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Color degree and heterochromatic cycles in edge-colored graphs *

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Abstract

Given a graph G and an edge coloring C of G, a heterochromatic cycle of G is a cycle in which any pair of edges have distinct colors. Let $d^c(v)$, named the color degree of a vertex v, be the maximum number of distinct colored edges incident with v. In this paper, some color degree conditions for the existence of heterochromatic cycles are obtained.

Keywords: heterochromatic cycle, color neighborhood, color degree

1 Introduction and notation

We use [3] for terminology and notations not defined here. Let $G = (V, E)$ be a graph. An *edge-coloring* of G is a function $C : E \to N$(N is the set of nonnegative integers). If

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G is assigned such a coloring C, then we say that G is an *edge-colored graph*, or simply *colored graph*. Denote by (G, C) the graph G together with the coloring C and by $C(e)$ the *color* of the edge $e \in E$. The $CN(v)$ of v is defined as the set $\{C(e) : \text{e is incident with } v\}$. For a subgraph H of G, let $C(H) = \{C(e) : e \in E(H)\}$ and $c(H) = |C(H)|$. Given a subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$.

A subgraph H of G is called *heterochromatic*, or *rainbow*, or *colorful* if any pair of edges in H have distinct colors. Existences of heterochromatic subgraphs have been studied since long time ago. In particular, there are many results on heterochromatic hamiltonian cycles and heterochromatic matchings. A problem of heterochromatic hamiltonian cycles in a colored complete graphs was mentioned in [9] by Erdös, Nešetřil and Rödl. This problem was also studied by Hahn and Thomassen(see [11]), Rödl and Winkler(see [10]), Albert,Frieze and Reed (see [1]), respectively. For the heterochromatic matchings, see the references [12, 14, 15, 16]. It can be easily seen that the heterochromatic matchings in colored bipartite graphs are in another terminology matchings in 3-partite 3-uniform hypergraphs.

If we regard an uncolored graph G as a colored graph (G, C) in which all edges have different colors, then G contains a cycle of length at least l if and only if (G, C) contains a heterochromatic cycle of length at least l. The problem of deciding whether there is a cycle of length at least l in an (uncolored) graph is NP-complete. Therefore the problem of deciding whether there is a heterochromatic cycle of length at least l in a colored graph is NP-complete, too.

For a vertex set $S \subseteq V(G)$, a *color neighbourhood* of S is defined as a set $T \subseteq N(S)$ such that there are $|T|$ distinct colored edges between S and T that are incident with distinct vertices of T. A *maximum color neighborhood* $N^c(S)$ of S is a color neighborhood of S with maximum size. Given a set S and a color neighborhood T of S, denote by $C(S, T)$ a set of $|T|$ distinct colors on the $|T|$ edges between S and distinct vertices of T. In particular, if $S = \{v\}$, we denote $d^c(v) = |N^c(v)|$ and call it the *color degree* of v. Clearly $d^c(v) = |CN(v)|$.

For $l \geq 3$, let HC_l denote a heterochromatic cycle with length l. The existence of heterochromatic cycles has been studied in [4] by Broersma, Li, Woegingerr and Zhang and they obtained the following results.

Theorem 1[4]. *Let G be a colored graph of order n such that $c(G) \geq n$. Then G contains a heterochromatic cycle of length at least $\frac{2c(G)}{n-1}$.*

Theorem 2[4]. *Let G be a colored graph of order $n \geq 4$, such that $|CN(u) \cup CN(v)| \geq n - 1$ for every pair of vertices u and v of G. Then G contains at least one HC_3 or one HC_4.*

2 The main results

We are interested in Dirac type conditions (*i.e.*, minimum color degree conditions) for existence of heterochromatic cycle, in particular the shortest heterochromatic cycles (heterochromatic girth) and the longest heterochromatic cycles (heterochromatic circumference).

