The Bousquet-Conway directed animals

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Abstract

Bousquet-Mélou and Conway in [3] found algebraic equations for the area generating function of directed animals on an infinite family of regular, non planar, two-dimensional lattices by using equivalences with hard particle models. We give in this paper a bijective proof of their results which is a generalization of Viennot's heaps of pieces [6, 8]. Thanks to this proof we can get some exact enumeration formulas for the number of configurations with area n which could not be deduced directly from the algebraic equation and were not contained in [3]. Moreover we give an extension of these results to another infinite family of regular, non planar, two-dimensional lattices.

1 Introduction

Directed animals were introduced by physicists to study directed site percolation models. A *directed animal* A is a finite set of vertices on an acyclic infinite periodic lattice L such that any vertex of A can be reached from a distinguished vertex, called the source, through an oriented path of L having all its vertices in A. Animals are defined up to a translation on the lattice. For the square (resp. triangular) 2-dimensional lattice the vertices are labeled by the elements $(x, y) \in \mathbb{Z} \times \mathbb{N}$ such that x and y have the same parity. From each vertex (x, y) there are 2 (resp. 3) emerging edges leading to the vertices $(x-1, y+1)$ and $(x+1, y+1)$ (resp. $(x-1, y+1)$, $(x+1, y+1)$ and $(x, y+2)$). Examples of a directed animal on the square and triangular lattices are given on Figure 1.

The *area* of an animal is its number of vertices. The *width* (resp. the *length*) of an animal is $\max_{\alpha,\beta \in A} |x_\alpha - x_\beta|$ (resp. $\max_{\alpha \in A} y_\alpha$). On Figure 1 the area of the animal on the square lattice is 19, the width 9 and the length 8. Physicists are interested in the following numbers: a_n the number of animals with area n, l_n the average width of an animal with area n and $\mathcal{L}_n$ the average length of an animal with area n. Physicists expect the following asymptotic behavior

$$a_n \sim \mu^n n^{-\theta}, \quad l_n \sim n^{\nu_\perp}, \quad \mathcal{L}_n \sim n^{\nu_{//}},$$

where θ, $\nu_\perp$, $\nu_{//}$ are critical exponents, that is they do not depend on the lattice. For the square (resp. triangular) lattices is was shown that $\mu = 3$, (resp. 4) and that $\theta = \nu_\perp = 1/2$. But $\nu_{//}$ is still conjectured to be $9/11$. There exist two main approaches to solve directed animals problems: equivalences with hard particle models [1, 2, 3, 4] and purely combinatorial approaches: bijections with one-dimensional paths [5] and heaps of pieces [6, 7, 8].

Bousquet-Mélou and Conway in [3] studied directed animals on an infinite family of regular, non planar, two-dimensional lattices:

Definition 1.1 *We define an oriented lattice L_j indexed by an integer $j \geq 2$.*

- *If $j = 2k+1$, the vertices of L_j are labeled by the elements of $\mathbb{Z} \times \mathbb{N}$. From each vertex (x, y) there are j emerging edges leading to the vertices $(x+r, y+1)$ for $-k \leq r \leq k$.*
- *If $j = 2k$, the vertices of L_j are labeled by the elements $(x, y) \in \mathbb{Z} \times \mathbb{N}$ such that x and y have the same parity. From each vertex (x, y) there are j emerging edges leading to the vertices $(x+2r+1, y+1)$ for $-k \leq r \leq k-1$.*

The case $j = 2$ corresponds to the square lattice. Example of animals on L_j for $j = 3, 4, 5$ are given on Figure 2. By using equivalences with hard-particle models, Bousquet-Mélou and Conway proved that

Theorem 1.2 *Bousquet-Mélou and Conway, 1996 [3]*

Let $p_{j,n}$ be the number of animals with area n on L_j. The algebraic equation of the area generating function $p_j(t) = \sum_{n \geq 1} p_{j,n} t^n$ (p for short), is:

$$t^2(1+t)^{j-1}(1+(j+1)p)^{j+1} - (1+t+(j-1)p)^{j-1}(t-2p)^2 = 0,$$

and it simplifies in the case $j = 2k+1$:

$$t(1+t)^k(1+(2k+2)p)^{k+1} + (1+t+2kp)^k(t-2p) = 0.$$

Note that $p_j(t)$ is algebraic of degree at most j if j is even and $(j+1)/2$ if j is odd. Thus the number of animals with area n on L_j is asymptotically equivalent to $C_{L_j} n^{-1/2} \mu_{L_j}^n$, where C_{L_j} is not depending on n and $(1+\mu_{L_j})^{j-1}(1+j)^{j+1} = 4\mu_{L_j}^{j+1}(j-1)^{j-1}$.

The aim of this paper is to give a bijective proof of Bousquet-Conway's algebraic equation for the area generating function and exact enumeration formulas. We present the bijective method in Section 2. In Sections 3 and 4, we apply this method to find the algebraic equation of the area generating function and exact enumeration formulas for animals on L_j, separating the cases j odd or even. In Section 5, we give an extension of these results to another infinite family of graphs T_j for $j \geq 2$, where the case $j = 2$ corresponds to the triangular lattice.

2 The bijective method

The idea is to decompose an animal into smaller "special" animals. This approach is a generalization of Jean-Guy Penaud's work [6] on the square lattice ($j = 2$). We want to point out that Penaud's bijection is equivalent to Viennot's bijection with the heap of pieces [8]. We first define an order relation $>_{L_j}$.

