Bijections between affine hyperplane arrangements and valued graphs

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Abstract. We show new bijective proofs of previously known formulas for the number of regions of some deformations of the braid arrangement, by means of a bijection between the no-broken-circuit sets of the corresponding integral gain graphs and some kinds of labelled binary trees. This leads to new bijective proofs for the Shi, Catalan, and similar hyperplane arrangements.

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1. Introduction

An *integral gain graph* is a graph whose edges are labelled invertibly by integers; that is, reversing the direction of an edge negates the label (the *gain* of the edge). The *affinographic hyperplane arrangement*, $\mathcal{A}[\Phi]$, that corresponds to an integral gain graph Φ is the set of all hyperplanes in $\mathbb{R}^n$ of the form $x_j - x_i = g$ for edges (i, j) with $i < j$ and gain g in Φ. (See [8, Section IV.4.1, pp. 270–271] or [4].)

In recent years there has been much interest in real hyperplane arrangements of this type, such as the Shi arrangement, the Linial arrangement, and the composed-partition or Catalan arrangement. For all these families, the characteristic polynomials and the number of regions have been found. See for example [6]. For the Shi arrangement, Athanasiadis [1] gave a bijection between the regions and the parking functions.

In this paper we give bijective proofs of the number of regions for some of these arrangements by establishing bijections between the no-broken-circuit (NBC) sets and types of labelled trees and forests, which can be counted directly. This means that we use the fact that the number of regions is equal to the number of NBC sets.

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2. Basic definitions

An *integral gain graph* $\Phi = (\Gamma, \varphi)$ consists of a graph $\Gamma = (V, E)$ and an orientable function $\varphi : E \to \mathbb{Z}$, called the *gain mapping*. Orientability means that, if (i, j) denotes an edge oriented in one direction and (j, i) the same edge with the opposite orientation, then $\varphi(j, i) = -\varphi(i, j)$. We have no loops but multiple edges are permitted. For the rest of the paper, we denote the vertex set by $V = \{1, 2, \ldots, n\} =: [n]$ with $n \geq 1$. We use the notations (i, j) for an edge with endpoints i and j, oriented from i to j, and $g(i, j)$ for such an edge with gain g; that is, $\varphi(g(i, j)) = g$. (Thus $g(i, j)$ is the same edge as $(-g)(j, i)$. The edge $g(i, j)$ corresponds to a hyperplane whose equation is $x_j - x_i = g$.) A *circle* is a connected 2-regular subgraph, or its edge set. Writing a circle C as a word $e_1 e_2 \cdots e_l$, the gain of C is $\varphi(C) := \varphi(e_1) + \varphi(e_2) + \cdots + \varphi(e_l)$; then it is well defined whether the gain is zero or nonzero. A subgraph is called *balanced* if every circle in it has gain zero. We will consider most especially balanced circles.

Given a linear order $<_O$ on the set of edges E, a *broken circuit* is the set of edges obtained by deleting the smallest element in a balanced circle. A set of edges, $N \subseteq E$, is a *no-broken-circuit set* (NBC set for short) if it contains no broken circuit. This notion from matroid theory (see [2] for reference) is very important here. We denote by $\mathcal{N}$ the set of NBC sets of the gain graph. It is well known that this set depends on the choice of the order, but its cardinality does not.

We can now transpose some ideas or problems from hyperplane arrangements to gain graphs. For any integers a, b, n, let K_n^{ab} be the gain graph built on vertices $V = [n]$ by putting on every edge (i, j) all the gains k, for $a \leq k \leq b$. These gain graphs are expansion of the complete graph and their corresponding arrangements are called sometimes deformations of the braid arrangement, truncated arrangements or affinographic arrangements. We have four main examples coming from well known hyperplane arrangements. We denote by B_n the gain graph K_n^{00} and call it the *braid gain graph*, by L_n the gain graph K_n^{11} and call it the *Linial gain graph*, by S_n the gain graph K_n^{01} and call it the *Shi gain graph* and finally by C_n the gain graph K_n^{-11} and call it the *Catalan gain graph*.

3. Height

We introduce the notion of height function on an integral gain graph on the vertex set $[n]$. A height function h defines two important things for the rest of the paper: the induced gain subgraph $\Phi[h]$ of a gain graph Φ and an order O_h on the set of vertices extended lexicographically to the set of edges.

