Europ. J. Combinatorics (2002) **23**, 567–572
doi:10.1006/eujc.2002.0580
Available online at http://www.idealibrary.com on IDEAL®

Bases in Orlik–Solomon Type Algebras

DAVID FORGE

Let M be a matroid on $[n]$ and $\mathcal{E}$ be the graded algebra generated over a field k generated by the elements $1, e_1, \ldots, e_n$. Let $\Im(M)$ be the ideal of $\mathcal{E}$ generated by the squares $e_1^2, \ldots, e_n^2$, elements of the form $e_i e_j + a_{ij} e_j e_i$ and 'boundaries of circuits', i.e., elements of the form $\sum \chi_j e_{i_1} \ldots e_{i_{j-1}} e_{i_{j+1}} \ldots e_{i_m}$, with $\chi_j \in k$ and $e_{i_1}, \ldots, e_{i_m}$ a circuit of the matroid with some special coefficients. The χ-algebra $\mathcal{A}(M)$ is defined as the quotient of $\mathcal{E}$ by $\Im(M)$. Recall that the class of χ-algebras contains several studied algebras and in first place the Orlik–Solomon algebra of a matroid. We will essentially construct the reduced Gröbner basis of $\Im(M)$ for any term order and give some consequences.

1. INTRODUCTION

In a vector space, a (*central*) *hyperplane arrangement* is a finite collection of codimension 1 subspaces. The *matroid of an hyperplane arrangement* can be defined by saying that a subset of the arrangement is independent if and only if the codimension of its intersection is equal to its cardinality. Manifolds defined as complements of complex hyperplane arrangements are important in the Aomoto–Gelfand theory of A-hypergeometric functions. In [5] the cohomology algebra of a manifold of this form is shown to be isomorphic to the Orlik–Solomon (OS) algebra of the matroid of the arrangement. This result has motivated further research on OS algebras. It is known that for OS algebras of matroids the set of 'no broken circuits' (NBC) gives a basis. We refer the reader to [6, 9] for more details on OS-algebras and to [2, 8] for good sources of matroid and oriented matroid theory.

In Section 2, we recall the construction of χ-algebras [4] as the quotient of an algebra $\mathcal{E}$ by an ideal $\Im(M)$. This is a generalization of OS algebras for which the set of NBC gives also a basis. We also recall two commutative examples of χ-algebras: an algebra defined for an arrangement of hyperplane [7] and an algebra defined for an oriented matroid [3]. A χ-algebra is defined by the quotient of an algebra $\mathcal{E}$ by an ideal $\Im(M)$ defined from the circuits of M. In Section 3, we construct the reduced Gröbner basis of the ideal $\Im(M)$ for any term order (Theorem 3.5). This gives as a corollary a universal Gröbner basis which is shown to be minimal. Finally we remark that the bases given by the NBC are also the bases corresponding to the reduced Gröbner bases for the different term orders.

2. χ-ALGEBRAS

Let M be a simple matroid of rank r on ground set $[n] := \{1, 2, \ldots, n\}$. We say that a subset $U \subseteq [n]$ is *unidependent* if it contains exactly one circuit, denoted by $C(U)$. For any $i \in C(U)$ the subset $U \setminus i$ is independent. This property characterizes unidependents among dependents: *a dependent D is unidependent if and only if there is $i \in D$ such that $D \setminus i$ is independent*.

Let I be an independent of M. We say that an element $i \in [n]$ is *active* with respect to I if $I \cup i$ contains a circuit with smallest element i. An independent set with at least one active element is said to be *active*, and *inactive* otherwise. We denote by $\alpha(I)$ the smallest active element with respect to an active independent I. Inactive independents are often called NBC in the literature, since a subset of $[n]$ is an inactive independent if and only if it contains NBC, where a *broken circuits* are the sets obtained by removing the smallest element from a circuit.