Definition 2.1 *Let S be a set of vertices of L_j, $\alpha = (x_\alpha, y_\alpha)$ and $\beta = (x_\beta, y_\beta)$ two vertices of S. We say that $\alpha >_{L_j} \beta$ if and only if*

- $|x_\alpha - x_\beta| \leq k$ *and* $y_\beta = y_\alpha + 2y + 1$ *with* $y \geq 0$, *if* $j = 2k + 1$ *or*
- $x_\beta = x_\alpha + 2x + 1$ *with* $0 \leq x \leq k - 1$, *if* $j = 2k$ *or*
- *There exist* $\alpha_1, \ldots, \alpha_l \in S$, *with* $l \geq 1$, *such that* $\alpha >_{L_j} \alpha_1 >_{L_j} \ldots >_{L_j} \alpha_l >_{L_j} \beta$.

Definition 2.2 *Let S be a set of vertices on L_j and α a vertex of S. Then* $\mathbf{dom}_S(\alpha)$ *is the set of vertices in S "dominated" by α:*

$$\mathbf{dom}_S(\alpha) = \{\alpha\} \cup \{\ \beta \in S \mid \alpha >_{L_j} \beta\}$$

An example is given on the Figure 3. The set of vertices dominated by $(1, 1)$ are drawn with larger black discs.

Definition 2.3 *Two sets of vertices S and S' on L_j are* **equivalent** *if and only if there exists a bijection $f : S \to S'$ such that for all $\alpha, \beta \in S$, $\alpha >_{L_j} \beta$ if and only if $f(\alpha) >_{L_j} f(\beta)$.*

Definition 2.4 *A* **pyramid** *is a set of vertices S such that there exists $\alpha_S \in S$ such that $\mathbf{dom}_S(\alpha_S) = S$. The vertex α_S is called the* **source** *of the pyramid.*

Then we define some operations which will be useful in the decomposition of our animals.

Definition 2.5 *An* $(h, l)-$**translation** *of S is the set of vertices*

$$\mathcal{T}_{l,h}(S) = \{\ (x + l, y + 2h) \mid (x, y) \in S\ \}.$$

Note that l must be even when j is even.

Definition 2.6 *A* **heap** *of a pyramid S is the set of vertices $H(S)$ created by:*

$H(S) \leftarrow S$
While there exists $\alpha \in H(S)$ *such that*
  $\forall \beta$, $(\beta \in H(S),\ \beta >_{L_j} \alpha) \Rightarrow y_\beta < y_\alpha - 2$
    $H(S) \leftarrow (H(S) \setminus \{\alpha\}) \cup \{(x_\alpha, y_\alpha - 2)\}$

An example of this operation is given on Figure 4. Note that the heap operation will create an animal with source α_S and that a pyramid S is a directed animal if and only if $H(S) = S$.

Lemma 2.7 *S and $H(S)$ are equivalent. Moreover, S and S' are equivalent if and only if there exist h, l, such that $H(S) = \mathcal{T}_{h,l}(H(S'))$.*

Proof. Immediate □

Definition 2.8 *The* **concatenation** *of two sets of vertices S and A is*

$$S \oplus A \ = \ H(S \ \cup \mathcal{T}_{0,Y(S)}(A)).$$

for $Y(S) > \max_{\alpha \in S} \{y_\alpha\}$. Note that the concatenation operation is not commutative, but associative.

An example of this operation is given on Figure 5.

Lemma 2.9 *Given two sets of vertices S and A, there exists a mapping $f : S \cup A \to S \oplus A$ such that for all $(x, y) \in S$, $f((x, y)) = (x, y)$ and for all $(x, y) \in A$, there exists $i \geq 0$ such that $f((x, y)) = (x, y + 2i)$. If α is the source of A, then*

$$(S \oplus A) \backslash \mathbf{dom}_{S \oplus A}(f(\alpha_A)) = S.$$

and there exists h such that $\mathcal{T}_{0,2h}(\mathbf{dom}_{S \oplus A}(f(\alpha_A))) = A$.

Proof. Immediate as the set of vertices dominated by $f(\alpha_A)$ in $S \oplus A$ is equivalent to A. □

Then we are able to decompose any animal A (see Figure 6 for an example). Let x_1 (resp. x_2) be the smallest (resp. largest) abscissa of a vertex in A with ordinate 1. Then we will decompose an animal A into the animals $A_{x_1}, \ldots, A_{x_2}$ and the source $(0, 0)$. Note that the animals A_x can be empty for $x_1 < x < x_2$.