Definition 1. A *height function* on a set V is a function h from V to $\mathbb{N}$ (the natural numbers including 0) such that $h^{-1}(0) \neq \emptyset$. The *corner* of the height function is the smallest element of greatest height.

Let Φ be a connected and balanced integral gain graph on a set V of integers. The *height function* of the gain graph is the unique function h_Φ such that for every edge $g(i, j)$ we have $h_\Phi(j) - h_\Phi(i) = g$. (Such a function exists if and only if Φ is balanced.) The *corner of* Φ is the corner of h_Φ.

We say that an edge $g(i, j)$ is *coherent with* h if $h(j) - h(i) = g$.

Definition 2. Let Φ be a gain graph also on $V = [n]$ and h be a height function on V. The subgraph $\Phi[h]$ of Φ *selected by* h is the gain subgraph on the same vertex set V whose edges are the edges of Φ that are coherent with h.

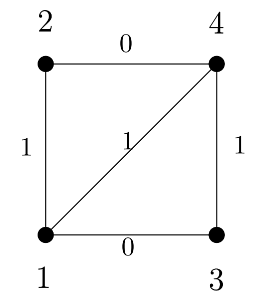

FIGURE 1. The gain graph $K_4^{01}[h]$ for $h(2) = h(4) = 1$ and $h(1) = h(3) = 0$

Definition 3. Given a height function h on the set V, the order O_h on the set $V = [n]$ is defined by $i <_{O_h} j$ if and only if $h(i) > h(j)$ or $(h(i) = h(j)$ and $i < j)$. The order O_h is extended lexicographically to an order O_h on the set of edges coherent with the height function.

For example if $n = 4$, $a = 0$, $b = 1$, and the height function h has $h(2) = h(4) = 1$ and $h(1) = h(3) = 0$, we get the order $2 <_{O_h} 4 <_{O_h} 1 <_{O_h} 3$. The corresponding $K_4^{01}[h]$ is given in Figure 1. Note that only 5 of the 12 edges are coherent with the height function.

4. NBC sets and NBC trees in gain graphs

Given a linear order $<_O$ on the set of edges E, a *broken circuit* is the set of edges obtained by deleting the smallest element in a balanced circle. An NBC set in a gain graph Φ is basically an edge set, as it arises from matroid theory. We usually assume an NBC set is a spanning subgraph, i.e., it contains all vertices. Thus, an NBC tree is a spanning tree of Φ. Sometimes we wish to have non-spanning NBC sets, such as the components of an NBC forest; then we write of NBC *subtrees*, which need not be spanning trees. The set of the NBC sets of Φ with respect to an order O is denoted $\mathcal{N}_O(\Phi)$.

Given a height function h, a gain graph Φ and a linear order $<_{O_h}$ on the edges, they determine the set of NBC sets of the subgraph $\Phi[h]$ relative to the order $<_{O_h}$, denoted by $\mathcal{N}_O(\Phi[h])$. As always, this set depends on the choice of the order but its cardinality does not.

Lemma 4. *Given an NBC tree A of height function h ($h = h_A$) with corner c, the forest $A \setminus c$ is a disjoint union of NBC subtrees of height functions h_1,...,h_k, and the orders O_{h_i} are restrictions of the order O_h.* $\square$

It is known from matroid theory that the NBC sets of the semimatroid of an affine arrangement $\mathcal{A}$, with respect to a given ordering $<_O$ of the edges, correspond to the regions of the arrangement [6, Section 9]. The semimatroid of $\mathcal{A}[\Phi]$ is the frame (previously "bias" in [8]) semimatroid of Φ, which consists of the balanced edge sets of the gain graph Φ ([8, Sect. II.2] or [4]). Thus, the NBC sets of that semimatroid are spanning forests of Φ. Therefore $|\mathcal{N}_O(\Phi)|$ equals the number of regions of $\mathcal{A}[\Phi]$.

We show that the total number of NBC trees in an integral gain graph Φ equals the sum, over all height functions h, of the number of NBC trees in $\Phi[h]$.

Let Φ be connected. Then we can decompose $\mathcal{N}_O(\Phi)$ into disjoint subsets $\mathcal{N}_O(\Phi[h])$, one for each height function h that is coherent with Φ (that means that $\Phi[h]$ is also connected).

