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BIPARTITE GRAPHS WITH EVERY MATCHING IN A CYCLE

AMAR D / FLANDRIN E / GANCARZEWICZ G / WOJDA A P

Unité Mixte de Recherche 8623
CNRS-Université Paris Sud-LRI

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CNRS – Université de Paris Sud
Centre d'Orsay
LABORATOIRE DE RECHERCHE EN INFORMATIQUE
Bâtiment 650
91405 ORSAY Cedex (France)

Bipartite graphs with every matching in a cycle

Denise Amar, LaBRI, Bordeaux I, France
e-mail: amar@labri.u-bordeaux.fr
Evelyne Flandrin, LRI, Orsay, France
e-mail: fe@lri.fr
Grzegorz Gancarzewicz*, A. Paweł Wojda†
Wydział Matematyki Stosowanej AGH, Kraków, Poland
e-mail: gancarz@uci.agh.edu.pl, wojda@uci.agh.edu.pl

Abstract

We give sufficient Ore type conditions for a balanced bipartite graph to contain every matching in a hamiltonian cycle or a cycle not necessarily hamiltonian. Moreover for the hamiltonian case we prove that the condition is almost best possible.

Résumé

Nous donnons ici deux conditions de degré de type Ore, suffisantes pour que, dans un graphe biparti équilibré, tout couplage soit contenu dans un cycle hamiltonien ou dans un cycle quelconque. Nous prouvons de plus que, dans le cas hamiltonien, la condition est presque la meilleure possible.

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*This work was carried out in part while GG was visiting , Orsay, France

†This work was carried out in part while APW was visiting LIFO at University of Orleans, France

1 Introduction

In 1972, M. Las Vergnas obtained the following results [5]:

Theorem 1 *Let $G = (B, W, E)$ be a balanced bipartite graph of order $2n$. If for any $x \in B$, $y \in W$, $xy \notin E$ we have $d(x) + d(y) \geqslant n + 2$, then every perfect matching in G is contained in a hamiltonian cycle.*

For the existence of a perfect matching, he gave the sufficient condition:

Theorem 2 *Let $G = (B, W, E)$ be a balanced bipartite graph of order $2n$ and let $q \geqslant 2$. If for any $x \in B$, $y \in W$, $xy \notin E$ we have $d(x) + d(y) \geqslant n + q$, then every matching of cardinality q is contained in a perfect matching.*

Using these two results he obtained the following Corollary:

Corollary 3 *Let $G = (B, W, E)$ be a balanced bipartite graph of order $2n$ and let $q \geqslant 2$. If for any $x \in B$, $y \in W$, $xy \notin E$ we have $d(x) + d(y) \geqslant n + q$, then every matching of cardinality q is contained in a hamiltonian cycle.*

About cycles through matchings in general graphs K.A. Berman proved in [1] the following result conjectured by R. Häggkvist in [3].

Theorem 4 *Let G be graph of order n. If for any x, $y \in V(G)$, $xy \notin E$ we have $d(x) + d(y) \geqslant n + 1$, then every matching lies in a cycle.*

Theorem 4 has been improved by B. Jackson and N.C. Wormald in [4]. R. Häggkvist [3] gave also a sufficient condition for a general graph to contain any matching in a hamiltonian cycle. We give this theorem below in a slightly improved version obtained in [6].
Let $\mathcal{G}_n$ be the family of graphs $G = \overline{K}_{\frac{n+2}{3}} * H$, where H is any graph of order $\frac{2n-3}{3}$ containing a perfect matching, if $\frac{n+2}{3}$ is an integer, and $\mathcal{G}_n = \emptyset$ otherwise ($*$ denotes the join of graphs).

Theorem 5 *Let G be a graph of order $n \geq 3$, such that for every pair of nonadjacent vertices x and y $d(x) + d(y) \geq \frac{4n-2}{3}$. Then every matching of G lies in a hamiltonian cycle, unless $G \in \mathcal{G}_n$.*

We give sufficient conditions in a balanced bipartite graph for a matching to be contained in an hamiltonian cycle or a cycle not necessarily hamiltonian. Moreover, for the hamiltonian case we prove that the condition is almost best possible. Results are presented in section 3 and will be proved in the sections 4 and 5.

2 Definitions

Let $G = (B, W, E)$ be a balanced bipartite graph and M a matching in G.

A subgraph H of G is said to be a *Θ-graph compatible with* M if H is a union of two cycles C_1 and C_2 satisfying the conditions:

1. The intersection of C_1 and C_2 is a path R of length at least one.
2. Every edge of M is an edge of H.
3. Every edge of M incident with a vertex of R lies in R.
4. $|V(R)|$ is even and the end vertices, say x and y, of R are in different partite sets.

We denote $P: \quad x\, C_1 \setminus C_2\, y$, $Q: \quad x\, C_2 \setminus C_1\, y$ and $H = (P, Q, R)$.

The notion of the Θ-graph is based on the paper of K. Berman [1]. On Figure (1) there is an example of a Θ-graph.

A subgraph H of G is said to be *a strict Θ-graph compatible with* M if H is a Θ-graph (P, Q, R) such that if we label the vertices of the paths

$$
\begin{aligned}
P &: \; xp_1 \dots p_\alpha y \\
Q &: \; xq_1 \dots q_\beta y \\
R &: \; xr_1 \dots r_\gamma y
\end{aligned}
$$

then $q_1 \in V(H) \setminus V(M)$, $p_\alpha \in V(H) \setminus V(M)$, $xr_1 \in M$, and $r_\gamma y \in M$.

On Figure (2) there is an example of a strict Θ-graph.

If on a path $\pi: \quad x_1 x_2 \dots x_k$ of $G = (B, W, E)$ is given an orientation from x_1 to x_k, π is said to be a BB-path if $x_1 \in B$, $x_k \in B$, a WW-path if $x_1 \in W$, $x_k \in W$, a BW-path if $x_1 \in B$, $x_k \in W$ and a WB-path if $x_1 \in W$, $x_k \in B$.

Let C be a cycle or path with an arbitrary orientation and $x \in V(C)$, then x^- is *the predecessor of* x and x^+ is its *successor* according to the orientation of C.

Let A be a subgraph of G, v a vertex of G, then $d_A(v)$ is equal to the number of neighbors of v in A, and for $S \subset V(G)$, we put $e(S, A) = \sum_{v \in S} d_A(v)$.

A path P is *an even path* if $|V(P)|$ is even and is *an odd path* if $|V(P)|$ is odd.

For notation and terminology not defined above a good reference should be [2].