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Decompose(input: A, x_1, x_2) output: {x_1, x_2, A_{x_1}, ..., A_{x_2}}
  For x from x_1 to x_2 do
    If there exists y such that (x, y) ∈ A
      Let y_x = min{y > 0 | (x, y) ∈ A} and α = (x, y_x)
        S ← dom_A(α)
        A ← A \ S
        A_x ← H(S)
    else A_x ← ∅
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We can re-compose the animal by

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Recompose(inputs: x_1, x_2, A_{x_1}, ..., A_{x_2}) output:{A, x_1, x_2}
  A ← (0, 0)
  For x from x_2 down to x_1 do
    A ← A ⊕ A_x
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Lemma 2.10

$$\text{Recompose}(\text{Decompose}(A, x_1, x_2)) = \{A, x_1, x_2\},$$

$$\text{Decompose}(\text{Recompose}(x_1, x_2, A_{x_1}, \ldots, A_{x_2}) = \{x_1, x_2, A_{x_1}, \ldots, A_{x_2}\}.$$

Proof. Immediate with Lemma 2.9. □

3 The odd case

3.1 Decomposition

Definition 3.1 *An* **angle animal** *on L_j is an animal with no vertices (x, y) such that $x < 0$. Let $m_{j,n}$ be the number of angle animals with area n on L_j.*

Proposition 3.2 *The area generating function $m_{2k+1}(t) = \sum_{n\geq 1} m_{2k+1,n}t^n$ (m for short) satisfies the algebraic equation*

$$m = \frac{t((1+2m)^{k+1}+1)}{2}. \tag{1}$$

Proof. In the decomposition algorithm of the animal A, we set $x_1 = 0$ and $x_2 = \max\{i|(i, 1) \in A\}$. The animals A_x $(0 \leq x \leq x_2 - 1)$ are empty or angle animals with a source with even or odd ordinate. The animal A_{x_2} is an angle animal. Therefore an angle animal A is a vertex (t) or at least a vertex and another vertex with ordinate 1 $(tm\sum_{i=0}^{k}(1+2m)^i)$. See Figure 7. □

Definition 3.3 *An* **i-angle animal** *$(1 \leq i \leq k)$ on L_{2k+1} is a (possibly empty) animal which does not have vertices (x, y) with $x < 0$, or $(0 \leq x \leq i-1$, y odd). Let $m^{(i)}_{2k+1,n}$ be the number of i-angle animals with area n on L_{2k+1}.*

Proposition 3.4 *The area generating function $M^{(i)}_{2k+1}(t) = 1+\sum_{n\geq 1} m^(i)_{2k+1,n}t^n$ ($M^{(i)}$ for short) follows the algebraic equation:*

$$M^{(i)} = 1 + t + t\frac{(1+2m)^{k-i+1}-1}{2}\prod_{j=1}^{i} M^{(j)}. \tag{2}$$

Proof. In the decomposition algorithm of the animal A, we set $x_1 = 0$ and $x_2 = \max_i\{(i, 1) \in A\}$. The animals A_x $(0 \leq x \leq i-1)$ are (possibly empty) $(i-x)$-angle animals. The animals A_x $(i \leq x \leq x_2 - 1)$ are empty or angle-animals with a source with even or odd ordinate. The animal A_{x_2} is an angle animal. Therefore an i-angle animal A is empty (1) or a vertex (t) or at least a vertex and another vertex with ordinate 1 $(t\prod_{j=1}^{i} M^{(j)} m \sum_{j=0}^{k-i}(1+2m)^j)$. See Figure 8. □

Lemma 3.5 *Let $M(t) = 1 + 2m(t)$, then*

$$M^{(i)} = \frac{M(M^{i-1}+1)}{M^i+1} \tag{3}$$

Proof. We use induction on i. For $i = 1$, we have from (2)

$$M^{(1)} = 1 + t + \frac{t}{2}(M^k - 1)M^{(1)}$$

and from (1), we get $M^k = \frac{M-1-t}{tM}$, which gives us :

$$M^{(1)} = \frac{2t(1+t)M}{tM + t + t^2 + t^2 M} = \frac{2M}{M+1}.$$

Now let's suppose the lemma true for all $l \leq j$ for some $j \geq 1$:

$$M^{(j+1)} = 1 + t + t\frac{M^{k-j} - 1}{2}\prod_{i=1}^{j+1} M^{(i)} = 1 + t + t\frac{M^k - M^j}{M^j + 1}M^{(j+1)}$$

that is to say

$$M^{(j+1)} = \frac{M(M^j + 1)}{M^{j+1} + 1}. \ \square$$

Proposition 3.6 *Let $p_{2k+1,n}$ be the number of animals with area n on L_{2k+1}. The generating function $p_{2k+1}(t) = \sum_{n\geq 1} p_{2k+1,n}t^n$ (p for short) follows the algebraic equation:*

$$p = t + tp(1 + \sum_{i=1}^{k-1}\prod_{l=k-i+1}^{k} M^{(i)} + \frac{1}{2}\prod_{l=1}^{k} M^{(l)}(\frac{(1+2m)^{k+1} - 1}{2m} + k + 1) \qquad (4)$$

Proof.

In the decomposition algorithm of the animal A, we set $x_1 = \min\{i|(i,1) \in A\}$ and $x_2 = \max\{i|(i,1) \in A\}$. The animal A_{x_1} is a plain animal. The animals A_x ($x_1 + 1 \leq x \leq \min(k, x_1 + k)$) are $(k + 1 + x_1 - x)$-angle animals. If $|x_2 - x_1| > k$, then the animals A_x ($x_1 + 1 \leq x \leq x_2 - 1$) are empty or angle animals with a source with even or odd ordinate and the animal A_{x_2} is an angle animal. An example of a decomposition is on Figure 6 for L_3. Hence an animal A is

- a vertex (t)
- or an animal such that for all i, j with $(i, 1)$ and $(j, 1) \in A$ then $|i - j| \leq k$ ($tp + tp\sum_{j=-k}^{k-1}\prod_{l=k}^{\max(1,j+1)} M^{(i)}$).
- or an animal such that $\exists i, j$ with $(i, 1)$ and $(j, 1) \in A$ and $|i - j| > k$ ($tpm\prod_{l=1}^{k} M^{(l)}\sum_{j=-k}^{-1}\sum_{i=k+j+1}^{k}(1 + 2m)^{i-k-j-1}$).