We begin with a study of the existence of a heterochromatic cycle. Existence of a heterochromatic cycle can be insured by Theorem 1 when $c(G) \geq n$. Under color degree conditions, we have

Theorem 3. *Let G be a colored graph with order $n \geq 3$. If $d^c(v) \geq \frac{n+1}{2}$ for every $v \in V(G)$, then G has a heterochromatic cycle.*

For the shortest heterochromatic cycles (heterochromatic girth), we get results on HC_3 or HC_4 with minimum color degree conditions.

Theorem 4. *Let G be a colored graph with order $n \geq 3$. If for every $v \in V(G)$, $d^c(v) \geq (\frac{4\sqrt{7}}{7} - 1)n + 3 - \frac{4\sqrt{7}}{7}$, then G has either an HC_3 or an HC_4.*

Note that $\frac{4\sqrt{7}}{7} - 1 \approx 0.515\cdots$ and $3 - \frac{4\sqrt{7}}{7} \approx 1.488\cdots$.

Theorem 5. *Let G be a colored graph with order $n \geq 3$. If for every $v \in V(G)$, $d^c(v) \geq \frac{\sqrt{7}+1}{6}n$, then G has an HC_3.*

Note that $\frac{\sqrt{7}+1}{6} \approx 0.608\cdots$. In fact, we think that the bound in Theorem 5 is not sharp. We propose the following conjecture.

Conjecture. *Let G be a colored graph with order $n \geq 3$. If $d^c(v) \geq \frac{n+1}{2}$ for every $v \in V(G)$, then G has an HC_3.*

We have the following example to show that if the above conjecture is true, it would be best possible. For any even integer n, let $B_{n/2,n/2}$ be an edge-proper-colored complete balance bipartite graph with order n. Then for every vertex v of $B_{n/2,n/2}$, it holds that $d^c(v) = \frac{n}{2}$, and $B_{n/2,n/2}$ has no HC_3.

It is natural to ask the following problem about the existence of the heterochromatic cycles:

Does there exists a function $f(n)$ such that for any colored graph G with order n, if $d^c(v) \geq f(n)$ for every vertex $v \in V(G)$, then G contains a heterochromatic cycle?

The following two propositions show that the function $f(n)$ must be greater than $\log_2 n$.

Proposition 1. *For any non-negative integer k, there exists an edge colored bipartite graph B with order $n = 2^k$ such that $d^c(v) = k = \log_2 n$, for every vertex $v \in V(B)$, and B has no heterochromatic cycles.*

To show Proposition 1, we construct the following example by induction.

Let G_1 be an edge e with colors $C(e) = 1$. Given a G_i for $i \geq 1$, define G_{i+1} as follows. First we construct a graph $G_i^{'}$ which is a copy of G_i. Then add the edges between $v \in V(G_i)$ and $v^{'} \in V(G_i^{'})$, in which $v^{'}$ is the copy of v in $G_i^{'}$. And color the new edge with color $i + 1$.

Then put $B = G_i$ which is an edge colored bipartite graphs with order $n = 2^i$. Thus $d^c(v) = i = \log_2 n$ for every vertex $v \in B$. Clearly B has no any heterochromatic cycles.

Proposition 2. *For any non-negative integer k, there exists an edge colored complete graph K with order $n = 2^k$ such that $d^c(v) = k = \log_2 n$, for every vertex $v \in V(K)$, and K has no heterochromatic cycles.*

We construct graphs in a way slightly different with the above example. Let G_1^* be an edge e with colors $C(e) = 1$. Given a G_i^* for $i \geq 1$, we construct G_{i+1}^* as follows. Let the graph G_i^{**} be a copy of G_i^*. For any $u \in V(G_i^*), u^{'} \in V(G_i^{**})$, we add the new edge $uu^{'}$ and let $C(uu^{'}) = i + 1$.

Then $K = G_i^*$ is a colored complete graph with order $n = 2^i$. It gives $d^c(v) = i = \log_2 n$ for every vertex $v \in K$. Clearly, K has no heterochromatic cycles.

