PERIODIC UNIQUE BETA-EXPANSIONS: THE SHARKOVSKIĬ ORDERING

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To O. M. Sharkovskiĭ on the occasion of his 70th birthday

ABSTRACT. Let $\beta \in (1,2)$. Each $x \in [0, \frac{1}{\beta-1}]$ can be represented in the form

$$x = \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k},$$

where $\varepsilon_k \in \{0,1\}$ for all k (a β-expansion of x). If $\beta > \frac{1+\sqrt{5}}{2}$, then, as is well known, there always exist $x \in (0, \frac{1}{\beta-1})$ which have a unique β-expansion.

In the present paper we study (purely) periodic unique β-expansions and show that for each $n \geq 2$ there exists $\beta_n \in [\frac{1+\sqrt{5}}{2}, 2)$ such that there are no unique periodic β-expansions of smallest period n for $\beta \leq \beta_n$ and at least one such expansion for $\beta > \beta_n$.

Furthermore, we prove that $\beta_k < \beta_m$ if and only if k is less than m in the sense of the Sharkovskiĭ ordering. We give two proofs of this result, one of which is independent, and the other one links it to the dynamics of a family of trapezoidal maps.

1. HISTORY OF THE PROBLEM AND FORMULATION OF RESULTS

This paper continues the line of research related to the combinatorics of representations of real numbers in non-integer bases ([12, 13, 15, 21]).

Let $\beta \in (1,2)$ be our parameter and let $x \in I_\beta := [0, \frac{1}{\beta-1}]$. Then x has at least one representation of the form

$$x = \pi_\beta(\varepsilon_1, \varepsilon_2, \dots) := \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k}, \quad \varepsilon_k \in \{0,1\}, \tag{1.1}$$

(use, e.g., the greedy algorithm) which we call *a β-expansion of x* and write $x \sim (\varepsilon_1, \varepsilon_2, \dots)_\beta$.

Let us recall some key results regarding β-expansions. Firstly, if $1 < \beta < G := \frac{1+\sqrt{5}}{2}$, then each $x \in \left(0, \frac{1}{\beta-1}\right)$ has a continuum of β-expansions [12]. On the other hand, for any $\beta > G$, there exist infinitely many x which have a unique β-expansion (see [9, 13]), although almost all $x \in I_\beta$ still have a continuum of β-expansions [21].

More specifically, put $x \sim (010101\dots)_\beta = \frac{1}{\beta^2-1}$. Then both x and βx have a unique β-expansion [13].

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Let X_β denote the set of x which have a unique β-expansion (these numbers are sometimes called univoque numbers with respect to β, see [10] for example). Denote by

$$(\mathfrak{m}_k)_{k=0}^{\infty} = 0110\ 1001\ 0110\ 1001\dots$$

the Thue-Morse sequence, i.e., the fixed point of the morphism $0 \to 01, 1 \to 10$ (see, e.g., the survey paper [6] for the many wonderful properties of this famous sequence). Let now $\beta_{KL} \approx 1.78723$ denote the *Komornik-Loreti constant*, i.e., the unique positive solution of the equation

$$\sum_{k=1}^{\infty} \mathfrak{m}_k x^{-k} = 1.$$

This constant proves to be the smallest β such that $1 \in X_\beta$ – see [14]. Note that in [4] it was shown that β_{KL} is transcendental.

The main result of [13] asserts that the set X_β is

1. infinite countable if $G < \beta < \beta_{KL}$;
2. a continuum of zero Hausdorff dimension if $\beta = \beta_{KL}$; and
3. a continuum of Hausdorff dimension strictly between 0 and 1 if $\beta_{KL} < \beta < 2$.

Notice that for $\beta > \beta_{KL}$ the set X_β is not necessarily a Cantor set and may have a somewhat complicated topology – see [15] for more detail and also for the case of an arbitrary $\beta > 1$.

Let Σ_β denote the set of all 0-1 sequences which are unique β-expansions, and σ_β denote the shift on Σ_β, i.e., $\sigma_\beta(\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) = (\varepsilon_2, \varepsilon_3, \dots)$. Let $\pi_\beta : \Sigma_\beta \to X_\beta$ be the projection given by (1.1)). It is obvious that π_β is a bijection.