See Figure 9. $\square$

3.2 Algebraic equation

Corollary 3.7

$$p=(t+1)m-(k+1)tp+2kmp \tag{5}$$

Proof. From the previous lemma we get : $\prod_{j=1}^{k} M^{(j)} = 2M^k/(M^k+1)$, $\prod_{j=k-i+1}^{k} M^{(j)} = M^i(M^{k-i}+1)/(M^k+1)$.

Hence we can rewrite the equation of p (4) :

$$\begin{aligned} p &= t+tp(\sum_{i=0}^{k-1}\frac{M^k+M^i}{M^k+1}+\frac{M^k}{M^k+1}(\frac{M^{k+1}-1}{M-1}+(k+1)) \\ &= t+\frac{tp}{M^k+1}((2k+1)M^k+\frac{M^{2k+1}-1}{M-1}). \end{aligned}$$

From (1), we know that $M^{k+1}=(M-1-t)/t$, therefore

$$0=t(1+t)(M-1)^2+p(2kt(M-1)^2-2t(M-1)-2(k+1)(M-1)t^2).$$

If we replace M by $1+2m$, simplify and divide by $4mt$, we get $(t+1)m+2kpm-p-(k+1)tp=0$. This completes the proof. □

Thanks to this result we can write m as a function of p and with (1) we get the algebraic equation of p:

Theorem 3.8 *(Bousquet-Conway)*

$$(1+t+2kp)^k(t-2p)+t(1+t)^k(1+2(k+1)p)^{k+1}=0. \tag{6}$$

3.3 Enumeration formulas

Proposition 3.9 *The number of angle animals with area n is*

$$m_{2k+1,n}=\frac{1}{2n}\sum_{i=0}^{n}\binom{n}{i}\binom{(k+1)i}{n-1} \tag{7}$$

Proof. From (1) and thanks to Lagrange inversion we can write

$$< t^n, m(t) >= \frac{1}{n!}F_n(0)=\frac{1}{n}< t^{n-1}, G_n(t) >, \quad \text{with } F_n(t)=\frac{d^{n-1}}{t^{n-1}}G_n(t),$$

$$\begin{aligned} G_n(t) &= (1+(1+2t)^{k+1})^n/2^n \\ &= \frac{1}{2^n}\sum_{i=0}^{n}\sum_{l=0}^{(k+1)i}\binom{n}{i}\binom{(k+1)i}{l}(2t)^l, \end{aligned}$$

$$< t^{n-1}, G_n(t) >= \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\binom{(k+1)i}{n-1}2^{n-1}. \quad \square$$

Corollary 3.10

$$(k+1)p(t) = m(t) + ktm'(t) \tag{8}$$

Therefore

$$(nk+1)m_{2k+1,n} = (k+1)p_{2k+1,n} \tag{9}$$

Proof. If we differentiate the equation (1) we get

$$m' = (1 + (1+2m)^{k+1})/2 + t(k+1)(1+2m)^k m'.$$

When we replace $(1+2m)^k$ by $(2m-t)/(t(1+2m))$ and multiply by t

$$tm' = m(1+2m) + t2mkm' - t^2(k+1)m' = 2m(m+tkm') + m - t^2(k+1)m'.$$

If we multiply the equation by k and add m on both sides

$$\begin{aligned} m + ktm' &= (k+1)m + 2mk(m+tkm') - t^2k(k+1)m' \\ &= (k+1)(1+t)m + 2mk(m+tkm') - t(k+1)(m+tkm') \end{aligned}$$

From the previous equation and (4) we get the result. □

Theorem 3.11 *The number of directed animals with area n on L_{2k+1} is*

$$p_{2k+1,n} = \frac{nk+1}{2n(k+1)} \sum_{i=0}^{n} \binom{n}{i} \binom{(k+1)i}{n-1} \tag{10}$$

4 The even case

4.1 Decomposition

Definition 4.1 *An* **angle animal** *is an animal with no vertices (x,y) such that $x<0$. Let $m_{2k,n}$ be the number of angle animals with area n on L_{2k}.*

Proposition 4.2 *The area generating function $m_{2k}(t) = \sum_{n\geq 1} m_{2k}(n)t^n$ (m for short) follows the algebraic equation*

$$m = \frac{t((1+m)^{2k+1}+1)}{m+2}. \tag{11}$$

Proof. In the decomposition algorithm of the animal A, we set $x_1 = 0$ and $x_2 = max\{i|(i,1) \in A\}$ (note that x_2 has to be odd). The animals A_x $(0 \leq x \leq x_2 - 1)$ are empty or angle animals. The animal A_{x_2} is an angle animal. An angle animal A is a vertex (t) or at least a vertex and another vertex with ordinate 1 $(tm\sum_{i=0}^{k-1}(1+m)^{2i+1})$. See Figure 10.□