3 Result

Theorem 6 *Let $G = (B, W, E)$ be a balanced bipartite graph of order $2n$.*

1. *If for any $x \in B$, $y \in W$, $xy \notin E$ we have*

$$d(x) + d(y) > \frac{4n}{3},$$

then every matching M in G is contained in a hamiltonian cycle.

2. *If $n > 4$ and for any $x \in B$, $y \in W$, $xy \notin E$ we have*

$$d(x) + d(y) \geqslant \frac{5n}{4},$$

then every matching M in G is contained in a cycle of G.

The first part of the theorem is almost best possible in the sense that if one decreases the sum of degrees of more than $\frac{2}{3}$ then the theorem is no more true. $\overline{K}_{l,l}$ denotes the balanced bipartite graph of order $2l$ with empty edge set. Let $G = \overline{K}_{p+1,p+1} \star K_{2p+1,2p+1}$. G is a balanced bipartite graph of order $2n = 2(3p+2)$. Let M be a perfect matching of $K_{2p+1,2p+1}$. It is evident that there is no hamiltonian cycle containing M and that the minimum sum of degrees of two nonadjacent vertices is $\frac{4n-2}{3}$.

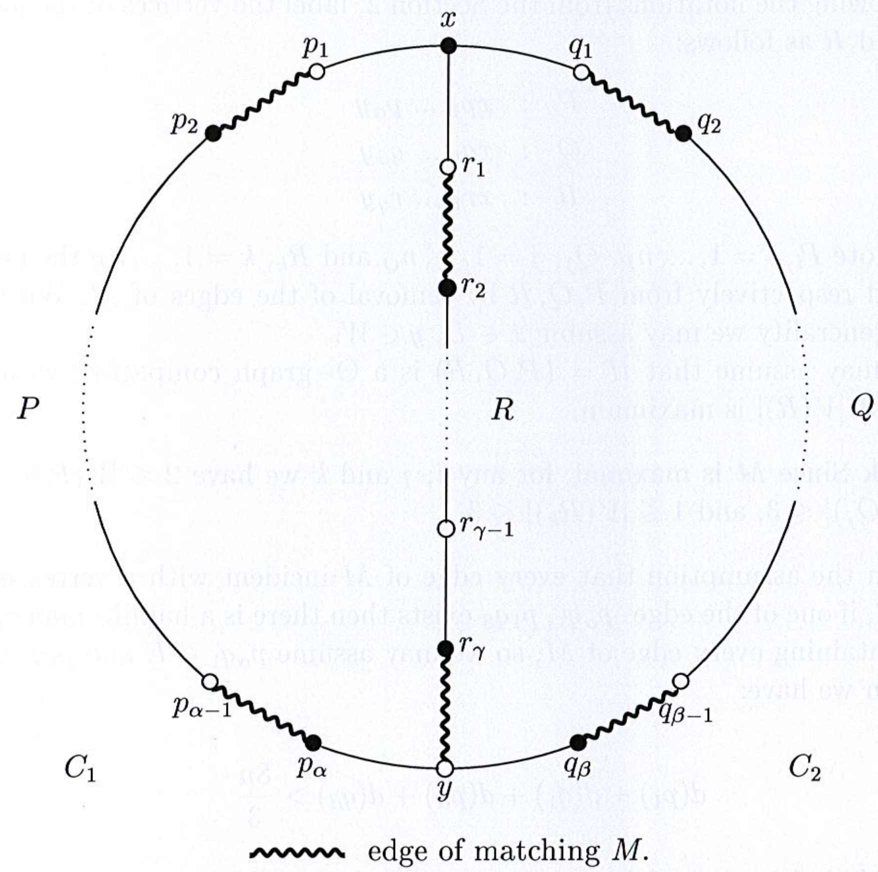
Let now G' be the graph obtained from G by replacing $\overline{K}_{p+1,p+1}$ by $\overline{K}_{p,p}$. Then G' is a balanced bipartite graph which satisfies the hypothesis of the part (1) of Theorem 6 and by consequence there is a hamiltonian cycle which contains M. Notice however that M is not contained in any perfect matching of G', and the degree constraint in part (2) of Theorem 6 is clearly not sufficient to imply that any matching can be extended into a perfect matching.

4 Proof of the part (1) of the Theorem 6:

Let $G = (B, W, E)$ be a bipartite graph satisfying the conditions of part (1) of the Theorem 6 and let us suppose that there is a matching M in G such that there is no hamiltonian cycle through M. Without loss of generality we may suppose that:

(i) M is maximal, i.e. M is the only matching which contains M.

(ii) G is maximal without a hamiltonian cycle through M (any addition of an edge uv, $u \in B$, $v \in W$, $uv \notin E$ creates a hamiltonian cycle containing M)



A Θ-graph compatible with M and containing all the vertices of the graph G.

Figure (1).

So we have a hamiltonian path $P_H: \quad up_1...p_{2n-2}v$ containing M. Since $uv \notin E$ we have $d(u) + d(v) > \dfrac{4n}{3}$ and this implies that we have at least two vertices p_i, p_{i+1} satisfying up_{i+1}, $vp_i \in E$. Then the hamiltonian cycle:

$$C': \quad up_{i+1}p_{i+2}...vp_ip_{i-1}...u$$

contains all edges of the path P_H except p_ip_{i+1}. Since there is no hamiltonian cycle containing M in G we have $p_ip_{i+1} \in M$. Now take the cycles: C_1 : $up_{i+1}p_{i-1}...u$ and C_2 : $\quad vp_ip_{i+1}p_{i+2}...v$. The subgraph $H = C_1 \cup C_2$ is a Θ-graph compatible with M and containing all the vertices of the graph G. We can see an example of such Θ-graph which in not a strict Θ-graph on the Figure (1).

Following the notations from the Section 2, label the vertices of the paths P, Q and R as follows:

$$\begin{aligned} P &: xp_1 \ldots p_\alpha y \\ Q &: xq_1 \ldots q_\beta y \\ R &: xr_1 \ldots r_\gamma y \end{aligned}$$

and denote P_i, $i = 1, \ldots, n_P$, Q_j, $j = 1, \ldots, n_Q$ and R_k, $k = 1, \ldots, n_R$ the paths obtained respectively from P, Q, R by removal of the edges of M. Without loss of generality we may assume $x \in B$, $y \in W$.

We may assume that $H = (P, Q, R)$ is a $\Theta-$graph compatible with M such that $|V(R)|$ is maximum.