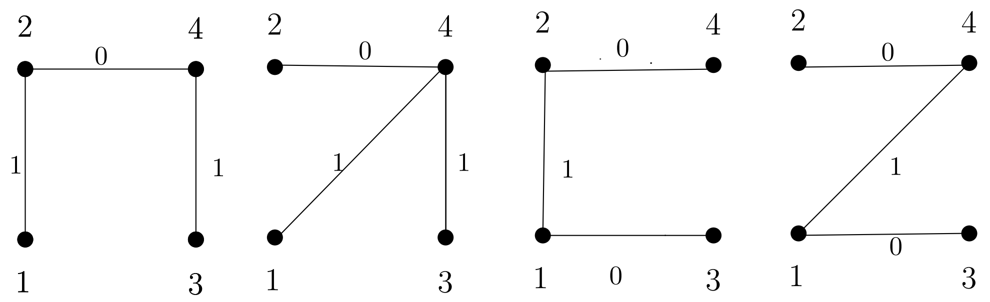

FIGURE 2. The NBC trees of the gain graph $K_4^{01}[h]$

We have now:

$$\mathcal{N}_O(\Phi) = \biguplus_h \{\mathcal{N}_O(\Phi[h]) \mid h \text{ is coherent with } \Phi\}.$$

Therefore, the total number of NBC trees of all $\Phi[h]$ with respect to all possible height functions h equals the number of NBC trees of Φ.

For example, the NBC trees corresponding to the gain graph $K_4^{01}[h]$ from Figure 1 are given in Figure 2.

5. $[a, b]$-gain graphs and their NBC trees

Let a and b be two integers such that $a \leq b$. The interval $[a, b]$ is the set $\{i \in \mathbb{Z} \mid a \leq i \leq b\}$. We consider the gain graph K_n^{ab} with vertices labelled by $[n]$ and with all the edges $g(i, j)$, such that $i < j$ and $g \in [a, b]$. These gain graphs, K_n^{ab}, are called $[a, b]$-gain graphs. The arrangements that correspond to these gain graphs, called deformations of the braid arrangement, have been of particular interest. The braid arrangement corresponds to the special case $a = b = 0$. Other well studied cases are $a = -b$ (extended Catalan), $a = b = 1$ (Linial) and $a = b - 1 = 0$ (Shi).

We will describe the set of NBC trees of $K_n^{ab}[h]$ for a given height function h. The idea is that, as mentioned above, the height function h defines an order O_h on a balanced subgraph. We will then be able to describe the NBC sets coherent with h for the order O_h.

Proposition 5. *Let a and b be integers such that $a \leq b$. Let h be a height function of corner c and let Φ be a spanning tree of $K_n^{ab}[h]$. Suppose c is incident to the edges $g_i(c, v_i)$, $1 \leq i \leq k$, and let Φ_i be the connected component of $\Phi \setminus c$ containing c_i (which is a subtree). Then Φ is an NBC tree if and only if all the Φ_i are NBC trees and each v_i is the O_h-smallest vertex of Φ_i adjacent to c in $K_n^{ab}[h]$.*

Proof. Everything comes from the choice of the order O_h for the vertices and the edges. If we have a vertex v in Φ_i such that $v <_{O_h} v_i$ for which the edge $(c, v) \in K_n^{ab}[h]$ exists then this edge is smaller than all the edges of $\Phi_i + c$. Such an edge then closes a balanced circle being the smallest edge of the circuit which is not possible.

In the other direction, if Φ is not an NBC tree then there is an edge (x, y) in $K_n^{ab}[h]$ closing a balanced circle by being the smallest edge of the circuit. Since the Φ_i are by hypothesis are NBC trees the vertices x and y cannot be in the same Φ_i. They cannot be in two different Φ_i either since the smallest edge would contain c necessarily. The last solution is that one of the vertex, say x, is c and that the other vertex y is in a Φ_i. Since the edge (c, v_i) will be in the circuit we need to have $(x, y) <_{O_h} (c, v_i)$. This implies the condition of the proposition. $\square$

6. $[a, b]$-GAIN GRAPHS WITH $a + b = 0$ OR 1

We start this Section by a Lemma that will help us for our recursive construction.