Since $\pi_\beta(\{\varepsilon : \varepsilon_1 = 0\}) \subset \left[0, \frac{1}{\beta}\right]$ and $\pi_\beta(\{\varepsilon : \varepsilon_1 = 1\}) \subset \left[\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}\right]$, the set X_β has an empty intersection with the middle interval $\left[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\right]$. Thus, we have the following commutative diagram:

$$\begin{tikzcd}
\Sigma_\beta \arrow[r, "\sigma_\beta"] \arrow[d, "\pi_\beta"] & \Sigma_\beta \arrow[d, "\pi_\beta"] \\
X_\beta \arrow[r, "F_\beta"] & X_\beta
\end{tikzcd}$$

where

$$F_\beta(x) = \begin{cases} \beta x, & 0 \le x < \frac{1}{\beta}, \\ \beta x - 1, & \frac{1}{\beta(\beta-1)} < x \le \frac{1}{\beta-1}. \end{cases}$$

To state the main theorem of the present paper, we recall that the *Sharkovskiĭ ordering* on the natural numbers is as follows:

$$\begin{array}{cccccccccccc}
 & 3 & \rhd & 5 & \rhd & 7 & \rhd & \cdots & \rhd & 2m+1 & \rhd & \cdots \\
\rhd & 2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & \cdots & \rhd & 2\cdot(2m+1) & \rhd & \cdots \\
\rhd & 4\cdot 3 & \rhd & 4\cdot 5 & \rhd & 4\cdot 7 & \rhd & \cdots & \rhd & 4\cdot(2m+1) & \rhd & \cdots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
\rhd & 2^n\cdot 3 & \rhd & 2^n\cdot 5 & \rhd & 2^n\cdot 7 & \rhd & \cdots & \rhd & 2^n\cdot(2m+1) & \rhd & \cdots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
 & & & \cdots & \rhd & 8 & \rhd & 4 & \rhd & 2 & \rhd & 1,
\end{array}$$

where the relation $a \ \rhd \ b$ indicates that a comes before b in the ordering.

Remark 1.1. This is a complete ordering on $\mathbb{N} := \{1, 2, 3, \ldots\}$ since $(n, m) \mapsto 2^n(2m + 1)$ is a bijection $(\mathbb{N} \cup \{0\})^2 \to \mathbb{N}$.

Theorem 1.2. (Sharkovskiĭ's Theorem) *Let f be a continuous map of the real line. If $k \rhd l$ in Sharkovkiĭ's ordering and if f has a point of smallest period k, then f also has a point of smallest period l.*

This was originally proved in [20], see also [11].

Now we are ready to state the main theorem of the present paper. Put

$$\begin{aligned} U_n &= \{\beta \in (1,2) : \Sigma_\beta \text{ contains a periodic sequence of smallest period } n\}. \\ &= \{\beta \in (1,2) : F_\beta : X_\beta \to X_\beta \ \text{ has an } n\text{-cycle.}\} \end{aligned}$$

(By the result quoted above, $U_2 = (G, 2)$, for instance.)

Theorem 1.3. *There exist real numbers β_n in $(1, 2)$ such that $U_n = (\beta_n, 2)$ for any $n \geq 2$. Furthermore, $\beta_n < \beta_m$ if and only if $n \lhd m$ in the sense of the Sharkovskiĭ ordering.*

Remark 1.4. We are going to give explicit formulae for the β_n via the fragments of the Thue-Morse sequence as well as the first n-cycle to appear – see Proposition 2.16 below. Note that for $n = 2^k$ this result is essentially contained in [13], where it is also shown that $\beta_{2^k} \nearrow \beta_{KL}$.

We will make use *inter alia* of results in combinatorics on words. In particular a set of binary sequences studied in [1] and denoted by Γ (see (2.2)) will play a rôle in some proofs. In this respect one could compare the combinatorial part of the main result of [13] with the "Théorème" and "Corollaire" on page 37 of [2].

The structure of this paper is as follows: the next section will be devoted to the proof of Theorem 1.3. In Section 3 we discuss possible links of this claim with the classical theory of one-dimensional continuous maps. (Note that Reference [5] showed a link between kneading sequences of unimodal continuous maps and unique β-expansions of 1.)