0195–6698/02/070567 + 06 $35.00/0

Fix a set $E = \{e_1, \dots, e_n\}$. Let $\mathcal{E}$ be the graded algebra over a field k generated by the elements $1, e_1, \dots, e_n$ and satisfying the relations $e_i^2 = 0$ for all $e_i \in E$ and $e_j \cdot e_i = a_{i,j} e_i \cdot e_j$ with $a_{i,j} \in k \setminus 0$ for all $i < j$. Both the free exterior algebra and the free commutative algebra with squares zero generated by the elements of E are such algebras (take $a_{i,j} = -1$ resp. $a_{i,j} = 1$ for all $i < j$) and will be the only ones to be used in the examples. When writing a set in the form $X = \{i_1, i_2, \dots, i_m\}$ we always suppose w.l.o.g. that we have $i_1 < i_2 < \dots < i_m$. Given a subset $X = \{i_1, i_2, \dots, i_m\} \subset [n]$ we will denote by e_X the corresponding (pure) element $e_{i_1} \cdot e_{i_2}, \dots, e_{i_m}$. Fix a mapping $\chi : 2^{[n]} \to k$. We define the *χ-boundary* of an element e_X by

$$\partial e_X = \sum_{\ell=1}^{\ell=m} (-1)^{\ell} \chi(X \setminus i_\ell) e_{X \setminus i_\ell}.$$

We extend ∂ to $\mathcal{E}$ by linearity.

Let $\Im_\chi(M)$ be the (right) ideal of $\mathcal{E}$ generated by the χ-boundaries $\{\partial e_C : C \text{ circuit}\}$. We say that

$$\mathcal{A}_\chi(M) = \mathcal{E}/\Im_\chi(M)$$

is a χ-algebra if χ satisfies the following two properties:

(UC1) $\chi(I) \neq 0$ *iff I is independent,*
(UC2) *for any unidependent U of M there is $a \in k \setminus 0$, such that*

$$\partial e_U = a(\partial e_{C(U)}) e_{U \setminus C(U)}.$$

It can be observed that (UC2) implies that $\chi(U) = 0$ for a unidependent U containing no basis of M. Values of χ on other dependents are irrelevant and can always be chosen null. For convenience, we will also note e_X for the residue class of e_X in $\mathcal{A}_\chi(M)$. Note that a χ-algebra is defined by the matroid M, the algebra $\mathcal{E}$ and the function χ.

EXAMPLE 2.1. *The OS algebra of a matroid* [6]. Let M be a matroid on $[n]$. The OS algebra $OS(M)$ is the quotient of $\mathcal{E}$, the graded exterior algebra of the vector space $\sum_{i=1}^{n} ke_i$, by the ideal generated by boundaries of circuits of M.

The OS algebra of M, $OS(M)$, is the χ-algebra obtained for M, the algebra $\mathcal{E}$ as above and χ defined for $X \subseteq [n]$ by $\chi(X) = 1$ for every independent.

EXAMPLE 2.2. *The Orlik–Terao algebra of a set of vectors* [7]. Let $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ be a set of vectors in a vector space over k. The Orlik–Terao algebra $OT(\mathcal{V})$ is the quotient of $\mathcal{E}$, the commutative graded algebra over the field k generated by the elements $1, e_1, \dots, e_n$, with squares zero, by the ideal generated by the elements of $\mathcal{E}$ of the form $\sum_{j=1}^{j=m} \lambda_{i_j} e_{i_1} e_{i_2} \dots e_{i_{j-1}} e_{i_{j+1}} \dots e_{i_m}$ for any minimal non-trivial linear dependency $\sum_{j=1}^{j=m} \lambda_{i_j} v_{i_j} = 0$ among the vectors of $\mathcal{V}$.

The Orlik–Terao algebra, $OT(\mathcal{V})$, is the χ-algebra obtained as follows. Let M be the matroid of linear dependencies of the vectors in $\mathcal{V}$ and $\mathcal{E}$ be the algebra as above. We fix a basis B_F for any flat F of the matroid M. Then for $I = \{i_1, i_2, \dots, i_k\}$ independent in M we define $\chi(I)$ as the determinant $\det(v_{i_1}, v_{i_2}, \dots, v_{i_k})$ with respect to $B_{\mathrm{cl}(I)}$.