Here we obtain a bound for the longest heterochromatic cycles (heterochromatic circumference) and we think it may not be the best.

Theorem 6. *Let G be a colored graph with order $n \geq 3$. If $d^c(v) \geq d \geq \frac{3n}{4} + 1$ for every $v \in V(G)$, then G has an HC_l such that $l \geq d - \frac{3n}{4} + 2$.*

The proofs of the main results in Theorems 3,4,5 and 6 will be given in Section 3.

3 Proofs of the main results

Proof of Theorem 5.

By contradiction. Suppose G is a colored graphs with $d^c(v) \geq \frac{\sqrt{7}+1}{6}n$ for every vertex v of G, and G contains no heterochromatic triangles. Let v be an arbitrary vertex of G. Choose a maximum color neighborhood $N^c(v)$ of v. And assume that $T = N^c(v) = \{v_1, v_2, \cdots, v_k\}$, where $k = d^c(v)$. Since G has no heterochromatic triangles, if $e = v_iv_j \in E(G[T])$, $1 \leq i, j \leq k$, then $C(e) = vv_i$ or $C(e) = vv_j$.

Give an orientation of $G[T]$ by the following rule: For an edge $e = v_iv_j$, if $C(e) = vv_i$, then the orientation of v_iv_j is from v_j to v_i; Otherwise the orientation is from v_i to v_j. After the orientation, the oriented graph is denoted by D. For any vertex $u \in V(D)$, let $N_D^+(u)$ denote the outneighbors of u in D and $d_D^+(u) = |N_D^+(u)|$.

Lemma 1.1. *Let $q \geq 3$. If there exists a directed cycle $\overrightarrow{C_q}$ in D, then C_q is a heterochromatic cycle of G.*

Proof. Without loss of generality, we assume that the directed cycle of D is $\overrightarrow{C_q} : v_1 \to v_2 \to \cdots \to v_q \to v_1$. Then by the above orientation rule, we conclude that $C(v_iv_{i+1}) = C(vv_{i+1})$ for $1 \leq i \leq q-1$ and $C(v_qv_1) = C(vv_1)$. Since $T = N^c(v)$ is a maximum color neighborhood of v, we have that $C(vv_i) \neq C(vv_j)$ for $i \neq j$. Thus C_q is a heterochromatic cycle of G. □.

Lemma 1.2[17]. *If $\alpha = 3 - \sqrt{7} = 0.3542\cdots$, then any digraph on m vertices with minimum outdegree at least αm contains a directed triangle.*

Since G has no heterochromatic triangles, by Lemma 1.1, D has no directed triangles. Then by Lemma 1.2, we conclude that there exists a vertex v_i in D such that $d_D^+(v_i) < \alpha d^c(v)$. Let $G_0 = G[T \cup \{v\}]$, and denote a maximum color neighborhood of v_i in graph G_0 by $N_{G_0}^c(v_i)$. Then by the orientation rule, $|N_{G_0}^c(v_i)| = |N_D^+(v_i)| + |v| = |d_D^+(v_i)| + 1 < \alpha d^c(v) + 1$. Let $N^c(v_i)$ be a maximum color neighborhood of v_i in G. Then it follows that $|N^c(v_i) \backslash (T \cup \{v\})| \geq d^c(v_i) - |N_{G_0}^c(v_i)| > d^c(v_i) - \alpha d^c(v) - 1$. It follows that

$$n \geq |N^c(v_i) \backslash (T \cup \{v\})| + |T| + |v| > d^c(v_i) + (1-\alpha) d^c(v) \geq (2-\alpha)\frac{\sqrt{7}+1}{6} n = n.$$

This contradiction completes the proof of Theorem 5. □

Proof of Theorem 3.

