v_k)d_H(u) + 1$, where $d_H(u)$ represents the average degree in H of the vertices in $N_H(v_k)$. Since $n = |V(H)| + |V(L)|$ and $|V(L)| \geq 3k - 3$, we have the following inequality:

$$\begin{aligned} n &\geq (\delta(G) - k - 2r + 1)(\delta(G) - 3k + r + 3) + 3k - 2 \\ &\geq (\delta(G) - k - 2r + 1)(3k - 1 - 3k + r + 3) + 3k - 2 \\ &\geq 2(n/2 - k - 2r + 1) + r(3k - 1 - k - 2r + 1) + 3k - 2 \\ &= n + k - 4r + r(2k - 2r). \end{aligned}$$

This is possible only if $r = k - 1$. Therefore the minimization of the cycle lengths implies v_k is adjacent to precisely $\{v_i^-, v_i, v_i^+\}$ for $(1 \leq i \leq k-1)$, and $d_H(v_k) \geq 2$. If $d_H(v_k) \geq 3$, then there are vertices $u_1, u_2, u_3 \in N_H(v_k)$ with disjoint neighborhoods except for $\{v_k\}$ and with average degree in H at least $\delta(G) - 3k + k - 1 + 3 = \delta(G) - 2k + 2$. This implies

$$|V(H)| \geq 3(\delta(G) - 2k + 2) + 1$$

$$\geq 2(n/2 - 2k + 2) + (3k - 1 - 2k + 2) + 1$$

$$= n - 3k + 6 > n - |V(L)|,$$

a contradiction, so we assume that $N_H(v_k) = \{u_1, u_2\}$. Also, $d_H(u_1), d_H(u_2) \geq 2$, and so there exist vertices $w_1, w_2 \in V(H) - \{v_k\}$ such that $u_i w_i \in E(H)$. Since there is no short cycle in H containing v_k, $d_H(u_1) + d_H(w_2) \leq |V(H)| - 2$ and $d_H(u_2) + d_H(w_1) \leq |V(H)| - 2$. Hence $d_L(v_k) + d_L(u_1) + d_L(u_2) + d_L(w_1) + d_L(w_2) \geq 5\delta(G) - 2|V(H)| + 2 \geq 2n + 3k - 1 - 2|V(H)| + 2 = 2|V(L)| + 3k + 1 \geq 9k - 5$. Since $(9k - 5)/(k - 1) > 9$, there is some C_i, say C_1, such that there are at least 10 edges between $\{v_k, u_1, u_2, w_1, w_2\}$ and $V(C_1)$.