Remark Since M is maximal, for any i, j and k we have $2 \leqslant |V(P_i)| \leqslant 3$, $2 \leqslant |V(Q_j)| \leqslant 3$, and $1 \leqslant |V(R_k)| \leqslant 3$.

From the assumption that every edge of M incident with a vertex of R lies in R, if one of the edges $p_\alpha q_1$, $p_1 q_\beta$ exists then there is a hamiltonian cycle in G containing every edge of M, so we may assume $p_\alpha q_1 \notin E$ and $p_1 q_\beta \notin E$ and then we have:

$$d(p_1) + d(q_1) + d(p_\alpha) + d(q_\beta) > \frac{8n}{3} \tag{1}$$

4.1 Neighbors of $p_1, p_\alpha, q_1, q_\beta$ on Q and P.

Claim 1 *If $p_1 q_l \in E$ and $l > 1$ (p_1 and q_1 are in the same partite set), then $q_l q_{l+1} \in M$. Moreover for $i = 2, \ldots n_Q$, $e(p_1, Q_i) \leqslant 1$ and if $e(p_1, Q_i) = 1$, then $e(q_\beta, Q_i) = 0$.*

Proof of the Claim 1:

In fact if $p_1 q_l \in E$, then $H' = (P', Q', R')$ with $P' :\ q_l p_1 p_2 \ldots p_\alpha y$, $Q' :$ $q_l q_{l+1} \ldots q_\beta y$ and $R' :\ q_l q_{l-1} \ldots q_1 x r_1 r_2 \ldots r_\gamma y$ is a $\Theta-$graph compatible with M with $|V(R')| > |V(R)|$ unless $q_l q_{l+1} \in M$.

So let us suppose that $p_1 q_l \in E$ and $q_l q_{l+1} \in M$, with $q_l \in Q_{i_0}$. Then $q_l \in B$ for $p_1 \in W$. The vertex q_{l-1} is the only vertex of $V(Q_{i_0})$ in W.

If $q_\beta q_{l-1} \in E$ then the cycle:

$$C' :\quad q_{l-1} q_{l-2} \ldots q_1 x r_1 \ldots r_\gamma y p_\alpha p_{\alpha-1} \ldots p_1 q_l q_{l+1} \ldots q_\beta q_{l-1} \tag{2}$$

is a hamiltonian cycle of G containing M and the Claim 1 is proved.

□

Claim 2 $1 \leqslant e(p_1, Q_1) \leqslant 2$ *and if* $e(p_1, Q_1) = 1$ *then* $e(q_\beta, Q_1) \leqslant 1$. *If* $e(p_1, Q_1) = 2$ *then* $e(q_\beta, Q_1) = 0$.

Proof of the Claim 2:

Since $x \in N(p_1) \cap Q_1$ we have $e(p_1, Q_1) \geqslant 1$. Observe that $|V(Q_1)| = 2$ or $|V(Q_1)| = 3$. When $|V(Q_1)| = 2$ then q_β may be adjacent to q_2 and $e(q_\beta, Q_1) \leqslant 1$. If $|V(Q_1)| = 3$ and $p_1 q_2 \in E$ then $e(q_\beta, Q_1) = 0$, because otherwise the cycle C' given by the formula (2) for $l = 2$ is a hamiltonian cycle of G containing M and the Claim 2 is proved.

□

Claim 3 *1. If* Q_{i_0} *is a* $BB-$*path and* Q_{j_0} *is a* $WW-$*path,* $2 \leqslant i_0,\ j_0 \leqslant n_Q$*, then*

$$e(\{p_1, q_\beta\}, Q_{i_0} \cup Q_{j_0}) \leqslant 3 = \frac{|V(Q_{i_0})| + |V(Q_{j_0})|}{2}. \tag{3}$$

2. If Q_k*,* $2 \leqslant k \leqslant n_Q$ *is a* BW*-path or a* WB*-path then*

$$e(\{p_1, q_\beta\}, Q_k) \leqslant 1 = \frac{|Q_k|}{2}. \tag{4}$$

3. In any case

$$e(\{p_1, q_\beta\}, Q_1) \leqslant 2. \tag{5}$$

Proof of the Claim 3

For any i, since the matching M is maximal we have $|Q_i| = 3$, iff Q_i is a $BB-$path or a $WW-$path and $|Q_i| = 2$, iff Q_i is a $BW-$path or a $WB-$path. Consider a BB-path Q_{i_0} and a $WW-$path Q_{j_0}, $(2 \leqslant i_0,\ j_0 \leqslant n_Q)$. From the Claim 1 for $2 \leqslant i_0,\ j_0 \leqslant n_Q$, we have $e(\{p_1, q_\beta\}, Q_{i_0}) \leqslant 1$ and $e(\{p_1, q_\beta\}, Q_{j_0}) \leqslant 2$. These proves the inequality (3). If $|Q_i| = 2$ from the Claim 1 we have (4). The inequality (5) is an immediate consequence of the Claim 2.

□

Let us denote $\nu_3(Q)$ the number of odd paths Q_i and $\nu_2(Q)$ the number of even paths Q_k, $1 \leqslant i, k \leqslant n_Q$.

As $|V(Q)|$ is even, the number of $BB-$paths is equal to the number of WW-paths and so $\nu_3(Q)$ is even i.e. $\nu_3(Q) = 2\mu$. Clearly $|V(Q)| = \beta + 2 = 3\,\nu_3(Q) + 2\,\nu_2(Q) = 6\mu + 2\,\nu_2(Q)$.

Now we shall estimate $e(\{p_1, q_\beta\}, Q)$. From Claims 1 — 3 we have:

$$\begin{aligned} e(\{p_1, q_\beta\}, Q) &= \sum_{|V(Q_i)|=3} e(\{p_1, q_\beta\}, Q_i) + \sum_{|V(Q_k)|=2} e(\{p_1, q_\beta\}, Q_k) \\ &\leqslant 3\mu + \nu_2(Q) + 1 = \frac{\beta}{2} + 2. \end{aligned} \tag{6}$$

Similarly we obtain the following three inequalities:

$$e(\{q_1, p_\alpha\}, Q) \leqslant \frac{\beta}{2} + 2, \tag{7}$$

$$e(\{p_1, q_\beta\}, P) \leqslant \frac{\alpha}{2} + 2, \tag{8}$$

$$e(\{q_1, p_\alpha\}, P) \leqslant \frac{\alpha}{2} + 2. \tag{9}$$

4.2 Neighbors of $p_1, p_\alpha, q_1, q_\beta$ on R

Observe that for any $k = 1, ..., n_R$ we have $1 \leqslant |V(R_k)| \leqslant 3$. If $xr_1 \in M$ then $R_1 = \{x\}$ and $|V(R_1)| = 1$. If $r_\gamma y \in M$ then $R_\gamma = \{x\}$ and $|V(R_\gamma)| = 1$. For $k = 2, ..., n_R - 1$ we have $2 \leqslant |V(R_k)| \leqslant 3$.