EXAMPLE 2.3. *A commutative algebra defined for an oriented matroid* [3]. Let OM be an oriented matroid on $[n]$. The commutative algebra $A(OM)$ is the quotient of $\mathcal{E}$, the commutative graded algebra over the field k generated by the elements $1, e_1, \dots, e_n$, with squares zero, by the ideal generated by the elements of $\mathcal{E}$ of the form $\sum_{i \in C} sg_C(i) e_{C \setminus i}$ for any signed circuit C of OM with signature sg_C.

The algebra $A(OM)$ is the χ-algebra obtained as follows. Let M be the underlying matroid of OM and $\mathcal{E}$ be the algebra as above. To define χ, we fix a basis signature independently in all restrictions of OM to a flat F of M (we recall that a basis signature of an oriented matroid is determined up to a factor ± 1). Then for I independent in M we define $\chi(I)$ as the sign of I in standard form for the chosen basis signature of the submatroid of OM on the geometric closure of I in M.

We say that a unidependent U is *inactive* if there is a (necessarily unique) active independent I such that $U = I \cup \alpha(I)$. Let $\mathcal{D}$ be the right ideal of $\mathcal{E}$ generated by the elements $\{e_C : C \text{ circuit}\}$. We will note $\mathcal{E}_i$ and $\Im_i$ the algebra quotient $\mathcal{E}/\mathcal{D}$ and its ideal quotient $\Im/\mathcal{D}$ respectively. We now rephrase the principal result of [4].

THEOREM 2.4 ([4]). *Let M be a matroid on $[n]$ and $\mathcal{A}_\chi(M)$ be a χ-algebra. Then the set $\{e_I : I$ inactive independent of $M\}$ is a basis of $\mathcal{A}_\chi(M)$ and the set $\{\partial e_U : U$ inactive unidependent of $M\}$ is a basis of $\Im_i$.*

3. REDUCED AND UNIVERSAL GRÖBNER BASIS

For general definitions on Gröbner bases, see [1]. We begin by adapting some of them to our context. Let M be a matroid, $\mathcal{E}$ be an algebra and $\mathcal{A}_\chi(M)$ a χ-algebra as defined in the previous section. A total order $\prec$ of the set of monomials (which is a standard basis of $\mathcal{E}$):

$$\mathbb{T} := \{e_X : X = \{i_1, \ldots, i_m\} \subset [n], i_1 < \cdots < i_m\},$$

is said to be a *term order* of $\mathbb{T}$ if $e_\emptyset = 1$ is the minimal element and

$$\forall e_X, e_Y, e_Z \in \mathcal{E}, \quad (e_X \prec e_Y) \cdot (e_X \cdot e_Z \neq 0) \cdot (e_Y \cdot e_Z \neq 0) \Longrightarrow e_{X \cup Z} \prec e_{Y \cup Z}.$$

EXAMPLE 3.1. A permutation $\pi \in S_n$ defines a linear re-ordering of the elements of $[n]$, $\pi^{-1}(1) <_\pi \pi^{-1}(2) <_\pi \cdots <_\pi \pi^{-1}(n)$. Consider the ordering $e_{\pi^{-1}(1)} \prec_\pi e_{\pi^{-1}(2)} \prec_\pi \cdots \prec_\pi e_{\pi^{-1}(n)}$. The corresponding degree lexicographic ordering in $\mathbb{T}$ is a term order, denoted here by $\prec_\pi$.

Given a term order $\prec$, and a non-zero element $f \in \mathcal{E}$, we may write