We claim that there are 2 disjoint cycles containing v_1 and v_k respectively in the graph spanned by $\{v_k, u_1, u_2, w_1, w_2\} \cup V(C_1)$. Note that these cycles must be short. (If one of them is not short, the other one is shorter than C_1, which contradicts the choice rule (1).) Note also that v_k is adjacent to precisely $\{v_1^-, v_1, v_1^+\}$. If $|V(C_1)| = 5$, then the only possible adjacency of u_1 or u_2 is v_1, for otherwise there would be a smaller cycle than C_1 that contains v_k. Since none of the vertices has more than 3 adjacencies in C_1, we can assume that $u_1 v_1 \in E(G)$, and so $w_1 v_1 \notin E(G)$, since this would contradict the minimality of the cycle lengths. There must be a matching between $\{w_1, w_2\}$ and $V(C_1) - \{v_1\}$ since there are at least 4 such edges, and from this matching 2 disjoint cycles containing v_1 and v_k can be formed. Next consider the case when $|V(C_1)| = 4$, and let v_1^* be the fourth vertex on C_1. The minimality of the length of the cycles implies that the only possible adjacencies of u_1 and u_2 are v_1 and v_1^*. With no loss of generality we can assume that $u_1 v_1 \in E(G)$ (otherwise, the cycle C_1 could be replaced by a cycle C_1' containing v_k that falls into a previous case), and so $w_1 v_1 \notin E(G)$. There must be a matching between w_1 and one of $\{u_2, w_2\}$ and the set $V(C_1) - \{v_1\}$, since there are at least 5 such edges. As before, this gives the required 2 disjoint cycles, so we are left with the case when C_1 is a triangle. Since there are 10 edges between C_1 and $\{v_k, u_1, w_1, u_2, w_2\}$, we can assume with no loss of generality that u_1 has at least one adjacency in C_1. If $u_1 v_1^- \in E(G)$, then by replacing the cycle C_1 with the cycle $(v_k, v_1^- u_1, v_k)$, we have a new minimum length cycle set with v_k replacing v_1. The structure of minimum cycle sets implies $v_1 u_1 \in E(G)$, so we can assume that $v_1u_1 \in E(G)$. If $v_1u_1, v_1w_1 \in E(G)$, (or equivalently $v_1u_2, v_1w_2 \in E(G)$) then there are 2 disjoint triangles, namely $v_1u_1w_1v_1$ and $v_kv_1^-v_1^+v_k$. Hence, we can assume that v_1 has at least 1 and at most 2 adjacencies in $\{u_1, w_1, u_2, w_2\}$. This implies that there are at least 5 edges between $\{v_1^-, v_1^+\}$ and $\{u_1, w_1, u_2, w_2\}$. Therefore, there are two independent edges e_1 and e_2 between these sets with one endvertex of e_i in $\{u_i, w_i\}$ for $i = 1, 2$. The required 2 cycles can be formed using the edges e_1, e_2 and v_1u_1. This completes the proof of Theorem 1.

3 Packing to Partition

In this section we will prove Theorem 2.

Proof: Choose the cycles C_i such that $\sum_{i=1}^{k} |V(C_i)|$ is maximum. Let $L = G[\cup_{i=1}^{k} V(C_i)]$ and $H = G - L$. Let $V(H_0)$ be a connected component of H.

Claim 1 *(a)* $|N(H_0) \cap V(C_i)| \leq 1$ *for* $1 \leq i \leq k$.
(b) $H = H_0$

Proof: (a) Suppose $|N(H_0) \cap V(C_i)| \geq 2$ for some i. With no loss of generality we can assume that $i = 1$. Select two vertices $u_1, u_2 \in V(C_1)$ and vertices $h_1, h_2 \in V(H_0)$ (possibly $h_1 = h_2$) such that $h_iu_i \in E(G)$ for $i = 1, 2$, $v_1 \notin C_1(u_1, u_2)$, and $N_{H_0}(C_1(u_1, u_2)) = \emptyset$. Let $P = C_1(u_1, u_2) = [u_1^+, \cdots, u_2^-]$, $p = |V(P)|$, and $u \in V(P)$. Then, for $h \in V(H_0)$,

$$n \leq d_G(h) + d_G(u) \leq |V(H)| - 1 + \lceil (|V(L)| - p)/2 \rceil + d_L(u).$$

This implies that $d_L(u) \geq (|V(L)| + p + 1)/2$. Therefore, we can replace P in the cycle C_1 by a path in H_0 containing h_1 and h_2, and all of the vertices $u \in V(P)$ can be inserted into $C_1[u_2, u_1] \cup (\cup_{i=2}^{k} C_i)$, since $d_L(u) \geq (|V(L)| + p + 1)/2$. This gives a new collection of k cycles that contradicts the maximality of the lengths of the cycles $\{C_1, C_2, \cdots, C_k\}$.

(b) Suppose $H \neq H_0$ and choose two vertices $v \in V(H_0), v' \in V(H) - V(H_0)$. Then $n \leq d(v) + d(v') \leq |V(H_0)| - 1 + d_L(v) + |V(H)| - |V(H_0)| - 1 + d_L(v')$. By (a) we obtain $2k \geq d_L(v) + d_L(v') \geq |V(L)| + 2 \geq 3k + 2$, a contradiction.