Definition 4.3 *An i-***angle animal** *$(1 \leq i \leq k-1)$ on L_{2k} is a (possibly empty) animal which does not have vertices (x,y) with $x<0$, or $x = 2l-1$, for $1 \leq l \leq i$. Let $m^{(i)}_{2k,n}$ be the number of $i-$angle animals with area n on L_{2k}.*

Proposition 4.4 *The area generating function* $M_{2k}^{(i)}(t) = 1 + \sum_{n\geq 1} m_{2k,n}^{(i)} t^n$ *($M^{(i)}$ for short) follows the algebraic equation:*

$$M^{(i)} = 1 + t + t\frac{(1+m)((1+m)^{2(k-i)} - 1)}{2+m} \prod_{j=1}^{i} M^{(j)}. \tag{12}$$

Proof. In the decomposition algorithm of the animal A, we set $x_1 = 0$ and $x_2 = max\{i|(i,1) \in A\}$ (note that x_2 has to be odd). The animals A_{2x} $(0 \leq x \leq i-1)$ are empty or $(i-x)$-angle animals. The animals A_{2x+1} $(0 \leq x \leq i-1)$ are empty. The animals A_x $(2i \leq x \leq x_2 - 1)$ are empty or angle animals. The animal A_{x_2} is an angle animal. Hence an i-angle animal A is empty (1) or a vertex (t) or at least a vertex and another vertex with ordinate 1 ($tm \prod_{j=1}^{i} M^{(j)} \sum_{j=0}^{k-i-1} (1+m)^{2i+1}$). See Figure 11. □

Corollary 4.5 *Let* $M(t) = 1 + m(t)$ *then*

$$M^{(i)} = \frac{M^2(M^{2i-1}+1)}{M^{2i+1}+1} \tag{13}$$

Proof. We use induction on i. For $i = 1$, from (12)

$$M^{(1)} = 1 + t + t\frac{M(M^{2(k-1)} - 1)}{1+M} M^{(1)}$$

and from the equation (11) we know that $M^{2k-1} = (M^2 - 1 - t)/tM^2$. Therefore we get $M^{(1)} = M^2(1+M)/(1+M^3)$.

Now let's suppose the lemma true for all $j \leq i$ for some $0 < i < k-1$. Then we have $\prod_{j=1}^{i} M^{(j)} = (M^{2i}(1+M))/(1+M^{2i+1})$.

From the previous equation, (12) and (11), we get

$$M^{(i+1)} = 1 + t + t\frac{M(M^{2(k-1)} - M^{2i})}{1 + M^{2i+1}} = 1 + t + \frac{M^2 - 1 - t - tM^{2i+3}}{1+M^{2i+1}} M^{(i+1)}.$$

which gives the result. □

Proposition 4.6 *The generating function* $p_{2k}(t)$ *for the "plain" animals (p for short) follows the algebraic equation:*

$$p = tp + tp \sum_{j=-k}^{k-1} \prod_{i=k-1}^{\max(1,j+1)} M^{(i)} + tp \prod_{i=1}^{k-1} M^{(i)} \left(\frac{(1+m)^2((1+m)^{2k} - 1)}{m(m+2)^2} - \frac{k}{m+2} \right). \tag{14}$$

Proof. In the decomposition algorithm of the animal A, we set $x_1 = \min\{i|(i,1) \in A\}$ and $x_2 = \max\{i|(i,1) \in A\}$. The animal A_{x_1} is a "plain" animal. The animals A_{2x+x_1} $(1 \leq x \leq \min(k-1, k-(x_1+1)/2))$ are $(k-x)$-angle animals and the animals A_{2x+x_1+1} $(0 \leq x \leq \min(k-1, k-(x_1+1)/2))$ are empty. If $|x_2 - x_1| \geq 2k$, then the animals A_x $(x_1 + 2k \leq x \leq x_2 - 1)$ are empty or angle animals and the animal A_{x_2} is an angle animal. Hence an animal A is

- a vertex (t)
- or a animal such that for all i, j with $(i,1)$ and $(j,1) \in A$ then $|i-j| \leq 2k$ $(tp + tp \sum_{j=-k}^{k-1} \prod_{i=k-1}^{\max(1,j+1)} M^{(i)})$.
- or a animal such that there exist i, j with $(i,1)$ and $(j,1) \in A$ and $|i-j| > 2k$ $(tp \prod_{j=1}^{k-1} M^{(j)} \sum_{j=-k}^{-1} m \sum_{i=k+j}^{k-1} (1+m)^{2(i-k-j)})$.

See Figure 12. □

4.2 Algebraic equation

Corollary 4.7

$$(1+m)^2(1+t+(2k-1)p) = (1+t)(1+(2k+1)p). \tag{15}$$

Proof. If we plug the equation (13) in the equation of p (??), with $M = 1+m$:

$$p = t + \frac{tp}{1+M^{2k-1}}(2kM^{2k-1} + \frac{M^{4k}-1}{M^2-1})$$

and from(11) we know that $M^{2k-1} = (M^2-1-t)/tM^2$ and that $M^{4k} = (M^2-1-t)^2/t^2M^2$. Therefore

$$p = t + \frac{p}{M^2-1-t+t^2M^2}(2kt(M^2-1-t) + M^2 - 1 - 2t - t^2).$$

which gives $M^2(1+(2k-1)p) = (1+t)(1+(2k+1)p)$. □

Theorem 4.8 *(Bousquet-Conway)*

$$(2p-t)^2(1+t+(2k-1)p)^{2k-1} = t^2(1+t)^{2k-1}(1+(2k+1)p)^{2k+1} \tag{16}$$

Proof. When we take the $(2k+1)^{th}$ power of the equation (15), and multiply it by t^2 :