It is easy to check that if $|V(R_i)| = 2$ then $e(\{p_1, p_\alpha\}, R_i) \leqslant 1$ and if $|V(R_j)| = 3$ then $e(\{p_1, p_\alpha\}, R_j) \leqslant 2$.

If $|V(R_j)| = 1$ then $e(\{p_1, p_\alpha\}, R_j) = 1$.

Denote by $\nu_3(R)$ the number of paths R_i with three vertices, by $\nu_2(R)$ the number of paths R_i with two vertices and by $\nu_1(R)$ the number of paths R_k with one vertex.

Observe that $\nu_1(R) + \nu_3(R)$ is even and $\gamma + 2 = 3\,\nu_3(R) + 2\,\nu_2(R) + \nu_1(R)$
We have:

$$\begin{aligned} e(\{p_1, p_\alpha\}, R) &= \sum_{|V(R_j)|=3} e(\{p_1, p_\alpha\}, R_j) + \sum_{|V(R_i)|=2} e(\{p_1, p_\alpha\}, R_i) \\ &+ \sum_{|V(R_k)|=1} e(\{p_1, p_\alpha\}, R_k) \\ &\leqslant 2\,\nu_3(R) + \nu_2(R) + \nu_1(R) \\ &= \frac{2\gamma + 4 + \nu_1(R) - \nu_2(R)}{3} \\ &\leqslant \frac{2\gamma + 6}{3}. \end{aligned} \tag{10}$$

Similarly we have:

$$e(\{q_1, q_\beta\}, R) \leqslant \frac{2\gamma+6}{3}. \tag{11}$$

Now we shall estimate the sum $d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta)$.
From (6) — (11) we have:

$$d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) =$$

$$e(\{p_1, q_\beta\}, Q) + e(\{q_1, p_\alpha\}, Q) + e(\{q_1, p_\alpha\}, P) + e(\{p_1, q_\beta\}, P)$$

$$+e(\{p_1, p_\alpha\}, R) + e(\{q_1, q_\beta\}, R) - 2\, e(\{p_1, q_1, p_\alpha, q_\beta\}, \{x, y\})$$

$$\leqslant \alpha + \beta + 8 + \frac{4\gamma+12}{3} - 8 = \frac{3\alpha+3\beta+4\gamma}{3} + 4.$$

As $\alpha \geqslant 2$ and $\beta \geqslant 2$, we obtain the following inequality:

$$d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) \leqslant \frac{4(\alpha+\beta+\gamma)+8}{3} = \frac{8n}{3},$$

which contradicts (1) and the proof is finished.

□

5 Proof of the part (2) of the Theorem 6

Let $G = (B, W, E)$ be a balanced bipartite graph with $|B| = |W| = n$, $n > 4$ satisfying the conditions of the Theorem 6.

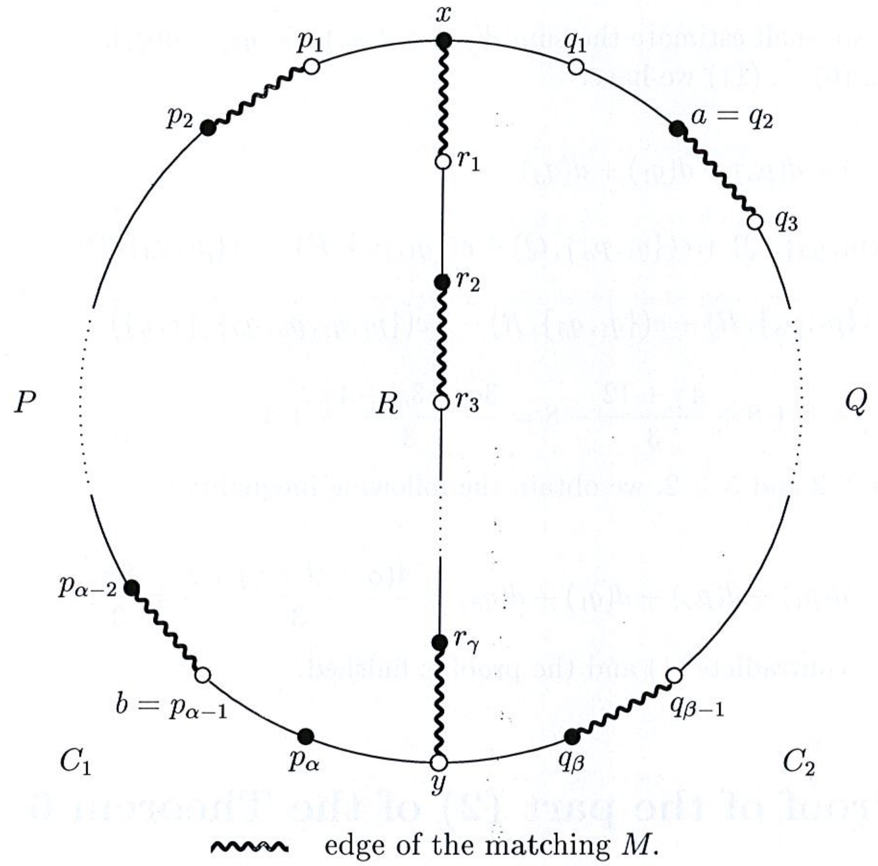
Observe that since $n > 4$, we have $\frac{5n}{4} \geqslant n + 2$. From the assumptions of the Theorem 6 we have:

$$d(x) + d(y) \geqslant \frac{5n}{4} \geqslant n + 2, \tag{12}$$

for any $x \in B$, $y \in W$, $xy \notin E$.

Let M be a matching in G. We may assume that M is a maximal matching. If M is a perfect matching then from Theorem 1 it is contained in a hamiltonian cycle. We can assume that M is not a perfect matching and we consider a maximal counterexample, i.e. a balanced bipartite graph G and a maximal matching M such that:

1. There is no cycle in G containing M.
2. For every pair of vertices (p, q), $p \in B$, $q \in W$, $pq \notin E$, $p, q \notin V(M)$, then M is contained in a cycle in $G \cup (pq)$.



A strict Θ-graph compatible with M.

Figure (2).

Observe that since M is not a perfect matching then we have at least two vertices p, q such that $p, q \notin V(M)$.