We may assume $N(H) \cap V(C_i) = \{u_i\}, 1 \leq i \leq r$, and $N(H) \cap V(C_i) = \emptyset, r+1 \leq i \leq k$. Further, we may assume that $|N(u_i) \cap V(H)| \geq 2, 1 \leq i \leq s$ and $|N(u_i) \cap V(H)| = 1$ for $s+1 \leq i \leq r$. Let $U = \{u_1, \ldots, u_r\}$.

Claim 2 *$u_i \neq v_i$ for $1 \leq i \leq s$*

Proof: If $u_i = v_i$ for some i, say $i = 1$, and $d_H(v_1) \geq 2$, then select $h_1, h_2 \in N_H(v_1)$. Since H is connected, there is a cycle C_1' containing v_1 and some vertices of H.

By the maximality of the lengths of the cycles,

$$d_L(v) \leq |V(C_1)| - 1 + \sum_{i=2}^{k} \lfloor \frac{1}{2} |V(C_i)| \rfloor \leq |V(L)| - 1 - 2(k-1)$$

for some vertex $v \in V(C_1) - v_1$. On the other hand, for h_1 we have $d(h_1) \leq |V(H)| - 1 + k$. Now $vh_1 \notin E(G)$ for any $v \in V(C_1) - v_1$ and $d(h_1) + d(v) \leq n - 1 - (k-1) = n - k$, a contradiction.

Claim 3 *For any $v \in V(H), |N(v) \cap V(L)| \geq 2$.*

Proof: Let $y \in V(C_i) - \{u_i\}$. Then v and y are nonadjacent. Now $|N(v) \cap N(y)| \geq 2$ since $d(v) + d(y) \geq n$ (Note that $N(v) \cap N(y) \subseteq U$).

Claim 4 $|V(H)| \geq 2$

Proof: For any $v \in V(H), k+1 \leq d(v) \leq |V(H)| - 1 + r \leq |V(H)| - 1 + k$.

Claim 5 $s \geq 2$

Proof: Suppose $s \leq 1$. Then $|V(H)| \leq r - 1$ by Claim 3. Note that $|V(H)| \cdot (k+1-(|V(H)|-1)) \leq |E(H,L)| \leq s \cdot |V(H)| + (r-s)$. (This inequality will be used several times.) Then $|V(H)|(k+2-|V(H)|) \leq s(|V(H)|-1)+r \leq |V(H)|-1+k$. Hence $|V(H)|^2 - |V(H)| - 1 = |V(H)|(|V(H)|-2)+|V(H)|-1 \geq k(|V(H)|-1) \geq r(|V(H)|-1) \geq (|V(H)|+1)(|V(H)|-1) = |V(H)|^2 - 1$. This is impossible.

Claim 6 $|V(H)| > r - s$

Proof: Suppose $|V(H)| \leq r - s \leq k - s$. Then $|V(H)|(k+2-|V(H)|) \leq s(|V(H)|-1)+r \leq (|V(H)|-1)(k-|V(H)|)+k$. This implies $2|V(H)| \leq |V(H)|$, but this contradicts Claim 4.

Claim 7 *For any* $y \in V(L) - U, d(y) = d_L(y) \geq |V(L)| - s + 1$.

Proof: Since u and y are nonadjacent for any $u \in V(H), \sum_{u \in V(H)} d(u) \leq |V(H)|(|V(H)| - 1) + s|V(H)| + r - s$. So $d(y) \geq n - (|V(H)| - 1) - s - \frac{r-s}{|V(H)|} > |V(L)| - s$ by Claim 6.

Claim 8 *There exist no disjoint subgraphs* $P, C'_2, \dots, C'_k$ *in* L *satisfying* P *is a path joining two vertices in* $\{u_1, \dots, u_s\}$ *and* $|V(P) \cap \{v_1, \dots, v_k\}| = 1$, C'_i *is a cycle and* $|V(C'_i) \cap \{v_1, \dots, v_k\}| = 1$ *for* $2 \leq i \leq k$, *and* $|(V(P) \cup \cup_{i=2}^k V(C'_i)) \cap U| \geq k - 1$.