The technique is similar to the proof of Theorem 5. By contradiction. Otherwise let G be a graph with $d^c(v) \geq \frac{n+1}{2}$ for every vertex v of G, and G has no heterochromatic cycles. Let v be an arbitrary vertex of G. Similarly we choose a maximum color neighborhood $N^c(v)$ of v. Since G contains no heterochromatic cycles, by the same orientation rule as above, we can get an oriented graph D_0. The following fact is clear.

Fact 2.1. *Every simple m-vertex digraph with minimum out-degree at least 1 has a directed cycle.*

By Lemma 1.1 and the above fact, we know that there exists a vertex v_j of D_0 such that $d_{D_0}^+(v_j) = 0$. Let $N^c(v_j)$ be a maximum color neighborhood of v_j in G. Then we conclude that $|N^c(v_j) \backslash (T \cup \{v\})| \geq d^c(v_j) - 1$. Thus it follows that

$$n \geq |N^c(v_j) \backslash (T \cup \{v\})| + |T| + |v| \geq d^c(v_j) - 1 + d^c(v) + 1 \geq 2(\frac{n+1}{2}) = n + 1.$$

This contradiction completes the proof of Theorem 3. $\square$

Proof of Theorem 4.

By contradiction. Suppose that G is a colored graph such that $d^c(v) \geq (\frac{4\sqrt{7}}{7} - 1)n + 3 - \frac{4\sqrt{7}}{7}$ for every vertex $v \in V(G)$, and G contains neither HC_3 and nor HC_4.

For an edge uv, let $N_1^c(u), N_1^c(v)$ denote a maximum color neighborhood of u, v, respectively, such that $v \in N_1^c(u)$ and $u \in N_1^c(v)$. Let $N^c(u,v)$ denote $N_1^c(u) \cup N_1^c(v)$ such that $|N_1^c(u) \cup N_1^c(v)|$ is maximum. And we choose an edge $uv \in E(G)$ such that $|N^c(u,v)|$ is maximum.

Assume that $N_1^c(u) = \{v, u_1, u_2, \cdots, u_s\}$ and $N_1^c(v) \backslash N_1^c(u) = \{u, v_1, v_2, \cdots, v_t\}$, in which $s = d^c(u) - 1$. Let $X = \{u_1, \cdots, u_s, v_1, \cdots, v_t\}$. Note that $|N^c(u,v)| = s + t + 2$. Consider the graph $G[X]$, and we have the following lemma.

Lemma 3.1. *Suppose $e \in E(G[X])$, then the following hold:*
(i) If $e = u_iu_j (1 \leq i, j \leq s)$, then $C(e) \in \{C(uu_i), C(uu_j)\}$.
(ii) If $e = v_iv_j (1 \leq i, j \leq t)$, then $C(e) \in \{C(vv_i), C(vv_j)\}$.
(iii) If $e = u_iv_j (1 \leq i \leq s, 1 \leq j \leq t)$ and $C(uu_i) \neq C(vv_j)$, then $C(e) \in \{C(uu_i), C(vv_j), C(uv)\}$.

Proof. Clearly (i) and (ii) hold, otherwise we can obtain an HC_3, which gets a contradiction.

If (iii) does not hold, then there exists an edge $e = u_iv_j$ $(1 \leq i \leq s, 1 \leq j \leq t)$ such that $C(uu_i) \neq C(vv_j)$ and $C(e) \notin \{C(uu_i), C(vv_j), C(uv)\}$. Since $v, u_i \in N_1^c(u)$, then $C(uu_i) \neq C(uv)$. Similarly, we obtain that $C(vv_j) \neq C(uv)$. Thus we can get an $HC_4 = uvv_ju_iu$, a contradiction. $\square$

Construct an oriented graph as follows.

(1). In graph $G[X]$, do the following operation: deleting the edges $e = v_iu_j$ if $C(e) = C(uv)$ or $C(uu_i) = C(vv_j)$, $1 \leq i \leq s$ and $1 \leq j \leq t$. After the operation, the graph is named $G_1[X]$.