Lemma 6. *If $a + b = 0$ or 1, the vertices v_i are the corners of the subtrees Φ_i (as in Proposition 5).*

Proof. In the case where $a + b = 0$, the interval $[a, b]$ is of the form $[-b, b]$ where b is a nonnegative integer. Similarly in the case where $a + b = 1$ the interval $[a, b]$ is of the form $[-b, b+1]$ where b is a positive integer. Therefore whenever a gain g is present in the graph it implies that all gains in the interval $[-|g| + 1, |g|]$ also exists in the graph. Therefore, if there exists a vertex v in Φ_i with $h(v) > h(v_i)$ then the edge (c, v) necessarily exists in $K_n^{ab}[h]$. In the case $h(v) = h(c_i)$ and $v < c_i$, the edge (c, v) also necessarily exists in $K_n^{ab}[h]$.

Let us suppose that c_i is not the corner of its tree. Then there exists v such that $h(v) > h(c_i)$ or $h(v) = h(c_i)$ and $v < c_i$. By taking the edge (c, c_i) along with the unique path $P(c_i, v)$ in this subtree we get a path P which is a broken circuit of $K_n^{ab}[h]$ (because the edge (c, v) is smaller in the order O_h than all the edges of P) and this contradicts the fact that $\mathcal{T}$ is an NBC tree of K_n^{ab}. Using Proposition 5, we get a contradiction and v_i has to be the corner of Φ_i. $\square$

Note that this will not be true as soon as $a+b = 2$ as in the Linial case. We now introduce our family of trees.

Definition 7. Let α and β be natural integers (including 0). An (α, β)*-rooted labelled tree* with n vertices is a rooted, labelled and weighted tree on the set of vertices $[n]$, such that each edge of the tree, (i, j) where i is the ancestor and j the descendant, is weighted with an integer from

- the interval $[1, \alpha]$ if $i < j$ and
- the interval $[1, \beta]$ if $i > j$.

Note that if one of the integers α or β is equal to 0 then the corresponding interval is empty. This just implies that such edges cannot exist. In the next theorem we go from the NBC trees of K_n^{ab} to (α, β)-trees by cutting the interval $[a, b]$ in two parts : the part $[a, 0]$ of the negative or null gains will correspond to α and the part $[1, b]$ of the positive gains will correspond to β.

Theorem 8. *If $b + a = 0$ or $b + a = 1$, the NBC trees of K_n^{ab} are in bijection with the $(1 - a, b)$-trees on $[n]$.*

Proof. We recursively decompose the NBC trees of K_n^{ab}. Let Φ be an NBC tree. Let c be its corner and let $c_1, c_2, \ldots, c_k$ be the neighbors of c with gains $g_1, g_2, \ldots, g_k$. We now construct a corresponding $(1-a, b)$-tree. The root of the $(1-a, b)$-tree is c, $c_1, c_2, \ldots, c_k$ are its children and the edges from c to c_i get the label g_i if it is strictly positive and $1 - g_i$ otherwise. The decomposition continues recursively on the trees with corners $c_1, c_2, \ldots, c_n$.

When we take out the vertex c from Φ, we get a forest of NBC trees, where each c_i is in a different tree. To prove that the decomposition is correct, we use Lemma 6 and we know that each c_i is the corner of its component. $\square$

A direct consequence of our Theorem is that :

Corollary 9. *If $b + a = 0$ or $b + a = 1$, the number of regions of $\mathcal{A}[K_n^{ab}]$ is equal to the number of $(1 - a, b)$-rooted labelled forests with n vertices.*

Proof. To get this consequence from the previous theorem, we use the facts that for any affine hyperplane arrangement the number of NBC sets is equal to the number of regions and that an NBC set is a union of NBC trees. See Proposition 9.4 of [6]. □

Theorem 10. [6] *The number of regions of* $\mathcal{A}[K_n^{ab}]$ *is*

$$an(an-1)\dots(an-n+2), \quad \textit{if} \;\; a+b=0,$$

and

$$(an+1)^{n-1}, \quad \textit{if} \;\; a+b=1.$$

To finish our proof of Theorem 10, we have to count the (α, β)-labelled trees and (α, β)-labelled forests.

Proposition 11. *The number of* (α, β)*-rooted labelled trees with* n *vertices is*

$$\prod_{i=1}^{n-1}[(\alpha-\beta)i+n\beta].$$

The number of (α, β)*-rooted labelled forests with* n *vertices is*

$$\prod_{i=1}^{n-1}[(\alpha-\beta)i+n\beta+1].$$

Proof. We suppose that $\alpha \geq \beta$. The other case is analogous. We first enumerate (α, β)-rooted labelled trees. We split the edges of the trees into two groups :

- The edges with labels $\beta+1, \dots, \alpha$.
- The edges with labels $1, 2, \dots, \beta$.