Proof: Let u_i and u_j be terminal vertices of P. Choose any $v \in N_H(u_i)$. Then there exists $v' \in N_H(u_j) - \{v\}$. Combining a path connecting v and v' in H and P, we get a cycle C'_1. Then all vertices in $V(L) - \cup_{i=1}^k V(C'_i) - U$ are insertible one by one, because of Claim 7. (Note that $d_L(y) \geq |V(L)| - s + 1 \geq |V(L)| - k + 1 \geq \frac{|V(L)|}{2} + \frac{3k}{2} - k + 1 \geq \frac{1}{2}|V(L)| + \frac{k}{2} + 1$.) This contradicts the maximality of the choice of $C_1, \dots, C_k$. (We may miss one vertex in U, but they contain two vertices in H.)

Choose disjoint cycles $D_1, \dots, D_k$ in L such that $\sum_{i=1}^k |V(D_i)|$ is as small as possible subject to $u_i \in V(D_i), 1 \leq i \leq r$, and $|V(D_i) \cap \{v_1, \dots, v_k\}| = 1$. We may assume that $v_i \in V(D_i), 1 \leq i \leq k$. Let $L_0 = L - \cup_{i=1}^k V(D_i)$.

Claim 9 $N(v_1) \cap \{u_2, \dots, u_s\} \neq \emptyset$.

Proof: If $N(v_1) \cap \{u_2, \dots, u_s\} = \emptyset$, then $d(v_1) \leq |V(L)| - 1 - (s - 1) = |V(L)| - s$, a contradiction to Claim 7.

Consider the following configuration (we call it a *good configuration*):
$1 \leq i \leq s, 1 \leq j \leq s, i \neq j$
$w \in N_{D_j}(v_i), v_j \in D_j[u_j^+, w^-]$
$y = v_j^+$ or $y = v_j^-, y \in D_j[u_j^+, w^-], z \in D_i(v_i, u_i)$.
If $z \in N(v_j)$, we call it a *very good configuration.* Here and in the following $D_j[u_j^+, w^-]$ will be used as the abbreviation for $V(D_j[u_j^+, w^-])$.

Claim 10 *A good configuration always exists.*

Proof: Take $v_i = v_1$ and $w = u_j \in N(v_1) \cap \{u_2, \dots, u_s\}$ which are guaranteed by Claim 9. Define the orientation of D_1 so that $D_1(v_1, u_1) \neq \emptyset$.

Now consider a *good configuration* (if possible, a *very good configuration*). We may assume that $i = 1$ and $j = 2$.

Claim 11 $d_{D_1}(z) + d_{D_1}(y) + d_{D_1}(v_2) \leq 2|V(D_1)| + 1$

Proof: $N(y) \cap N(v_2) \cap (V(D_1) - \{u_1, v_1\}) = \emptyset$ (otherwise, we get a disjoint path P connecting u_1 and u_2 through v_1 and a cycle C_2' through v_2 in $G[V(D_1) \cup V(D_2)]$, contradicting Claim 8).

Note that $d_{D_1[v_1,u_1]}(z) = 2$ by the minimality of D_1. So, $d_{D_1}(z) + d_{D_1}(y) + d_{D_1}(v_2) \leq 2|D_1(u_1, v_1)| + 4 + |D_1(v_1, u_1)| + 2 \leq 2|V(D_1)| + 1$.

Claim 12 $d_{D_2}(z) + d_{D_2}(y) + d_{D_2}(v_2) \leq 2|V(D_2)|$

We have $d_{D_2}(v_2) = 2$ by the minimality of D_2. Next, $d_{D_2}(z) \leq |V(D_2)| - 1$, since $\{y, v_j\} \not\subseteq N(z)$, and $d_{D_2}(y) \leq |V(D_2)| - 1$.