$$f = a_1 e_{X_1} + a_2 e_{X_2} + \cdots + a_m e_{X_m},$$

where $a_i \in k \backslash 0$, and $e_{X_m} \prec \cdots \prec e_{X_1}$. We say that the $a_i e_{X_i}$, [resp. e_{X_i}] are the *terms* [resp. *powers*] of f. We say that $\mathrm{lp}_\prec(f) := e_{X_1}$ [resp. $\mathrm{lt}_\prec(f) := a_1 e_{X_1}$] is the *leading power* [resp. *leading term*] of f (with respect to $\prec$). Note that we can have $\mathrm{lp}_\prec(hg) \neq \mathrm{lp}_\prec(h)\mathrm{lp}_\prec(g)$ when $\mathrm{lp}_\prec(h)\mathrm{lp}_\prec(g) = 0$. Let $\Im$ be an ideal of $\mathcal{E}$ and let $\prec$ be a term order of $\mathbb{T}$. A subset of non-zero elements $\mathcal{G} \subset \Im$ is a *Gröbner basis* of the ideal $\Im$ with respect to $\prec$ iff, for all non-zero element $f \in \Im$, there exists $g \in \Im$ such that $\mathrm{lp}_\prec(g) = e_Y$ *divides* $\mathrm{lp}_\prec(f) = e_X (\Leftrightarrow Y \subset X)$. For any subset S of $\mathcal{E}$, we define the *leading power ideal of S with respect to* $\prec$, $\mathrm{Lp}_\prec(S)$, to be the ideal of $\mathcal{E}$ spanned by the elements $\{\mathrm{lp}_\prec(s) : s \in S\}$. Consider the subset of powers

$$\mathbb{T}_i := \{e_I : I \text{ independent}\} \qquad \text{and} \qquad \mathbb{T}_d := \{e_D : D \text{ dependent}\}.$$

Let $k[\mathbb{T}_i]$ and $k[\mathbb{T}_d]$ be the k-vector subspace of $\mathcal{E}$ generated by the bases $\mathbb{T}_i$ and $\mathbb{T}_d$, respectively. So $\mathcal{E} = k[\mathbb{T}_i] \oplus k[\mathbb{T}_d]$. With the notation of Section 2, we have that $k[\mathbb{T}_d] = \mathcal{D}$ and $k[\mathbb{T}_i] \cong \mathcal{E}_i$. Let $p_i : \mathcal{E} \to k[\mathbb{T}_i]$ be the first projection. We define the term orders of $\mathbb{T}_i$ in a similar way to term orders of $\mathbb{T}$. It is clear that the restriction of every term order of $\mathbb{T}$ to the subset $\mathbb{T}_i$ is also a term order of $\mathbb{T}_i$. We can also add to $k[\mathbb{T}_i]$ a structure of k-algebra with the product $\star : k[\mathbb{T}_i] \times k[\mathbb{T}_i] \to k[\mathbb{T}_i]$, determined by the equalities $e_I \star e_{I'} = p_i(e_I e_{I'})$ for all I, I' independents. Note that if $e_I \star e_{I'} \neq 0$, then $e_I \star e_{I'} = e_I e_{I'}$ ($\Leftrightarrow e_I e_{I'} \neq 0$ iff $I \cap I' = \emptyset$ and $I \cup I'$ is an independent set of M). So $\Im_i(M) := p_i(\Im(M))$ is an ideal of $k[\mathbb{T}_i]$.

PROPOSITION 3.2. *Let $\prec$ be a term order of $\mathbb{T}$. A Gröbner basis of $\Im_i(M)$ with respect to $\prec$ is also a Gröbner basis of $\Im(M)$ with respect to $\prec$.*

PROOF. Let $\mathcal{G}_i$ be a Gröbner basis of $\Im_i(M)$ with respect to the term order $\prec$. Pick a non-null element $f \in \Im(M)$. If we see $\Im(M)$ as a k-vector space it is clear that $\Im(M) = \Im_i(M) \oplus k[\mathbb{T}_d]$. So $e_X := \mathrm{lp}_\prec(f) \in \Im_i(M)$ if X is an independent set of M or $e_X \in k[\mathbb{T}_d]\backslash 0$ if X is a dependent set of M. If X is independent there is an element $g \in \mathcal{G}_i$ such that $\mathrm{lp}_\prec(g) = e_I$ such that $I \subset X$, so $\mathrm{lp}_\prec(g)$ divides $\mathrm{lp}_\prec(f)$ in $\Im(M)$. Suppose now that X is a dependent set of M. Then there is a circuit $C \subset X$. We know that $\partial e_C \in \Im_i(M)$ and if $\mathrm{lp}_\prec(\partial e_C) = e_Y$ then $Y \subset C \subset X$. So, $\mathrm{lp}_\prec(\partial e_C)$ divides $\mathrm{lp}_\prec(f)$ in $\Im(M)$ and $\mathcal{G}_i$ is also a Gröbner basis of $\Im(M)$. □