Our central result of that section is in the negative direction: we show that if $\beta > \beta_4$, then any continuous extension S_β of $F_\beta : X_\beta \to X_\beta$ has a k-cycle for any $k \geq 2$ provided S_β is monotonic on $[0, 1/\beta]$.

Section 4 is devoted to a different proof of our Theorem 1.3 via the classical Sharkovskiĭ theorem applied to a family of trapezoidal maps, and – again – some combinatorics on words and properties of the set Γ.

2. PROOF OF THE MAIN THEOREM

Definition 2.1. Put $\tau_\beta x = \beta x \bmod 1$, and $\varepsilon_n = [\beta \tau_\beta^{n-1}(x)]$ for $x \in [0,1]$ and $n \geq 1$. Then $(\varepsilon_1 \varepsilon_2 \dots)$ is called the *greedy expansion* of x in base β.

Put $\Sigma = \{0,1\}^{\mathbb{N}}$ and let $d(\beta) \in \Sigma$ denote the greedy expansion of 1 in base β. If $d(\beta)$ is finite, i.e., of the form $d(\beta) = \varepsilon_1 \dots \varepsilon_{n-1} 10^\infty$, then $d'(\beta) := (\varepsilon_1 \dots \varepsilon_{n-1} 0)^\infty$ - the *quasi-greedy expansion of* 1.

We say that the sequence (resp. the finite word) $\varepsilon = \varepsilon_1 \varepsilon_2 \dots$ is *lexicographically less* than the sequence (resp. the finite word with same length) ε' if $\varepsilon_k < \varepsilon'_k$ for the least k such that $\varepsilon_k \neq \varepsilon'_k$. Notation: $\varepsilon \prec \varepsilon'$. We write $\varepsilon \preceq \varepsilon'$ if either $\varepsilon \prec \varepsilon'$ or $\varepsilon = \varepsilon'$.

We will use the following simple remark.

Remark 2.2. The relation $\preceq$ is a total order on the set of infinite sequences (resp. on the set of words of given length). Further let ε and ε' be two infinite sequences. Let u and u' be their respective prefixes of length say ℓ, then $u \prec u'$ implies $\varepsilon \prec \varepsilon'$, and $\varepsilon \preceq \varepsilon'$ implies $u \preceq u'$.

We need the following auxiliary results on greedy and quasi-greedy expansions:

Lemma 2.3. (1) *If $d'(\beta)$ is defined, then it is also an expansion of 1 in base β.*

(2) *The equation $1 = \sum_{i=1}^\infty \varepsilon_j x^{-j}$, for some fixed $\varepsilon = \varepsilon_1 \varepsilon_2 \varepsilon_3 \dots \in \Sigma$, with at least one 0 and at least two 1s, always has a unique solution $\beta \in (1,2)$.*

(3) **(Monotonicity)** *Let $\beta, \tilde\beta \in (1,2)$. Then $\beta > \tilde\beta$ if, and only if $d(\beta) \succ d(\tilde\beta)$.*

(4) *Let $\beta, \tilde\beta \in (1,2)$ and assume that $d'(\beta)$ and $d'(\tilde\beta)$ are both defined and of the same smallest period. Then $\beta > \tilde\beta$ if, and only if $d'(\beta) \succ d'(\tilde\beta)$.*

(5) *Assume $\varepsilon \in \Sigma$ satisfies*

$$\sigma^j \varepsilon \begin{cases} \prec \varepsilon & \text{if } j \not\equiv 0 \bmod n; \\ = \varepsilon & \text{if } j \equiv 0 \bmod n. \end{cases}$$

Then there exists $\beta \in (1,2)$ such that $d'(\beta)$ is defined and equals ε.

Proof. (1) is trivial exercise, while (2) follows from letting $f(x) := \sum_{i=1}^\infty \frac{\varepsilon_i}{x^i} - 1$ and observing that $f(1) > 0$, $f(2) < 0$, and $f'(x) < 0$, $\forall x \in (1,2)$; (3) is proved in [19] and (4) is an easy consequence of it.