(2). Then give an orientation of $G_1[X]$: For an edge $xy \in E(G_1[X])$, if $C(xy) = C(uy)$ or $C(xy) = C(vy)$, then the orientation of xy is from x to y; Otherwise, by Lemma 3.1, $C(xy) = C(ux)$ or $C(xy) = C(vx)$, then the orientation of xy is from y to x.

After the orientation, the oriented graph is denoted by D_1. For any vertex $w \in V(D_1)$, let $N^+_{D_1}(w)$ denote the outneighbors of w in D_1 and $d^+_{D_1}(w) = |N^+_{D_1}(w)|$. Let $G_0 = G[X \cup \{u, v\}]$.

Lemma 3.2. *If there exists a directed triangle $\overrightarrow{C_3}$ in D_1, then C_3 is a heterochromatic triangle in G.*

Proof. Suppose that $\overrightarrow{C_3} : x \to y \to z \to x$ is a directed triangle in D_1. If $x, y, z \in N_1^c(u)$, then by the orientation rule, it holds that $C(xy) = C(uy), C(yz) = C(uz)$ and $C(zx) = C(ux)$. Then by the definition of $N_1^c(u)$, we conclude that $C(ux), C(uy), C(uz)$ are distinct pairwise. Thus, $C_3 = xyzx$ is a heterochromatic triangle of G.

Thus, without loss of generality, we assume that $x, y \in N_1^c(u)$ and $z \in N_1^c(v)$. By the orientation rule, $C(xy) = C(uy), C(yz) = C(vz)$, and $C(zx) = C(ux)$. By the definition of $N_1^c(u)$ and Lemma 3.1(iii), we have that $C(ux), C(uy)$ and $C(vz)$ are distinct pairwise, then it follows that $C_3 = xyzx$ is a heterochromatic triangle of G. $\square$

Let $\alpha = 3 - \sqrt{7}$. By Lemma 3.2, there is no directed triangles in D_1. Then by Lemma 1.2, there is a vertex w such that $d^+_{D_1}(w) < \alpha|V(D_1)| = \alpha(s+t) = \alpha(d^c(u)+t-1)$. Without loss of generality, assume that $w \in N_1^c(u)$. Denote a maximum color neighborhood of w in G_0 by $N^c_{G_0}(w)$. Note that, in the deleting operation, at most two colors of the edges incident with w are deleted. Thus it holds that $|N^c_{G_0}(w)| \leq |N^+_{D_1}(w)| + |v|(or|u|) + 2 = d^+_{D_1}(w) + 3$. Let $N^c(w)$ be a maximum color neighborhood of w in G. It follows that

$$|N^c(w)\backslash(X \cup \{u, v\})| \geq d^c(w) - |N^c_{G_0}(w)| > d^c(w) - \alpha(d^c(u) + t - 1) - 3.$$

If $d^c(w) - \alpha(d^c(u) + t - 1) - 3 > t$, then we consider the edge uw. It follows that

$$\begin{aligned} |N^c(u, w)| \quad &\geq |\{u_1, u_2, \cdots, u_s\} \cup \{v\}| + |N^c(w)\backslash(X \cup \{u, v\})| + |w| \\ &> s + t + 2 \\ &= |N^c(u, v)|, \end{aligned}$$

a contradiction with the choice of uv.

Thus $d^c(w) - \alpha(d^c(u) + t - 1) - 3 \leq t$, then $t \geq \frac{d^c(w)}{1+\alpha} - \frac{\alpha d^c(u)}{1+\alpha} + \frac{\alpha-3}{1+\alpha}$. It follows that

$$\begin{aligned} n \quad &\geq |X| + |u| + |v| + |N^c(w)\backslash(X \cup \{u, v\})| \\ &> d^c(u) + t - 1 + 2 + d^c(w) - \alpha(d^c(u) + t - 1) - 3 \\ &\geq (1-\alpha)d^c(u) + d^c(w) + (1-\alpha)(\frac{d^c(w)}{1+\alpha} - \frac{\alpha d^c(u)}{1+\alpha} + \frac{\alpha-3}{1+\alpha}) + \alpha - 2 \\ &\geq \frac{1-\alpha}{1+\alpha}d^c(u) + \frac{2}{1+\alpha}d^c(w) + \frac{3\alpha-5}{1+\alpha}. \end{aligned}$$