This implies that we have a path:

$$D: \quad qu_1u_2...u_lp \tag{13}$$

containing M and oriented from q to p.

Since $qp \notin E$ then from (12) there exists i such that $1 \leqslant i \leqslant l-1$, $qu_{i+1} \in E$ and $pu_i \in E$.

The cycle:

$$C' : \quad qu_{i+1}u_{i+2}...u_l p u_i u_{i-1}...u_1 q$$

can not contain the matching M, so $u_i u_{i+1} \in M$.
Consider paths:

$$\begin{aligned} P &: u_i p u_l \ ... \ u_{i+2} u_{i+1} \\ Q &: u_i u_{i-1} \ ... \ u_1 q u_{i+1} \\ R &: u_i u_{i+1} \end{aligned}$$

and observe that $H = (P, Q, R)$ is a strict $\Theta-$graph compatible with the matching M. (For an example of such strict $\Theta-$graph compatible with the matching M see Figure (2).)

Let u_s, $u_r \in V(D)$, $s < r$ be such that $pu_s \in E$, $qu_r \in E$, $u_s u_{s+1} \in M$, $u_{r-1}u_r \in M$ (remark that $s = i$, $r = i+1$ satisfies these conditions) and $r - s$ is maximum.

The graph $H = (P, Q, R)$:

$$\begin{aligned} P &: u_s p u_l u_{l-1} \ ... \ u_{r+1} u_r \\ Q &: u_s u_{s-1} \ ... \ u_1 q u_r \\ R &: u_s \ ... \ u_r \end{aligned}$$

is a strict $\Theta-$graph compatible with the matching M such that $|V(R)|$ is maximum.

Since there is no cycle containing M we have $E(P) \cap M \neq \emptyset$, $E(Q) \cap M \neq \emptyset$ and since H is a strict Θ-graph $|V(P)|$, $|V(Q)| \geqslant 6$.

We label the vertices of H as follows:

$$\begin{aligned} P &: x p_1 \ ... \ p_\alpha y \\ Q &: x q_1 \ ... \ q_\beta y \\ R &: x r_1 \ ... \ r_\gamma y \end{aligned}$$

We assume that $x \in B$, $y \in W$, $q = q_1 \in W$, $a = q_2 \in B$, $p = p_\alpha \in B$ and $b = p_{\alpha-1} \in W$.

Let G_M be the subgraph of G induced by $V(G) \setminus V(M)$ and let Z be the subgraph of G induced by $V(G) \setminus V(D)$. G_M and Z are independent i.e. $e(V(G_M), Z) = 0$.

Since $V(G) = V(P - \{y\}) \cup V(Q - \{x\}) \cup V(R - \{x,y\}) \cup V(Z)$ and the sets $V(P - \{y\})$, $V(Q - \{x\})$, $V(R - \{x,y\})$ and $V(Z)$ are vertex-disjoint for every vertex $v \in V(G)$, we have:

$$d(v) = d_{P-\{y\}}(v) + d_{Q-\{x\}}(v) + d_{R-\{x,y\}}(v) + d_Z(v). \tag{14}$$

Let $|M| = m$, $|V(M)| = 2m$, $|V(D \backslash M)| = 2\delta$ and $|V(Z)| = 2t$, then $n = m + \delta + t$.

Remark: As $p_\alpha \notin V(M)$, $q_1 \notin V(M)$, $|V(P)|$ and $|V(Q)|$ are even, then $\delta \geqslant 2$. (There are at least two vertices of $V(G) \backslash V(M)$ on P and on Q.)

Denote P_i $i = 1, ..., n_P$, Q_j, $j = 1, ..., n_Q$ and R_k, $k = 1, ..., n_R$ the paths obtained respectively from P, Q and $R \setminus \{x, y\}$ by removal of the edges of M.

Take an $i \in \{1, ..., n_P\}$. Observe that since M is maximal then if P_i is an odd path then $|V(P_i)| = 3$ and if P_i is an even path then $|V(P_i)| = 2$. Moreover if $|V(P_i)| = 3$ then P_i is a BB-path or a WW-path. If $|V(P_i)| = 2$ then P_i is a BW-path or WB-path. As $|V(P)|$ is even, the number of $BB-$paths is equal to the number of WW-paths. Let $\nu_3(P)$ be the number of odd paths P_i, $\nu_2^{BW}(P)$ the number of BW-paths P_i, $\nu_2^{WB}(P)$ the number of WB-paths P_i and $\nu_2(P) = \nu_2^{BW}(P) + \nu_2^{WB}(P)$ the number of even paths P_i.

The paths Q_i $i = 1, ..., n_Q$.and R_i. $i = 1, ..., n_R$. have the same properties as the paths P_i and in the same way as above, we define ν_2^{BW}, ν_2^{BW}, $\nu_2 = \nu_2^{BW} + \nu_2^{BW}$ and ν_3 for paths Q and R (in which the number of $BB-$paths is also equal to the number of WW-paths).

From the maximality of G and M the graph induced by $V(D) \setminus V(M)$ is independent. Thus since $bp_\alpha y$ is a WW-path we have:

$$n_P = \nu_3(P) + \nu_2(P) = |M \cap E(P)| + 1. \tag{15}$$

Similarly since xq_1a is a BB-path we have:

$$n_Q = \nu_3(Q) + \nu_2(Q) = |M \cap E(Q)| + 1. \tag{16}$$

Observe that on the path $R \setminus \{x, y\}$ we have:

$$n_R = \nu_3(R) + \nu_2(R) = |M \cap E(R)| - 1. \tag{17}$$

From (15) — (17) we have:

$$\sum_{i=2}^{3}(\nu_i(P)+\nu_i(Q)+\nu_i(R)) = m+1. \tag{18}$$

In every path odd P_i, there is one vertex of $V(D)\setminus V(M)$ and since $|V(R)|$ is even we have:

$$\nu_3(P) = |V(P\setminus M)|. \tag{19}$$

Similarly we have:

$$\nu_3(Q) = |V(Q\setminus M)|. \tag{20}$$

$$\nu_3(R) = |V(R\setminus M)|. \tag{21}$$

From (19) — (21) we have:

$$\nu_3(P)+\nu_3(Q)+\nu_3(R) = 2\delta. \tag{22}$$

5.1 Lower bound of the sums of degrees

If one of the edges ab, $p_\alpha q_1$, $p_1 q_\beta$ exists, we have a cycle in G containing every edge of M. For example if $p_1 q_\beta \in E$ then the cycle:

$$C: \quad p_1 q_\beta q_{\beta-1} ... q_1 x r_1 ... r_\gamma y p_\alpha .... p_1$$

is containing M.
We may assume $ab \notin E$, $p_\alpha q_1 \notin E$, $p_1 q_\beta \notin E$ and then:

$$d(a)+d(b) \geqslant \frac{5n}{4}, \tag{23}$$

$$d(q_\beta)+d(p_1) \geqslant \frac{5n}{4}, \tag{24}$$

$$d(p_\alpha)+d(q_1) \geqslant \frac{5n}{4}. \tag{25}$$

5.2 Upper bound of sum of degrees

5.2.1 Neighbors of a, b, p_α, q_1, q_β, p_1 on $R \setminus \{x, y\}$

1. Consider a $WB-$path R_i : vu on R, $u \in B$, $v \in W$, $v = u^-$, $uv \notin M$. Since there is no cycle containing every edge of M, the following inequalities are satisfied: $e(\{p_\alpha, p_1\}, R_i) \leqslant 1$, $e(\{q_1, q_\beta\}, R_i) \leqslant 1$ and $e(\{a, b\}, R_i) \leqslant 1$.

   Suppose that $e(\{a, b\}, R_i) = 2$, then av, $bu \in E$ and the cycle C given by the formula:

   $$C: \quad avv^- ... r_1 x p_1 ... p_{\alpha-2} b u u^+ ... r_\gamma y q_\beta ... a$$

   contains M, a contradiction.

   Now suppose that $e(\{p_1, p_\alpha\}, R_i) = 2$. In this case $p_1 u$, $p_\alpha v \in E(G)$ and the cycle C given by the formula:

   $$C: \quad p_\alpha v v^- ... r_1 x q_1 ... q_\beta y r_\gamma ... u p_1 p_2 ... p_\alpha$$

   contains M, a contradiction.

   The case $e(\{q_1, q_\beta\}, R_i) = 2$ is the same as $e(\{p_1, p_\alpha\}, R_i) = 2$ and so we have:

   $$e(\{a, b, p_1, p_\alpha, q_1, q_\beta\}, R_i) \leqslant 3. \tag{26}$$

2. Consider a $BW-$path R_i : uv on R, $u \in B$, $v \in W$, $v = u^+$, $uv \notin M$. The following inequalities holds: $e(\{p_1, p_\alpha\}, R_i) \leqslant 1$, $e(\{q_1, q_\beta\}, R_i) \leqslant 1$, $e(\{a, b\}, R_i) \leqslant 2$.

   Since a, $u \in B$ and b, $v \in W$ it is clear that $e(\{a, b\}, R_i) \leqslant 2$.

   Suppose that $e(\{p_1, p_\alpha\}, R_i) = 2.$, then $p_1 u$, $v p_\alpha \in E$ and the cycle C given by the formula:

   $$C: \quad p_\alpha v v^+ ... r_\gamma y q_\beta ... q_1 x r_1 .... u p_1 ... p_\alpha$$

   contains M, a contradiction.

   The case $e(\{q_1, q_\beta\}, R_i) = 2$ is the same as $e(\{p_1, p_\alpha\}, R_i) = 2$. Thus:

   $$e(\{a, b, p_1, p_\alpha, q_1, q_\beta\}, R_i) \leqslant 4. \tag{27}$$

3. Consider a $WW-$path $R_i: \ v_1uv_2$, $u \in B$, v_1, $v_2 \in W$, $u \in V(D \setminus M)$, $u = v_1^+ = v_2^-$. As $q_1 \notin V(M)$, $u \notin V(M)$ and M is maximal, we have $q_1u \notin E$. Since there is no cycle containing M, the following inequalities holds: $e(\{p_1, p_\alpha\}, R_i) \leqslant 2$, $e(\{a, q_\beta\}, R_i) \leqslant 2$.

   We will start by computing the $e(\{p_1, p_\alpha\}, R_i)$.

   If $p_1u \notin E$ then $e(\{p_1, p_\alpha\}, R_i) \leqslant 2$.

   Suppose now that $p_1u \in E$ and $e(p_\alpha, R_i) \neq 0$. $e(p_\alpha, R_i) \neq 0$ implies that $p_\alpha v_1 \in E$ or $p_\alpha v_2 \in E$.

   If $p_\alpha v_1 \in E$, then the cycle C given by the formula:

$$C: \quad p_\alpha v_1 v_1^- ... r_1 x q_1 ... q_\beta y r_\gamma .... u p_1 ... p_\alpha$$

   contains M, a contradiction.

   If $p_\alpha v_2 \in E$, then the cycle C given by the formula:

$$C: \quad p_\alpha v_2 v_2^+ ... r_\gamma y q_\beta ... q_1 x r_1 .... u p_1 ... p_\alpha$$

   contains M, a contradiction. So if $p_1u \in E$ we have $e(\{p_1, p_\alpha\}, R_i) = 1$.

   Thus in any case we have $e(\{p_1, p_\alpha\}, R_i) \leqslant 2$.

   Now we shall compute the $e(\{a, q_\beta\}, R_i)$. Observe that a and q_β can not be adjacent to two different vertices on R_i! Since a, u, $q_\beta \in B$ and v_1, $v_2 \in W$, we shall consider the existence of four edges: av_1, $q_\beta v_1$, av_2 and $q_\beta v_2$.

   Suppose that av_1, $q_\beta v_2 \in E$, then the cycle C given by the formula:

$$C: \quad a v_1 v_1^- ... r_1 x p_1 .... p_\alpha y r_\gamma ... v_2 q_\beta ... a$$

   contains M, a contradiction.

   If av_2, $q_\beta v_1 \in E$, then the cycle C given by the formula:

$$C: \quad a v_2 v_2^+ ... r_\gamma y p_\alpha .... p_1 x r_1 ... v_1 a$$

   contains M, a contradiction.