(5) follows from the fact that necessarily $\varepsilon_{pn} = 0$ for all $p \in \mathbb{N}$ (otherwise the condition in question is not satisfied), whence $\varepsilon_1 \dots \varepsilon_{n-1} 10^\infty$ is the greedy expansion of 1 in base β. □

Theorem 2.4. (Parry [19]**)** *Let $\varepsilon \in \Sigma$. Then $\varepsilon = d(\beta)$ for some $\beta \in (1,2)$ if, and only if, $\sigma^j \varepsilon \prec \varepsilon, \forall j \geq 1$.*

This is essentially proved in [19]. The following auxiliary lemmas will be needed later on.

Lemma 2.5. *There exist no $\beta, \tilde\beta \in (1,2)$ such that $d'(\beta) \prec d(\tilde\beta) \prec d(\beta)$.*

Proof. Assume $d'(\beta) := (\varepsilon_1 \dots \varepsilon_{n-1} 0)^\infty$ is defined and let $d(\tilde\beta) := \delta_1 \delta_2 \delta_3 \dots$ and suppose $d'(\beta) \prec d(\tilde\beta) \prec d(\beta)$. This immediately forces

$$\delta_1 \delta_2 \delta_3 \dots \delta_n = \varepsilon_1 \varepsilon_2 \varepsilon_3 \dots \varepsilon_{n-1} 0.$$

By Theorem 2.4, $\delta_{n+1}\delta_{n+2}\cdots = \sigma^n d(\tilde\beta) \prec d(\tilde\beta) = \varepsilon_1\ldots\varepsilon_{n-1}0\delta_{n+1}\ldots$. Hence

$$\delta_{n+1}\ldots\delta_{2n} \preceq \varepsilon_1\ldots\varepsilon_{n-1}0.$$

On the other hand $(\varepsilon_1\ldots\varepsilon_{n-1}0)^\infty = d'(\beta) \prec d(\tilde\beta)$ implies that

$$\delta_{n+1}\ldots\delta_{2n} \succeq \varepsilon_1\ldots\varepsilon_{n-1}0,$$

and hence

$$\delta_{n+1}\ldots\delta_{2n} = \varepsilon_1\ldots\varepsilon_{n-1}0.$$

By repeating this process we see that we are forced into the spurious conclusion that $d(\tilde\beta) = d'(\beta)$. □

Put

$$\mathcal{A}_\beta = X_\beta \cap \left(\frac{2-\beta}{\beta-1}, 1\right).$$

It is clear that $\mathcal{A}_\beta$ is invariant under F_β and moreover, it is an attractor for F_β (see [13] for more detail).

Let $\overline{\varepsilon}$ denote the mirror image of ε, i.e., $(\overline{\varepsilon})_n = 1-\varepsilon_n$.

Lemma 2.6. [13] *Let* $\beta \in (1,2)$. *Then*

$$\Sigma_\beta^{\mathcal{A}_\beta} := \pi_\beta^{-1}(\mathcal{A}_\beta) = \left\{\varepsilon \in \Sigma \ : \ \overline{d(\beta)} \prec \sigma^j\varepsilon \prec d(\beta), \quad \forall\, j \geq 0\right\} \tag{2.1}$$

if $d'(\beta)$ *is not defined. If it is, replace* $d(\beta)$ *and* $\overline{d(\beta)}$ *with* $d'(\beta)$ *and* $\overline{d'(\beta)}$ *respectively in the above.*

Remark 2.7. In [13] it was shown that if $\varepsilon \in \Sigma_\beta$ is periodic, then $\pi_\beta(\varepsilon) \in \mathcal{A}_\beta$.

Put

$$\Gamma := \{\varepsilon \in \Sigma : \overline{\varepsilon} \preceq \sigma^k\varepsilon \preceq \varepsilon, \forall k \geq 0\}. \tag{2.2}$$

It is obvious that all sequences in Γ begin with 1, and furthermore, if $\varepsilon \in \Gamma$ begins with 10, then $\varepsilon = (10)^\infty$. This set has been introduced and studied in detail in [1, 2], as well as the sets $\Gamma_\eta := \{\varepsilon \in \Sigma \ : \ \overline{\eta} \preceq \sigma^j\varepsilon \preceq \eta, \ \forall\, j \geq 0\}$ for $\eta \in \Gamma$. (Actually the sequences in the set Γ studied in [1, 2] satisfy the extra condition that they begin with 11, which only excludes from the present set Γ the sequence $(10)^\infty$.)