A Gröbner basis $\mathcal{G}$ of an ideal $\Im$ is called *reduced* (with respect to the term order $\prec$) if for every element $g \in \mathcal{G}$ we have $\mathrm{lt}_\prec(g) = \mathrm{lp}_\prec(g)$, and for every two distinct elements $g, g' \in \mathcal{G}$, no term of g' is divisible by $\mathrm{lp}_\prec(g)$. A (finite) subset $\mathcal{U} \subset \Im$ is called a *universal Gröbner basis* if $\mathcal{U}$ is a Gröbner basis of $\Im$ with respect to all term orders simultaneously.

PROPOSITION 3.3. *Let $\mathcal{G}$ be a Gröbner basis of the ideal $\Im(M)$ with respect to the term order $\prec$ of $\mathbb{T}$. Then*

$$\mathcal{B}_\mathcal{G} := \{e_X : X \subset [n], e_X \notin \mathrm{L}_{\mathrm{p}_\prec}(\mathcal{G}) = \mathrm{L}_{\mathrm{p}_\prec}(\Im(M))\}$$

is a basis of $\mathcal{A}_\chi(M)$.

We say that $\mathcal{B}_\mathcal{G}$ is the *canonical basis of the χ-algebra $\mathcal{A}_\chi(M)$ for the Gröbner basis $\mathcal{G}$ of the ideal $\Im(M)$.*

REMARK 3.4. From the preceding proposition we see that, for every term order $\prec$ of $\mathbb{T}$, there is a unique monomial basis of $\mathcal{A}_\chi(M)$ denoted by $\mathcal{B}_\prec$. We say that $\mathcal{B}_\prec$ is the *canonical basis of $\mathcal{A}_\chi(M)$*. On the other hand it is well known that the term order $\prec$ determines a unique reduced Gröbner basis of $\Im(M)$ denoted $(\mathcal{G}_r)_\prec$. From the definitions we can also deduce that $\mathcal{B}_\prec = \mathcal{B}_{\prec'} \Leftrightarrow (\mathcal{G}_r)_\prec = (\mathcal{G}_r)_{\prec'} \Leftrightarrow \mathrm{Lp}_\prec(\Im(M)) = \mathrm{Lp}_{\prec'}(\Im(M))$.

For a term order $\prec$ of $\mathbb{T}$ we say that $\pi_\prec \in S_n$, is the *permutation compatible* with $\prec$ if, for every pair $i, j \in [n]$, we have $e_i \prec e_j$ iff $i <_{\pi_\prec} j$ ($\Leftrightarrow \pi_\prec^{-1}(i) < \pi_\prec^{-1}(j)$). Let $\mathfrak{C}_{\pi_\prec}(M)$ be the subset of circuits of M such that $\inf_{<_{\pi_\prec}}(C) = \alpha_\pi(C)$ and $C \backslash \alpha_\pi(C)$ is inclusion minimal with this property. ($\alpha_\pi(C)$ is the minimum active element of $C \backslash \inf_{<_{\pi_\prec}}(C)$ where the order used for activity and taking inf is $<_{\pi_\prec}$.) In the following we may replace '$\pi_\prec$' by 'π' when no mistake can result.

THEOREM 3.5. *Let $\prec$ be a term order of $\mathbb{T}$ compatible with the permutation $\pi \in S_n$. Then the family $\mathcal{G}_{\mathrm{red}} := \{\partial e_C : C \in \mathfrak{C}_{\pi_\prec}(M)\}$ form a reduced Gröbner basis of $\Im(M)$ with respect to the term order $\prec$.*