$$t^2(1+m)^{2(2k+1)}(1+t+(2k-1)p)^{2k+1} = t^2(1+t)^{2k+1}(1+(2k+1)p)^{2k+1}$$

and from (11) we know that $(1+m)^{2k+1} = ((1+m)^2-1-t)/t$. Hence

$$((1+m)^2-1-t)^2(1+t+(2k-1)p)^{2k+1} = t^2(1+t)^{k+1}(1+(2k+1)p)^{2k+1}$$

By using (15) we find the algebraic equation

$$(2p-t)^2(1+t+(2k-1)p)^{2k-1} = t^2(1+t)^{2k-1}(1+(2k+1)p)^{2k+1}. \square$$

4.3 Enumeration formulas

Proposition 4.9 *The number of angle animals with area n is*

$$m_{2k,n} = \frac{1}{n}\sum_{i=0}^{2kn}\sum_{j=0}^{\lfloor\frac{i}{2k+1}\rfloor}(-1)^{i-(2k+1)j}\binom{n}{j}\binom{n+i-(2k+1)j-1}{n-1}\binom{i}{n-1}. \tag{17}$$

Proof. We use Lagrange inversion as in the odd case. □

We can not apply the technique used in the odd case to find an enumeration formula for $p_{2k,n}$ as (15) is not linear in m. Nonetheless we get an equation independant of k which relates $p_{2k}(t)$ and $m_{2k}(t)$ as in (8) for the odd case and gives a way to compute $p_{2k,n}$.

Corollary 4.10

$$p_{2k}(t) = t(1+t)m'_{2k}(t)/(1+m_{2k}(t)) \tag{18}$$

Therefore

$$p_{2k,n} = nm_{2k,n} + (n-1)m_{2k,n-1} - \sum_{i=1}^{n-1} m_{2k,i}p_{2k,n-i}. \tag{19}$$

Proof. Let $M = 1+m$. We know that $M^2 = 1+t+tM^{2k+1}$ from (11). If we derive the equation with respect to t we now get

$$2M'M = 1 + M^{2k+1} + (2k+1)tM'M^{2k}.$$

If we replace M^{2k} by $(M^2-t-1)/(tM)$ that is to say $M^2/t + (2k-1)tM'M = 1/t + (2k+1)M'(1+t)/(tM)$. With (15) we get the result. □

5 Generalization

Definition 5.1 *We define an oriented lattice T_j indexed by an integer $j \geq 2$. T_j is L_j plus the edges $((x,y),(x,y+2))$ for all the $(x,y) \in L_j$. The case $j=2$ corresponds to the triangular lattice.*

An example of animal on T_4 is given on Figure 13.

Definition 5.2 *Let S be a set of vertices on T_j, and $\alpha = (x_\alpha, y_\alpha)$ and $\beta = (x_\beta, y_\beta)$ two vertices of S. We say that $\alpha \succ \beta$ if and only $\alpha >_{L_j} \beta$ (in the sense of the definition 2.1) or $x_\alpha = x_\beta$ and $y_\alpha + 2y = y_\beta$ $(y \geq 0)$.*

Theorem 5.3 *The algebraic equation of the area generating function $p_j(t)$ (p for short) on T_j, is:*

$$t(1+(2k+2)p)^{k+1} + (1+2kp(1-t))^k(t-2p(1-t)) = 0,$$

if $j = 2k+1$ and

$$t^2(1+(2k+1)p(1-t))^{2k+1} + (1+(2k-1)p(1-t))^{2k}(t-2p(1-t))^2 = 0,$$

if $j = 2k$. Note that $p_j(t)$ is algebraic of degree at most j if j is even and $(j+1)/2$ if j is odd. Thus the number of animals with area n on T_j is asymptotically equivalent to $C_{T_j} n^{-1/2} \mu_{L_j}^n$, where $\mu_{T_j} = \mu_{L_j} + 1$.

Proof. (sketch for the odd case) The algebraic equation for the area generating function of the angle animals is $m = t(1+2m)((1+2m)^k+1)/2$, for the i-angle animals

$$M^{(i)} = 1 + tM^{(i)} + t\frac{(1+2m)^{k-i+1}-1}{2}\prod_{j=1}^{i} M^{(j)} = \frac{(1+2m)((1+2m)^{i-1}+1)}{(1+2m)^i+1},$$

and for the animals

$$p = t + tp + tp(1 + \sum_{i=1}^{k-1} \prod_{l=k-i+1}^{k} M^{(i)} + \frac{1}{2}\prod_{l=1}^{k} M^{(l)}(\frac{(1+2m)^{k+1}-1}{2m} + (k+1))).$$

We get from these $p = m - ktp + 2kmp(1-t)$. This gives the algebraic equation of p, when we plug this result in the algebraic equation of m. □

If we look at the equation, we get that the equations for T_j are the same as the ones of L_j when we replace t by $t/(1-t)$. It means that any animal on L_j can be transformed into some animal on T_j by duplicating any vertex any number of times (see [2] for the case $j = 2$). The algorithm **Duplicate** gives a way to duplicate any vertex (x, y) in A, i times. An example is given on Figure 14. The algorithm **reduce** explains how to get from an animal on T_j to an animal on L_j.