Define

$$V_n := \{\beta \in (1,2) \ : \ d'(\beta) \text{ exists, has smallest period } n, \text{ and } d'(\beta) \in \Gamma\},$$

and let $\beta_n := \inf U_n$ and $\gamma_n := \min V_n$. (Provided $U_n, V_n \neq \emptyset$.)

Proposition 2.8. *Let* $n \geq 2$. *Then* $U_n, V_n \neq \emptyset$ *and* $\beta_n = \gamma_n$.

Proof. For any $n \geq 2$, there always exists a $\beta \in (1,2)$ such that $d'(\beta) = (1\ldots10)^\infty$ (of smallest period n) by Lemma 2.3, (5) so $V_n \neq \emptyset$. Notice also that V_n is finite, so $\min V_n$ is well defined.

Let $\beta \in (1,2)$, $\beta > \gamma_n$. Then since $\gamma_n \in V_n$, $\sigma^j d'(\gamma_n) \preceq d'(\gamma_n) \prec d(\gamma_n) \prec d(\beta)$ (using monotonicity), and similarly $\sigma^j\overline{d'(\gamma_n)} \preceq d'(\gamma_n) \prec d(\gamma_n) \prec d(\beta)$, $\forall j \geq 0$. So we have $\overline{d(\beta)} \prec \sigma^j d'(\gamma_n) \prec d(\beta), \ \forall j \geq 0$ whence $d'(\gamma_n) \in \Sigma_\beta^{\mathcal{A}_\beta} \subset \Sigma_\beta$ by Lemma 2.6 provided $d'(\beta)$ does not exist. (In the event that $d'(\beta)$ does exist, this still holds via $\overline{d'(\beta)} \prec \sigma^j d'(\gamma_n) \prec d'(\beta), \ \forall j \geq 0$. Indeed if this were false then $d'(\beta) \preceq \sigma^j d'(\gamma_n) \preceq d'(\gamma_n) \prec d(\gamma_n) \prec d(\beta)$, but this is rendered absurd by Lemma 2.5.) Hence $d'(\gamma_n) \in \Sigma_\beta$, which implies $\beta \in U_n$, because $d'(\gamma_n)$ is periodic with smallest period n, and therefore $\gamma_n \geq \beta_n$, since β was arbitrary. Moreover, we now know that $U_n \neq \emptyset$.

We complete the proof by showing $\gamma_n \leq \beta_n$. Let $\beta \in U_n$, $\beta > \beta_n$. So there exists $\varepsilon \in \Sigma_\beta$ which is periodic with smallest period n. Moreover, because ε is periodic, it must represent some number (in base β) belonging to $\mathcal{A}_\beta$. I.e., $\varepsilon \in \Sigma_\beta^{\mathcal{A}_\beta}$. Hence by Lemma 2.6, $\overline{d(\beta)} \prec \sigma^j \varepsilon \prec d(\beta), \quad \forall\, j \geq 0$. (If $d'(\beta)$ exists, use the fact that $d'(\beta) \prec d(\beta)$.)

Put $a = \max\{\sigma^j \varepsilon, \sigma^j \overline{\varepsilon} \ \mid \ 0 \leq j \leq n-1\}$. So $\overline{a} \preceq \sigma^j a \preceq a$, $\forall j \geq 0$, i.e., $a \in \Gamma$. Moreover, since a is periodic with smallest period n, we must have,

$$\sigma^j a \begin{cases} \prec a & \text{if } j \not\equiv 0 \bmod n; \\ = a & \text{if } j \equiv 0 \bmod n. \end{cases}$$

Hence $a = d'(\tilde{\beta})$, for some $\tilde{\beta}$ (by Lemma 2.3, (5)), and so $\tilde{\beta} \in V_n$.