PROOF. From Proposition 3.2 it is enough to prove that $(\mathcal{G}_r)_\prec$ is a reduced Gröbner of $\Im_i(M)$. Let f be any element of $\Im_i(M)$, we have from Theorem 2.4 (we note $\mathfrak{U}_\pi$ the set of inactive unidependent for the order $<_\pi$) that $f = \sum_{U \in \mathfrak{U}_\pi} \xi_U \partial e_U$, $\xi_U \in k$. Let now remark that $\mathrm{lp}_\prec(\partial e_U) = e_{U \backslash \alpha_\pi(U)}$ and that these terms are all different. We have then clearly that

$\mathrm{lp}_\prec(f) = \sup_\prec\{\mathrm{lp}_\prec(\partial e_U)\}$. Given $U \in \mathfrak{U}_\pi(M)$ it is clear that $\alpha_\pi(C(U)) = \alpha_\pi(U)$. So, $C(U)\setminus\alpha_\pi(C(U)) \subset U\setminus\alpha_\pi(U)$. Let C' be a circuit of $\mathfrak{C}_\pi$ such that $C' \setminus \alpha_\pi(C') \subset C(U) \setminus \alpha_\pi(C(U))$. So we have that $\mathrm{lp}_\prec(\partial e_{C'})$ divides $\mathrm{lp}_\prec(\partial e_U)$, and $(\mathcal{G}_r)_\prec$ is a Gröbner basis of $\Im_i(M)$.

Suppose for a contradiction that $(\mathcal{G}_r)_\prec$ is not a reduced Gröbner basis: i.e., there exist two circuits C and C' in $\mathfrak{C}_\pi$ and an element $c \in C$ such that $e_{C'\setminus\alpha_\pi(C')}$ divides $e_{C\setminus c}(\Leftrightarrow C' \setminus \alpha_\pi(C') \subset C\setminus c)$. First we can say that $c \neq \alpha_\pi(C)$ because the sets $C'\setminus\alpha_\pi(C')$ and $C\setminus\alpha_\pi(C)$ are incomparable. This, in particular, implies that $\alpha_\pi(C) \in C'\setminus\alpha_\pi(C')$, and $\alpha_\pi(C') \prec \alpha_\pi(C)$. On the other hand we have $\alpha_\pi(C') \in \mathrm{cl}(C' \setminus \alpha_\pi(C')) \subset \mathrm{cl}(C \setminus c) = \mathrm{cl}(C \setminus \alpha_\pi(C))$, so $\alpha_\pi(C) \prec \alpha_\pi(C')$, a contradiction. □

COROLLARY 3.6. *$\mathcal{G}_u := \{\partial e_C : C \in \mathfrak{C}(M)\}$ form a minimal universal Gröbner basis of $\Im(M)$.*

PROOF. From Theorem 3.5, the reduced Gröbner basis constructed for the different orders $\prec$ are all contained in $\mathcal{G}_u$ which proves the universality. We prove the minimality by contradiction. Let $C_0 = \{i_1, \ldots, i_m\}$ be a circuit of M and let $\pi \in S_n$ be a permutation such that $\pi^{-1}(i_j) = j$, $j = 1, \ldots, m$. Then $\mathcal{G}'_u := \{\partial e_C : C \in \mathfrak{C} \setminus C_0\}$ is not a Gröbner basis since $\mathrm{lp}_{\prec_\pi}(\partial e_{C_0}) = e_{C_0\setminus i_1}$ is not in $\mathrm{Lp}_{\prec_\pi}(\mathcal{G}'_u)$. □

To finish we give a characterization of the NBC bases of the χ-algebras in terms of the Gröbner bases of their ideals. Consider a permutation $\pi \in S_n$ and the associated re-ordering $<_\pi$ of $[n]$. When the $<_\pi$-smallest element $\inf_{<_\pi}(C)$ of a circuit $C \in \mathfrak{C}(M)$, $|C| > 1$, is deleted, the remaining set, $C \setminus \inf_{<_\pi}(C)$, is called a *$\pi$-broken circuit* of M. We set

$$\boldsymbol{nbc}_\pi(M) := \{e_X : X \subset [n] \text{ contains no } \pi\text{-broken circuit of } M\}.$$

As the algebra $\mathcal{A}_\chi(M)$ does not depend on the ordering of the elements of M it is clear that π-$\boldsymbol{nbc}(M)$ is a NBC basis of $\mathcal{A}_\chi(M)$.