Duplicate(A, x, y, i)
  $S \leftarrow \mathbf{dom}_A(x, y)$
  $A \leftarrow A \setminus S$
  $A \leftarrow (A \cup_{j=0}^{i-1} \{(x, y+2j)\}) \oplus S$

Reduce(A)
  While $\exists\ (x, y) \in A$ s.t. $\forall (x', y') \in A$,
  if $(x', y') >_{T_j} (x, y)$ then $y' \leq y - 2$ do
    $S \leftarrow \mathbf{dom}_A(x, y)$
    $A \leftarrow (A \setminus S \setminus \{(x, y-2)\}) \oplus S$

Lemma 5.4 *There is a one-to-one correspondance between $P_{T_j}(n)$ the set of animals with area n on T_j and the union for $1 \leq i \leq n$ of the product of $C(n, i)$ the set of compositions of n into i parts and $P_{L_j}(i)$ the set of animals with area i on L_j.*

$$P_{T_j}(n) \leftrightarrow \cup_{i=1}^{n} C(n, i) \times P_{L_j}(i).$$

Proof. Given an animal on L_j with area i, and a composition on n into i parts $c = (c_1, \ldots, c_i)$. Number the vertices of the animal. For $1 \leq l \leq i$, duplicate the l^{th} vertex $c_l - 1$ times and get an animal with area n on T_j. Conversely given an animal on T_j with area n, asgign the weight 1 to all of its vertices. At each step of the reverse operation add 1 to the weight of the vertex (x, y). □

Theorem 5.5 *The number of directed animals with area n on T_{2k+1} is*

$$\sum_{i=1}^{n} \binom{n-1}{i-1} \frac{ik+1}{2i(k+1)} \sum_{l=0}^{i} \binom{i}{l} \binom{(k+1)l}{i-1}. \tag{20}$$

Proof. The previous lemma gives that $|P_{T_j}(n)| = \sum_{i=1}^{n} \binom{n-1}{i-1} |P_{L_j}(n)|$, associated with (10), we get the result. $\square$

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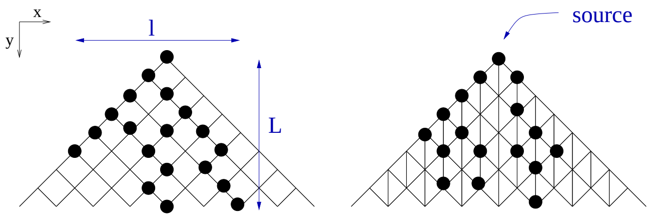

Figure 1: Animals on the square and triangular lattices

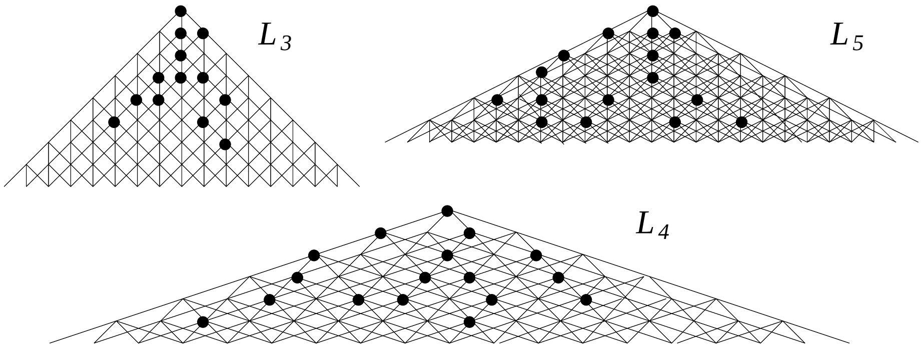

Figure 2: Examples of animals on the lattices L_3, L_4 and L_5.