Suppose that the first group has k edges. They form a decreasing forest on n vertices with k edges, such that the edges can have $(\alpha-\beta)$ different labels. The number of such forests is well known to be $|s(n, n-k)|(\alpha-\beta)^k$ where $s(n, k)$ is the Stirling number of the first kind.

The second group is a rooted labelled forest on n vertices with $n-k-1$ edges, such that the edges can have β different labels. The number of such forests is $(n\beta)^{n-k-1}$. The two groups have disjoint edges. Therefore, we deduce that the number of (α, β)-rooted labelled trees with n vertices and k edges in the first group is :

$$|s(n, n-k)|(\alpha-\beta)^k(n\beta)^{n-k-1}.$$

Therefore the number of (α,β)-rooted labelled trees with n vertices is :

$$\begin{aligned}
&\sum_{k=0}^{n} |s(n,n-k)|(\alpha-\beta)^k(n\beta)^{n-k-1}\\
&=\frac{(\alpha-\beta)^n}{n\beta}\sum_{k=0}^{n}|s(n,n-k)|\left(\frac{n\beta}{\alpha-\beta}\right)^{n-k}\\
&=\frac{(\alpha-\beta)^n}{n\beta}\prod_{i=0}^{n-1}\left(i+\frac{n\beta}{\alpha-\beta}\right)\\
&=\prod_{i=1}^{n-1}((\alpha-\beta)i+n\beta). \qquad \square
\end{aligned}$$

The proof for the number of (α,β)-rooted labelled forests is identical.

7. The special cases of the braid and the Shi arrangements

The first cases of $[a,b]$-gain graphs with $a+b=0$ or $a+b=1$ are obtained by taking $a=0$. The gain graph with $a+b=0$ and $a=0$ corresponds to the braid arrangement and the gain graph with $a+b=1$ and $a=0$ corresponds to the Shi arrangement. A recent bijective correspondence for the braid arrangement appears in the recent paper [5].

Corollary 12. *The NBC sets of the braid arrangement in dimension n are in one-to-one correspondence with the decreasing labelled trees on $n+1$ vertices.*

Proof. Theorem 8 tells us that the set of NBC trees of the braid arrangement (case $a=b=0$) is in one-to-one correspondence with the set of $(1,0)$-labelled trees with n vertices. Such labelled trees have no possible value on edges (i,j) when $i>j$ and have the value 1 on edges (i,j) when $i<j$ (and since there is no choice we can forget the value). This means that the correspondence of NBC trees of the braid arrangement is with the set of rooted labelled trees such that the label of the father is always smaller than the label of the son (such a tree is called an increasing tree). To get the bijection between the set of NBC sets and the set of increasing rooted labelled trees we just need to add vertex 0 and to connect it to the different increasing rooted labelled trees coming from the NBC trees (components). $\square$

Corollary 13. *The NBC sets of the Shi arrangement in dimension n are in one-to-one correspondence with the labelled trees on $n+1$ vertices.*

Proof. Theorem 8 tells us that the set of NBC trees of the Shi arrangement (case $a=0$ and $b=1$) is in one-to-one correspondence with the set of $(1,1)$-labelled trees with n vertices. Such labelled trees have the value 1 on edges (i,j) when $i<j$ as well as when $j>i$. As in the previous proof, since there is only one possible value it can be ignored. This means that the correspondence of NBC trees of the Shi arrangement is with the set of rooted labelled trees. To get the bijection between the set of NBC sets and the set of labelled trees on $n+1$ vertices we just need to add vertex $n+1$ and to connect the different rooted labelled trees coming from the NBC trees (components). $\square$

8. Conclusion

In this paper, we show that given a height function on a gain graph K_n^{ab} with $a + b = 0$ or 1, the corresponding NBC trees with n vertices and corner c are in bijection with some trees with n veritices and root c. In a following paper, we will show that this is still true in the Linial case; that is, $a = 0$ and $b = 2$ [3]. It would be interesting to investigate whether this is true for other deformations of the Braid arrangement.

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