COROLLARY 3.7. *Let $\mathcal{B}$ be a basis of $\mathcal{A}_\chi(M)$. Then are equivalent:*

(3.7.1) *$\mathcal{B}$ is the canonical basis $\mathcal{B}_\prec$, for some term order $\prec$ of $\mathbb{T}$.*
(3.7.2) *$\mathcal{B}$ is the π-NBC basis π-$\boldsymbol{nbc}(M)$, for some permutation $\pi \in S_n$.*
(3.7.3) *$\mathcal{B}$ is the canonical basis $\mathcal{B}_{\mathcal{G}_r}$, for some reduced Gröbner basis $\mathcal{G}_r$ of the ideal $\Im(M)$.*

PROOF. (3.7.1) $\Rightarrow$ (3.7.2). Let $\prec$ be a term order of $\mathbb{T}$. Since from Corollary 3.6 $\mathcal{G}_u$ is a universal Gröbner basis of $\Im(M)$ it is trivially a Gröbner basis relatively to $\prec$. We have already remarked that the leading term of ∂e_C is $e_{C\setminus\inf_{<_{\pi_\prec}}(C)}$. From Proposition 3.3 we conclude that $\mathcal{B}_\prec = \pi_\prec$-$\boldsymbol{nbc}(M)$.

(3.7.2) $\Rightarrow$ (3.7.3). Suppose that $\mathcal{B} = \pi$-$\boldsymbol{nbc}(M)$. Let $\prec_\pi$ be the degree lexicographic order of $\mathbb{T}$ determined by the permutation $\pi \in S_n$. Note that $\pi_{\prec_\pi} = \pi$. From Theorem 3.5 we know that $(\mathcal{G}_r)_{\prec_\pi} = \{\partial e_C : C \in \mathfrak{C}_{\prec_\pi}\}$ is the reduced Gröbner basis of $\Im(M)$ with respect to the term order $\prec_\pi$. Then $\mathcal{B}$ is the canonical basis for the reduced Gröbner basis $(\mathcal{G}_r)_{\prec_\pi}$.

(3.7.3) $\Rightarrow$ (3.7.1). This is a consequence of Proposition 3.3 and Remark 3.4. □

ACKNOWLEDGEMENTS

The author's research was supported by FCT through the projects SAPIENS/36563/99 and by DONET.

REFERENCES

1. T. Becker and V. Weispfenning, *Gröbner Bases. A Computational Approach to Commutative Algebra*, In cooperation with Heinz Kredel, *Graduate Texts in Mathematics* **141**, Springer, New York, 1993.
2. A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, *Oriented Matroids*, 2nd edn. *Encyclopedia Math. Appl.* **46**, Cambridge University Press, Cambridge, 1999.
3. R. Cordovil, A commutative algebra of an oriented matroid, *Discrete Comput. Geom.*, (2001).
4. D. Forge and M. Las Vergnas, Orlik–Solomon type algebras, *Europ. J. Combinatorics*, **22** (2001), 699–704.
5. P. Orlik and L. Solomon, Combinatorics and topology of complements of hyperplanes, *Invent. Math.*, **56** (1980), 167–189.
6. P. Orlik and H. Terao, *Arrangements of Hyperplanes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*], Springer, Berlin, 1992.
7. P. Orlik and H. Terao, Commutative algebras for arrangements, *Nagoya Math. J.*, **134** (1992), 65–73.
8. N. White (ed.), Theory of matroids. *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1986, Vol. 26.
9. S. Yuzvinsky, Orlik–Solomon algebras in algebra and topology, *Russian Math. Surveys*, **56** (2001), 293–364.

Received 26 October 2001 and accepted 14 March 2002

DAVID FORGE

Centro de Investigação Operacional,
Departamento de Estatística e Investigação Operacional,
Faculdade de Ciências, Universidade de Lisboa,
Bloco C2 - Piso 2,
Campo Grande,
1749-016 Lisboa, Portugal
E-mail: dforge@math.ist.utl.pt