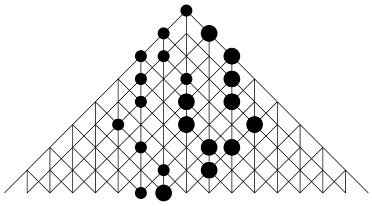

Figure 3: $\mathbf{dom_S}((\mathbf{1},\mathbf{1}))$ on L_3

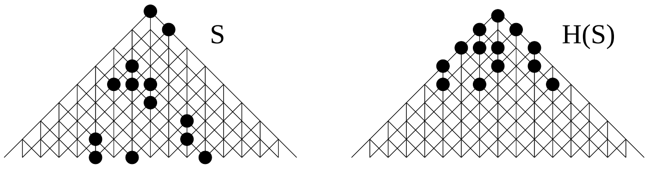

Figure 4: The heap operation on L_3

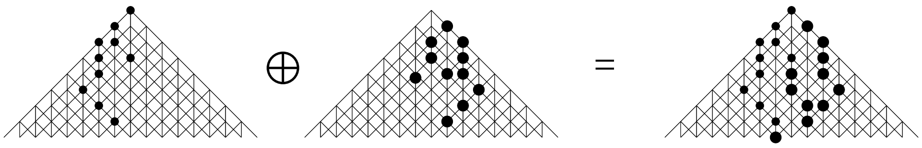

Figure 5: The concatenation operation on L_3

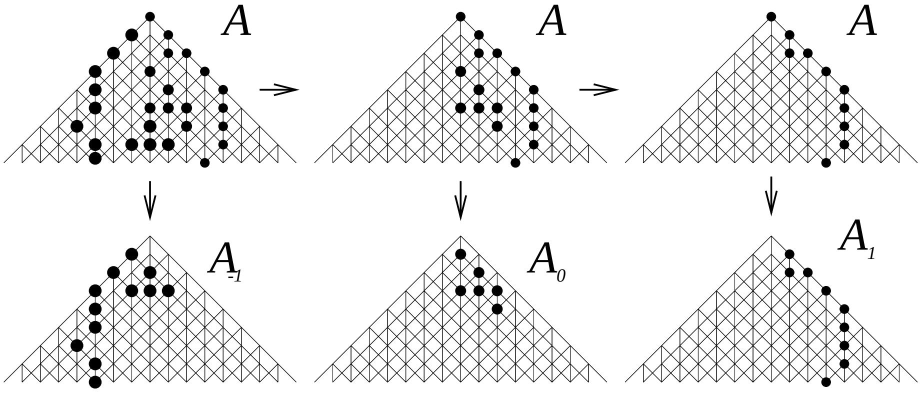

Figure 6: Example of a decomposition on L_3

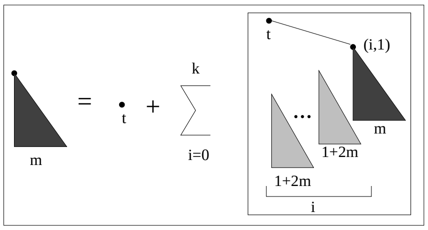

Figure 7: Decomposition of an angle animal on L_{2k+1}

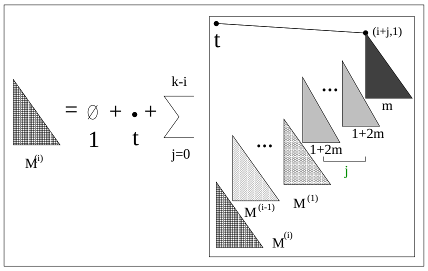

Figure 8: Decomposition of an i-angle animal on L_{2k+1}

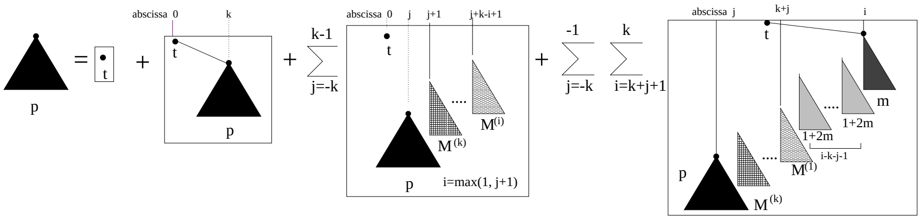

Figure 9: Decomposition of an animal

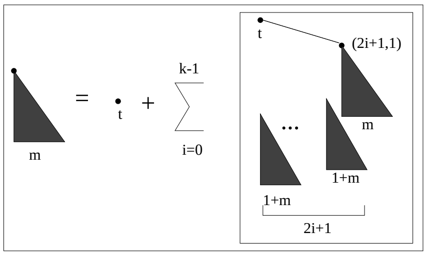

Figure 10: Decomposition of an angle animal on L_{2k}

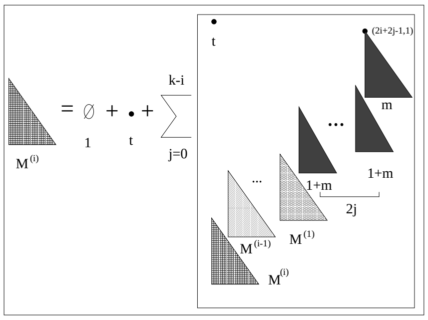

Figure 11: Decomposition of an i-angle animal on L_{2k}

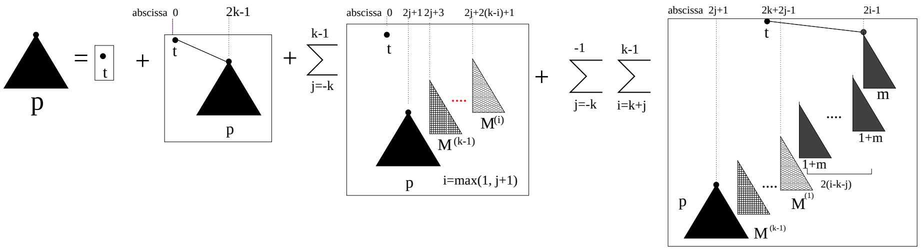

Figure 12: Decomposition of an animal on L_{2k}

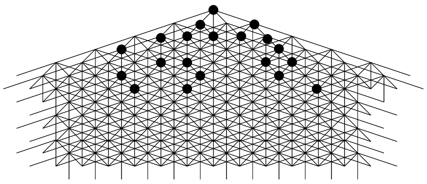

Figure 13: An animal on T_4

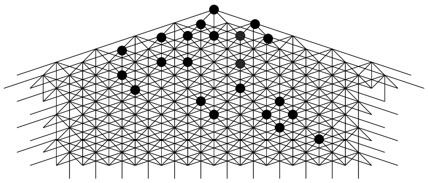

Figure 14: Duplication of the vertex $(2,2)$ twice