Finally, since $d'(\tilde{\beta}) = a \prec d(\beta)$, we must have $\tilde{\beta} \leq \beta$ by Lemma 2.5. I.e., for all $\beta \in U_n$, $\beta > \beta_n$, $\exists\ \tilde{\beta} \in V_n$ such that $\tilde{\beta} \leq \beta$. Hence $\gamma_n \leq \beta_n$, as claimed. □

Corollary 2.9. $U_n = (\beta_n, 2)$.

Proof. In the above it was shown that, if $\beta \in (1,2)$ with $\beta > \gamma_n$, then $\beta \in U_n$. Moreover, it was shown that $\gamma_n = \beta_n = \inf U_n$, hence $U_n = (\beta_n, 2)$, or $[\beta_n, 2)$. We now discount the second case. Let $\beta \in U_n$. So there exists a periodic sequence, with smallest period n, $\varepsilon \in \Sigma$ such that $\pi_\beta(\varepsilon) \in X_\beta$. Moreover the fact that ε is periodic ensures that $\pi_\beta(\varepsilon) \in \mathcal{A}_\beta$ (see [13]). Hence

$$\overline{d(\beta)} \prec \sigma^j \varepsilon \prec d(\beta), \quad \forall j \in [0, n-1]$$

by Lemma 2.6.

Putting $d(\beta) := d_1 d_2 d_3 \dots$, let k be the smallest number such that

$$\overline{d_1 d_2 d_3 \dots d_k} 1^\infty \prec \sigma^j \varepsilon \prec d_1 d_2 d_3 \dots d_k 0^\infty, \quad \forall j \in [0, n-1].$$

Now by Theorem 2.4, there exists $\beta' \in (1,2)$ such that $d(\beta') = d_1 d_2 d_3 \dots d_k 0^\infty$, whence

$$\overline{d(\beta')} \prec \sigma^j \varepsilon \prec d(\beta'), \quad \forall j \in [0, n-1],$$

i.e., $\beta' \in U_n$ by lemma 2.6. Now note that $d(\beta') \prec d(\beta)$ implies $\beta' < \beta$ by monotonicity. □

We now introduce a result that allows us to make the connection to the Sharkovskiĭ ordering.

Proposition 2.10. *If a_k denote the lexicographically least sequence in Γ of smallest period k, then*

$$\begin{array}{cccccccccccc}
 & a_3 & \succ & a_5 & \succ & a_7 & \succ & \cdots & \succ & a_{2m+1} & \succ & \cdots \\
\succ & a_{2\cdot 3} & \succ & a_{2\cdot 5} & \succ & a_{2\cdot 7} & \succ & \cdots & \succ & a_{2\cdot(2m+1)} & \succ & \cdots \\
\succ & a_{4\cdot 3} & \succ & a_{4\cdot 5} & \succ & a_{4\cdot 7} & \succ & \cdots & \succ & a_{4\cdot(2m+1)} & \succ & \cdots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
\succ & a_{2^n\cdot 3} & \succ & a_{2^n\cdot 5} & \succ & a_{2^n\cdot 7} & \succ & \cdots & \succ & a_{2^n\cdot(2m+1)} & \succ & \cdots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
 & & & & & \cdots & \succ & a_8 & \succ & a_4 & \succ & a_2.
\end{array}$$

Proof. This proposition is a consequence of Proposition 2.15 and Lemma 2.12 (d) and (f) – see below. ☐

Now by Lemma 2.3 (5), $a_k = d'(\beta)$ for some $\beta \in (1,2)$. Furthermore, by Lemma 2.3 (4), $\beta = \beta_k$. Hence, invoking monotonicity allows us to apply the required ordering on the β_k. This reduces the proof of Theorem 1.3 to proving Proposition 2.10.

To calculate β_k explicitly, given a_k, we need to solve $a_k = d'(\beta)$ for β. I.e., if $a_k = \alpha_1^{(k)}\alpha_2^{(k)}\ldots$ then β_k is the unique root in $(1,2)$ of the polynomial

$$x^k - \alpha_1^{(k)}x^{k-1} - \alpha_2^{(k)}x^{k-2} - \cdots - \alpha_{k-1}^{(k)}x - 1.$$

In order to prove Proposition 2.10, we will give a construction of the sequences (a_k). This construction was suggested in [2] (see also [1]). For the sake of completeness, we give here a self-contained proof extracted from these two references. We begin with a definition and two lemmas.