   So we have $e(\{a, q_\beta\}, R_i) \leqslant 2$ and since it may happen that $bu \in E$, we have:

$$e(\{a, b, p_1, p_\alpha, q_1, q_\beta\}, R_i) \leqslant 5. \tag{28}$$

4. Consider a $BB-$path $R_i: u_1vu_2$, u_1, $u_2 \in B$, $v \in W$, $v \in V(D \setminus M)$, $v = u_1^+ = u_2^-$. As $p_\alpha \notin V(M)$, $v \notin V(M)$ and M is maximal, we have $p_\alpha v \notin E$. Using the same arguments as in the case 3, since there is no cycle containing M, the following inequalities holds: $e(\{q_1, q_\beta\}, R_i) \leqslant 2$, $e(\{b, p_1\}, R_i) \leqslant 2$ and since it may happen that $av \in E$, we have:

$$e(\{a, b, p_1, p_\alpha, q_1, q_\beta\}, R_i) \leqslant 5. \tag{29}$$

By summing over all the paths R_i from (26) — (29) we have:

$$e(\{a, b, p_\alpha, q_1, q_\beta, p_1\}, R - \{x, y\}) \leqslant 3\nu_2(R) + \nu_2^{BW}(R) + 5\nu_3(R). \tag{30}$$

5.2.2 Neighbors of a, b, p_α, q_1, q_β, p_1 on $Q \setminus \{x\}$.

1. Consider the vertices $\{q_1, a\}$. Since there is no cycle containing M we have $e(\{p_1, q_\beta\}, \{q_1, a\}) \leqslant 1$, $aq_1 \in E$, $p_\alpha q_1$, $ab \notin E$ and thus:

$$e(\{a, b, p_1, p_\alpha, q_1, q_\beta\}, \{q_1, a\}) \leqslant 3. \tag{31}$$

2. Consider a BW-path $Q_i : uv$, $u \in B$, $v \in W$, $v = u^+$, $uv \notin M$. Since there is no cycle containing M we have $e(\{p_1, p_\alpha\}, Q_i) \leqslant 1$ and $e(\{a, b\}, Q_i) \leqslant 1$.

   Suppose that $e(\{p_1, p_\alpha\}, Q_i) = 2$, then p_1u, $p_\alpha v \in E$ and the cycle C given by the formula:

$$C: \quad p_1uu^-....q_1xr_1...r_\gamma yq_\beta....vp_\alpha p_{\alpha-1}....p_1$$

   contains M, a contradiction.

   If $e(\{a, b\}, Q_i) = 2$, then bu, $av \in E$ and the cycle C given by the formula:

$$C: \quad buu^-....avv^+....q_\beta yr_\gamma...r_1xp_1...b$$

   contains M, a contradiction.

   Observe that $e(\{q_1, q_\beta\}, Q_i) \leqslant 2$ and thus:

$$e(\{a, b, p_1, p_\alpha, q_1, q_\beta\}, Q_i) \leqslant 4. \tag{32}$$

3. Consider a WB-path $Q_i : vu$, $u \in B$, $v \in W$, $u = v^+$, $vu \notin M$. Since there is no cycle containing M, using similar arguments as in the case 2, for the vertices a, b, we have $e(\{q_1, p_\alpha\}, Q_i) \leqslant 1$, $e(\{p_1, q_\beta\}; Q_i) \leqslant 1$ and thus:

$$e(\{a, b, p_1, p_\alpha, q_1, q_\beta\}, Q_i) \leqslant 4. \tag{33}$$

4. Consider a WW-path $Q_i : v_1 u v_2$, $v_2 \neq y$, $u \in B$, v_1, $v_2 \in W$, $u = v_1^+ = v_2^-$.

   As $q_1 \notin V(M)$, $u \notin V(M)$ and M is maximal, we have $q_1 u \notin E$ and from this: $e(\{q_1, q_\beta\}, Q_i) \leqslant 2$.

   Since $v_2 \neq y$ and as R is maximal $p_\alpha v_2 \notin E$. Suppose that $p_\alpha v_2 \in E$ then the graph $H' = (P', Q, R')$ with:

$$\begin{aligned} P' &: \; xp_1 \,...\, p_\alpha v_2 \\ Q' &: \; xq_1 \,...\, v_2 \\ R' &: \; xr_1 \,...\, r_\gamma y q_\beta ....v_2 \end{aligned}$$

   is a strict Θ-graph compatible with M with $|V(R')| > |V(R)|$.

   Since there is no cycle containing M, using similar arguments as in the case 2, we have $e(\{p_1, p_\alpha\}, \{v_1, u\}) \leqslant 1$, $e(\{a, b\}, \{u, v_2\}) \leqslant 1$. From this $e(\{p_1, p_\alpha\}, Q_i) \leqslant 1$ and since it is possible that $av_1 \in E$ we have $e(\{a, b\}, Q_i) \leqslant 2$.

   From these inequalities we have:

$$e(\{a, b, p_1, p_\alpha, q_1, q_\beta\}, Q_i) \leqslant 5. \tag{34}$$

5. In the case 4 we have assumed that $v_2 \neq y$. If $v_2 = y$, then $i = N_Q$ and the path Q_{N_Q} is a WW-path $Q_{N_Q} : q_{\beta-1} q_\beta y$. In fact it is the same case as the case 4, but since $p_\alpha y \in E$, we have:

$$e(\{a, b, p_1, p_\alpha, q_1, q_\beta\}, Q_i) \leqslant 6. \tag{35}$$

6. Consider a BB-path $Q_i : u_1 v u_2$, u_1, $u_2 \in B$, $v \in W$, $v = u_1^+ = u_2^-$. Observe that since p_α, $v \notin V(M)$ and since M is maximal we have $p_\alpha v \notin E$.

Since there is no cycle containing M we have $e(\{a,b\},\{u_1,v\}) \leqslant 1$, $e(\{p_1,q_\beta\},\{v,u_2\}) \leqslant 1$.

Suppose that $e(\{a,b\},\{u_1,v\}) = 2$, then av, $bu_1 \in E$ and the cycle C given by the formula:

$$C: \quad bu_1u_1^{-}....avv^{+}....q_\beta yr_\gamma...r_1xp_1...b$$

contains M, a contradiction.

Suppose that $e(\{p_1,q_\beta\},\{v,u_2\}) = 2$, then p_1u_2, $q_\beta v \in E$ and the cycle C given by the formula:

$$C: \quad p_1u_2u_2^{+}....q_\beta vu_1....q_1xr_1...r_\gamma yp_\alpha...p_1$$

contains M, a contradiction.

Observe that $p_1u_1 \notin E$, because if $p_1u_1 \in E$, then the graph $H' = (P', Q, R')$ with:

$$\begin{aligned} P' &: u_1p_1 \ldots p_\alpha y \\ Q' &: u_1vu_2 \ldots q_\beta y \\ R' &: u_1u_1^{-} \ldots q_1xr_1 \ldots r_\gamma y \end{aligned}$$

is a strict Θ-graph compatible with M with $|V(R')| > |V(R)|$, a contradiction.