Since $d^c(v) \geq (\frac{4\sqrt{7}}{7} - 1)n + 3 - \frac{4\sqrt{7}}{7}$ for every vertex $v \in V(G)$ and $\alpha = 3 - \sqrt{7}$, the above inequality is

$$n > \frac{3-\alpha}{1+\alpha}[(\frac{4\sqrt{7}}{7} - 1)n + 3 - \frac{4\sqrt{7}}{7}] + \frac{3\alpha-5}{1+\alpha} \geq n.$$

This contradiction completes the proof of Theorem 4. $\square$

Proof of Theorem 6.

By contradiction, since $d^c(v) \geq \frac{3n}{4} + 1 > \frac{n+1}{2}$, by Theorem 4, G has a heterochromatic cycle. Then we choose a longest heterochromatic cycle HC_l with length l. If the conclusion fails, it holds that $l < d - \frac{3n}{4} + 2$. Note that now $d > \frac{3n}{4} + 1$.

Assume that $xy \in E(HC_l)$. Let $N^c(x), N^c(y)$ be a maximum color neighborhood of x, y, respectively. Then choose a set S_x such that:

(R_1). $S_x \in N^c(x) \backslash V(HC_l)$.
(R_2). For each $v \in S_x$, $C(xv) \notin C(HC_l)$.
(R_3). Subject to R_1, R_2, $|S_x|$ is maximum.

Similarly, choose a set S_y satisfying the following:

$(R_1^{'})$. $S_y \in N^c(y) \backslash V(HC_l)$.
$(R_2^{'})$. For each $v \in S_y$, $C(yv) \notin C(HC_l)$.
$(R_3^{'})$. Subject to $R_1^{'}, R_2^{'}$, $|S_y|$ is maximum.

Let $P = S_x \cap S_y$ and $p = |P|$. And we have the following lemmas.

Lemma 4.1. $p \geq 2d - n + 6 - 3l > 0$.

Proof. Clearly, we conclude that $|S_x| \geq d^c(x) - l - (l-3) \geq d + 3 - 2l$. Similarly, $|S_y| \geq d + 3 - 2l$. Then $p \geq |S_x| + |S_y| - (n-l) \geq 2d - n + 6 - 3l > 0$. □

Lemma 4.2. *If $u \in P$, then $C(ux) = C(uy)$.*

Proof. Otherwise, if $C(ux) \neq C(uy)$, since $C(ux), C(uy) \notin C(HC_l)$, we can get a heterochromatic cycle: $HC_l \cup \{xu, uy\} \backslash \{xy\}$ with length $l+1$, a contradiction. □

Lemma 4.3. *If $uv \in E(G[P])$, then $C(uv) \in \{C(ux), C(vy), C(HC_l) \backslash C(xy)\}$.*

Proof. If $uv \in E(G[P])$, then by Lemma 4.2, $C(ux) = C(uy)$ and $C(vx) = C(vy)$. Clearly, we have that $C(ux) \neq C(vy)$. So if $C(uv) \notin \{C(ux), C(vy), C(HC_l) \backslash C(xy)\}$. Then we can get a heterochromatic cycle: $HC_l \cup \{xu, uv, vy\} \backslash \{xy\}$ with length $l+2$, a contradiction. □

Construct an oriented graph as follows.

(a). In graph $G[P]$, do the following operation: deleting the edges uv if $C(uv) \in C(HC_l) \backslash C(xy)$. After the operation, the graph is named $G_1[P]$.