Definition 2.11. We denote by μ the map defined on Σ by: if $\varepsilon = (\varepsilon_n)_{n\geq 1}$ belongs to Σ, then

$$(\mu(\varepsilon))_1 := 1 \text{ and } \forall n \geq 1, \ \begin{cases} (\mu(\varepsilon))_{2n} & := \ \varepsilon_n \\ (\mu(\varepsilon))_{2n+1} & := \ 1-\varepsilon_n. \end{cases}$$

We denote by $L := (\mathfrak{m}_n)_{n\geq 1}$ the sequence obtained by shifting the (complete) Thue-Morse sequence $(\mathfrak{m}_n)_{n\geq 0}$.

Lemma 2.12. *The following properties of the map μ and of the sequence L hold true:*

(a) *For all $k \geq 0$, $\sigma^{2k+1}\mu = \sigma\mu\,\sigma^k$.*
(b) *For any sequence ε in Σ we have $\sigma(\overline{\mu(\varepsilon)}) = \sigma(\mu(\overline{\varepsilon})) = \overline{\sigma(\mu(\varepsilon))}$.*
(c) *Let ε be a sequence in Σ. If ε is periodic with smallest period $T > 0$ and $\varepsilon_T = 0$, then $\mu(\varepsilon)$ is periodic with smallest period $2T$. In particular, if ε is a sequence in Σ such that $\mu(\varepsilon)$ is periodic with smallest period U, then U cannot be odd, hence U is even, say $U = 2T$, and ε is periodic with smallest period T and satisfies $\varepsilon_T = 0$.*
(d) *If the sequences $\varepsilon = (\varepsilon_n)_{n\geq 1}$ and $\varepsilon' = (\varepsilon'_n)_{n\geq 1}$ in Σ satisfy $\varepsilon \prec \varepsilon'$, then $\mu(\varepsilon) \prec \mu(\varepsilon')$ and $\sigma(\mu(\varepsilon)) \prec \sigma(\mu(\varepsilon'))$.*
(e) *The sequence L is a fixed point of the map μ. Also, for any sequence $\varepsilon := (\varepsilon_n)_{n\geq 1}$, we have that $\mu^\infty(\varepsilon) := \lim_{n\to\infty}\mu^n(\varepsilon)$ exists and $\mu^\infty(\varepsilon) = L$.*
(f) *Let $\varepsilon := (\varepsilon_n)_{n\geq 1}$ be a sequence in Σ. If $\varepsilon \prec L$, then $\varepsilon \prec \mu(\varepsilon)$. If $\varepsilon \succ L$, then $\varepsilon \succ \mu(\varepsilon)$.*

Proof. The proofs of (a–d) are easy and left to the reader.

Let us prove (e). It is straightforward that the sequence $0(\mu(\varepsilon))_{n\geq 1}$ is the image of the sequence $0(\varepsilon_n)_{n\geq 1}$ under the morphism φ defined by $\varphi(0) = 01$, $\varphi(1) = 10$, introduced above. In particular, the sequence $L = (\mathfrak{m}_n)_{n\geq 1}$ is a fixed point of μ, since the complete Thue-Morse sequence $(\mathfrak{m}_n)_{n\geq 0}$ is a fixed point of the morphism φ. Also we have, for any sequence ε in Σ and any $k \geq 0$, the equality $0\mu^k(\varepsilon) = \varphi^k(0\varepsilon)$. Hence $0\mu^\infty(\varepsilon)$ exists and is equal to $\varphi^\infty(0\varepsilon)$ which is precisely the (complete) Thue-Morse sequence $(\mathfrak{m}_n)_{n\geq 0}$.