From the above we have: $e(\{p_1,q_\beta\},Q_i) \leqslant 1$, $e(\{a,b\},Q_i) \leqslant 2$ and since $e(\{q_1\},Q_i) \leqslant 2$ we have

$$e(\{a,b,p_1,p_\alpha,q_1,q_\beta\},Q_i) \leqslant 5. \tag{36}$$

By summing over all the paths Q_i from (31) — (36) we have:

$$e(\{a,b,p_1,p_\alpha,q_1,q_\beta\},Q\setminus\{x\}) \leqslant 4\nu_2(Q)+5\nu_3(Q)-1. \tag{37}$$

5.2.3 Neighbors of $a,b,p_\alpha,q_1,q_\beta,p_1$ on $P\setminus\{y\}$

Using the similar arguments as in Section (5.2.2) we have

$$e(\{a,b,p_1,p_\alpha,q_1,q_\beta\},P\setminus\{y\}) \leqslant 4\nu_2(P)+5\nu_3(P)-1. \tag{38}$$

5.2.4 Neighbors of $a, b, p_\alpha, q_1, q_\beta, p_1$ in Z

Since G_M and Z are independent we have:

$$d_Z(p_\alpha) = d_Z(q_1) = 0$$

and thus:

$$e(\{a, b, p_1, p_\alpha, q_1, q_\beta\}, Z) \leqslant 4t. \tag{39}$$

5.2.5 Neighbors of p_α and q_1 on $R \cup Q \cup P$

Using similar method as those in sections (5.2.1) — (5.2.3) we get the following inequalities:

$$e(\{p_\alpha, q_1\}, R \setminus \{x, y\}) \leqslant \nu_2^{BW}(R) + 2\nu_2^{WB}(R) + 2\nu_3(R). \tag{40}$$

$$e(\{p_\alpha, q_1\}, Q \setminus \{x\}) \leqslant \nu_2(Q) + 2(\nu_3(Q) - 1) + 1 = \nu_2(Q) + 2\nu_3(Q) - 1. \tag{41}$$

$$e(\{p_\alpha, q_1\}, P \setminus \{y\}) \leqslant \nu_2(P) + 2(\nu_3(P) - 1) + 1 = \nu_2(P) + 2\nu_3(P) - 1. \tag{42}$$

Now we shall estimate the sum of degrees. From (40) — (42) we have

$$d(p_\alpha) + d(q_1) \leqslant \nu_2^{BW}(R) + m + 2\delta - 1 = \nu_2^{WB}(R) + n - t + \delta - 1. \tag{43}$$

5.2.6 Conclusion

From (30), (37), (38) and (39) we have

$$\begin{aligned} d(a) + d(b) + d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) \leqslant \\ 4\Big(\sum_{i=2}^{3} (\nu_i(P) + \nu_i(Q) + \nu_i(R))\Big) \\ +\nu_3(P) + \nu_3(Q) + \nu_3(R) - 2 + 4t - \nu_2^{WB}(R). \end{aligned} \tag{44}$$

From (18), (22) and (44) we deduce:

$$d(a) + d(b) + d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) \leqslant \\ -2 + 4(m+1) + 2\delta + 4t - \nu_2^{WB}(R). \qquad (45)$$

Since $n = m + \delta + t$, from (45) we have:

$$d(a) + d(b) + d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) \leqslant \\ 4n + 2 - \nu_2^{WB}(R) - 2\delta. \qquad (46)$$

From (43) and (46) we can deduce that:

$$d(a) + d(b) + d(q_\beta) + d(p_1) + 2d(p_\alpha) + 2d(q_1) \leqslant 5n - t - \delta + 1. \qquad (47)$$

Observe that $\delta \geqslant 2$ and from (47) we have:

$$d(a) + d(b) + d(q_\beta) + d(p_1) + 2d(p_\alpha) + 2d(q_1) \leqslant 5n - 1. \qquad (48)$$

Now we shall give the lower bound of the sum of degrees. From (23) — (25) we have

$$4\frac{5n}{4} \leqslant d(q_\beta) + d(p_1) + d(a) + d(b) + 2d(p_\alpha) + 2d(q_1) \qquad (49)$$

Assuming that there does not exist a cycle which contains every edge of the matching M, we have obtained (48) and (49) and this implies that

$$5n \leqslant 5n - 1,$$

a contradiction. The part (2) of the Theorem 6 is proved.

□

5.3 Conjecture

Conjecture: Let $G = (B, W, E)$ be a balanced bipartite graph of order $2n$. If for any $x \in B$, $y \in W$, $xy \notin E$ we have

$$d(x) + d(y) \geqslant n + 2,$$

then every matching M in G is contained in a cycle of G.

Remark: If $|M| = n - 1$ and for any $x \in B$, $y \in W$, $xy \notin E$ $d(x) + d(y) \geqslant n + 2$, then M is contained in a hamiltonian cycle.

Suppose that G is not a complete graph (if G is complete then **Remark** is true.) Let $M \cup (pq)$, with $p \in B$, $q \in W$, $pq \notin E$ be a perfect matching containing M. From the Theorem 1 it is contained in a hamiltonian cycle C. Let $D: \quad qu_1u_2...u_{2l}p$ be a hamiltonian path in G deduced from C by deleting the edge pq. The edges u_1u_2, ... $u_{2i+1}u_{2i+2}$, ... $u_{2l-1}u_{2l}$ are edges of the matching M. Since $d(p) + d(q) \geqslant n + 2$ then there exists a k, such that $qu_{k+1} \in E$ and $pu_k \in E$. Observe that $p \in B$, $q \in W$, $u_k \in W$ and then k is even. The edge u_ku_{k+1} is not in M. The cycle $C: \quad qu_1, ...u_kpu_lu_{l-1}...u_{k+1}q$ is a hamiltonian cycle of G which contains M.

References

[1] K.A. Berman, *Proof of a conjecture of Haggkvist on cycles and independent edges*, Discrete Mathematics 46, (1983) 9 — 13.

[2] J.A. Bondy and U.S.R. Murthy, *Graph theory with applications,* The Macmillan Press LTD London (1976).

[3] R. Häggkvist, *On F-hamiltonian graphs*, in: J.A. Bondy and U.S.R. Murthy, eds., Graph Theory and Related Topics, Academic Press (1997) 219–231.

[4] B. Jackson and N.C. Wormald, *Cycles containing matchings and parwise compatible Euler tours*, J. Graph Theory 14 (1990) 127–138.

[5] M. Las Vergnas, *Thèse d'état*, Paris (1972).

[6] A.P. Wojda, *Hamiltonian cycles through matchings,* Demonstratio Mathematica 21 (1983) 547–553.

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