(b). Then give an orientation of $G_1[P]$: For an edge $uv \in E(G_1[P])$, if $C(uv) = C(xu)$, then the orientation of uv is from v to u; Otherwise, by Lemma 4.3, $C(uv) = C(xv)$, then the orientation of uv is from u to v.

After the orientation, the oriented graph is denoted by D_2. Let v_0 be a vertex in D_2 with minimum outdegree, $d^+_{D_2}(v_0)$. Clearly, $d^+_{D_2}(v_0) \leq \frac{p-1}{2}$. Let $N^c(v_0)$ denote a maximum color neighborhood of v_0 in G. And assume that $N^c(v_0) = V_1 \cup V_2 \cup V_3 \cup V_4$, in which

$$\begin{aligned}
V_1 &= \{v : v \in P \ and\ C(v_0v) \notin C(HC_l)\}, \\
V_2 &= \{v : v \in V(HC_l) \ and\ C(v_0v) \notin C(HC_l)\}, \\
V_3 &= \{v : v \in P \cup V(HC_l) \ and\ C(v_0v) \in C(HC_l)\}, \\
V_4 &= \{v : v \notin P \cup V(HC_l)\},
\end{aligned}$$

and $V_i \cap V_j = \phi$, for $1 \leq i \neq j \leq 4$. We can conclude that $|V_1| \leq d^+_{D_2}(v_0) + 1 \leq \frac{p-1}{2} + 1$ and $|V_3| \leq l$.

Lemma 4.4. $|V_1| + |V_2| \leq \frac{p-1}{2} + \frac{l-1}{2}$.

Proof. First, we conclude that $|V_2| \leq \frac{l-1}{2}$. Otherwise if $|V_2| > \frac{l-1}{2}$, by $C(xv_0) = C(yv_0) \notin C(HC_l)$, then there exists two consecutive vertices v_i, v_{i+1} of HC_l, such that $C(v_0v_i), C(v_0v_{i+1}) \notin C(HC_l)$ and $C(v_0v_i) \neq C(v_0v_{i+1})$. Thus we can get a heterochromatic cycle: $HC_l \cup \{v_iv_0, v_0v_{i+1}\} \backslash \{v_iv_{i+1}\}$ with length $l+1$, a contradiction. So if $|V_1| \leq \frac{p-1}{2}$, then $|V_1| + |V_2| \leq \frac{p-1}{2} + \frac{l-1}{2}$.

Moreover if $|V_1| = \frac{p-1}{2} + 1$, then $C(xv_0) \in C(v_0, V_1)$. By the definition of a maximum color neighborhood $N^c(v_0)$ of v_0 and $V_1 \cap V_2 = \phi$, we conclude that $C(xv_0) \notin C(v_0, V_2)$. Then if $|V_2| > \frac{l-3}{2}$, use the same method as above, we can get a heterochromatic cycle with length $l+1$, a contradiction. So it holds that $|V_2| \leq \frac{l-3}{2}$, thus $|V_1| + |V_2| \leq \frac{p-1}{2} + \frac{l-1}{2}$. □

Now we complete the proof of Theorem 6 as follows. Since $\sum_{i=1}^{4} |V_i| = d^c(v_0) \geq d$ and $V_i \cap V_j = \phi$, for $1 \leq i \neq j \leq 4$, then $|V_4| \geq d - \sum_{i=1}^{3} |V_i| \geq d - l - \frac{p-1}{2} - \frac{l-1}{2}$. Clearly $V_4 \subseteq V(G) \backslash (P \cup V(HC_l))$. So we have $d - l - \frac{p-1}{2} - \frac{l-1}{2} \leq n - p - l$. It follows that $p \leq 2(n-d) + l - 2$. We also have that $p \geq 2d - n + 6 - 3l$ by Lemma 4.1. Thus $l \geq d - \frac{3n}{4} + 2$. This contradiction completes the proof.

□

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