Claim 13 $d_{D_i}(z) + d_{D_i}(y) + d_{D_i}(v_2) \leq 2|V(D_i)| + 2$ *for* $s + 1 \leq i \leq k$

Proof: Suppose, $d_{D_i}(z) + d_{D_i}(y) + d_{D_i}(v_2) > 2|V(D_i)| + 2$ for some i. Then $d_{D_i}(z) \geq 3$. Choose $w_1, w_2 \in N_{D_i}(z)$ so that $v_i \in D_i[w_1, w_2)$ and $N(z) \cap D_i(w_1, w_2) = \emptyset$. Since $N(v_2) \cap N(y) \cap D_i(w_2, w_1) = \emptyset$, the claim follows.

Claim 14 $d_{D_i}(z) + d_{D_i}(y) + d_{D_i}(v_2) \leq 2|V(D_i)| + 1$ *for* $3 \leq i \leq s$

Proof: Suppose, $d_{D_i}(z) + d_{D_i}(y) + d_{D_i}(v_2) \geq 2|V(D_i)| + 2$ for some i. Choose w_1 and w_2 as in the proof of Claim 13. Then $\{w_1, w_2\} \subseteq N(z) \cap N(y) \cap N(v_2)$ and $|N(w) \cap \{z, y, v_2\}| = 2$ for any $w \in V(D_2) - \{w_1, w_2\}$. In particular, $D_2[w_1, w_2] \subseteq N(y) \cap N(v_2)$ and $D_2[w_2, w_1] \subseteq N(z)$.

CASE 1 $v_i = w_1$

Since (v_2, y, w_2) is a triangle, $D_i(w_2, w_1) = \emptyset$. This means that $v_2 u_i \in E(G)$ and $y v_i \in E(G)$ which is a very good configuration. By the choice of the configuration, v_2 and z are adjacent. Then (v_2, z, w_2) and (y, v_i, v_i^+) are triangles, which contradicts Claim 8.

CASE 2 $v_i \neq w_1$

If $D_i(w_1, w_2) \neq \{v_i\}$, then (v_2, w_2, z, w_1) and (y, v_i, v_i^+) or (y, v_i, v_i^-) are cycles. So $D_i(w_1, w_2) = \{v_i\}$. We may assume that $w_1 \neq u_i$. Then there is a very good configuration. By the choice of the configuration v_2 and z are adjacent. Then (v_2, z, w_2) and (y, v_i, v_i^-) are triangles. This contradicts Claim 8.

Claim 15 $d_{L_0}(z) + d_{L_0}(y) + d_{L_0}(v_2) \leq 2|V(L_0)|$

Proof: $N(y) \cap N(v_2) \cap L_0 = \emptyset$.

Claim 16 $d(z) + d(y) + d(v_2) \geq 2(|V(L)| - s + 1) + 3k - 2$

Proof: For any $v \in V(H)$ we have $d(v) + d(y) + d(v_2) \geq \sigma_2(G) + \delta(G) \geq n + 3k - 2$. Hence $\sum_{v \in V(H)} d(v) + |V(H)|(d(y) + d(v_2)) \geq |V(H)|(n + 3k - 2)$. Thus $d(y) + d(v_2) \geq n + 3k - 2 - \frac{1}{|V(H)|}\sum_{v \in V(H)} d(v) \geq n + 3k - 2 - (|V(H)| - 1 + s + \frac{r-s}{|V(H)|}) > |V(L)| - s + (3k-2)$. Therefore, $d(z) + d(y) + d(v_2) \geq 2(|V(L)| - s + 1) + 3k - 2$.

By Claims 11, 12, 13, 14 and 15, $d(z) + d(y) + d(v_2) \leq 2|V(D_1)| + 1 + 2|V(D_2)| + \sum_{i=3}^{s}(2|V(D_i)| + 1) + \sum_{i=s+1}^{k}(2|V(D_i)| + 2) + 2|V(L_0)| \leq 2|V(L)| + 1 + s - 2 + 2(k - s) \leq 2|V(L)| + 2k - s - 1$. By Claim 16, $2|V(L)| + 3k - 2s \leq 2|V(L)| + 2k - s - 1$. Hence, $k + 1 \leq s$, which is impossible. This completes the proof of Theorem 2.

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