In order to prove (f), note that, if $\varepsilon \prec \mu(\varepsilon)$, then, using (d), $\mu(\varepsilon) \prec \mu(\mu(\varepsilon))$. Hence $\varepsilon \prec \mu(\varepsilon) \prec \mu^2(\varepsilon)$, and by induction $\varepsilon \prec \mu(\varepsilon) \prec \mu^k(\varepsilon)$ for all $k \geq 2$. Letting k tend to infinity and using (e) gives that $\varepsilon \prec \mu(\varepsilon) \preceq L$, hence $\varepsilon \prec L$. Reversing the inequalities show that $\varepsilon \succ \mu(\varepsilon)$ implies that $\varepsilon \succ L$, which proves (f). $\square$

Lemma 2.13. *The set* Γ *has the following properties:*

(a) *Let* ε *be a sequence in* Γ. *Suppose that there exists* $d \geq 1$ *such that*

$$\varepsilon_{d+1}\,\varepsilon_{d+2}\cdots\varepsilon_{2d} = \overline{\varepsilon_1\,\varepsilon_2\cdots\varepsilon_d}$$

then the sequence ε *is periodic of period* $2d$, *i.e., we have*

$$\varepsilon = (\varepsilon_1\,\varepsilon_2\cdots\varepsilon_d\,\overline{\varepsilon_1\,\varepsilon_2\cdots\varepsilon_d})^\infty.$$

(b) *Let* ε *be a sequence in* Σ. *Then* ε *belongs to* Γ *if and only if* $\varepsilon_1 = 1$ *and* $\mu(\varepsilon)$ *belongs to* Γ.

(c) *If* ε *is a sequence in* Γ *such that* $\varepsilon \neq (10)^\infty$ *and* $\varepsilon \preceq 1(10)^\infty$, *then there exists a sequence* ε' *also in* Γ *such that* $\varepsilon = \mu(\varepsilon')$.

(d) *If* ε *is a periodic sequence in* Γ *such that* $\varepsilon \preceq 1(10)^\infty$, *then its smallest period is even.*

Proof. We first prove (a). Define, for $j \geq 0$, the *block* (or *word*) z_j by

$$z_j := (\varepsilon_{jd+1}\,\varepsilon_{jd+2}\cdots\varepsilon_{(j+1)d})$$

so that we can write

$$\varepsilon = (\varepsilon_1\,\varepsilon_2\cdots\varepsilon_d)\,(\varepsilon_{d+1}\,\varepsilon_{d+2}\cdots\varepsilon_{2d})\cdots(\varepsilon_{jd+1}\,\varepsilon_{jd+2}\cdots\varepsilon_{(j+1)d})\cdots = z_0\,z_1\,z_2\cdots$$

We prove by induction on $j \geq 0$ that $z_{2j} = z_0$ and $z_{2j+1} = \overline{z_0}$. The case $j = 0$ is exactly the hypothesis in (a). Suppose the result is true for some $j \geq 0$, i.e., that

$$\varepsilon = (z_0\,\overline{z_0})^{j+1}\,z_{2j+2}\,z_{2j+3}\cdots$$

Now

$$z_{2j+2}\,z_{2j+3}\cdots = \sigma^{2d(j+1)}(\varepsilon) \preceq \varepsilon = z_0\,\overline{z_0}\cdots \tag{2.3}$$

hence $z_{2j+2} \preceq z_0$, and

$$\overline{z_0}\,z_{2j+2}\,z_{2j+3}\cdots = z_{2j+1}\,z_{2j+2}\,z_{2j+3}\cdots = \sigma^{d(2j+1)}(\varepsilon) \succeq \overline{\varepsilon} = (\overline{z_0}\,z_0)^{j+1}\cdots$$

hence $\overline{z_0}\,z_{2j+2} \succeq \overline{z_0}\,z_0$, which gives $z_{2j+2} \succeq z_0$, hence $z_{2j+2} = z_0$. But the inequality (2.3) now implies $z_{2j+3} \preceq \overline{z_0}$. On the other hand

$$z_{2j+3}\,z_{2j+4}\cdots = \sigma^{d(2j+3)}(\varepsilon) \succeq \overline{\varepsilon} = \overline{z_0}\cdots$$

hence $z_{2j+3} \succeq \overline{z_0}$ which finally implies that $z_{2j+3} = \overline{z_0}$.

We now prove (b). Suppose that ε belongs to Γ. Since $\overline{\varepsilon} \preceq \varepsilon$, we have $\varepsilon_1 = 1$. Applying Lemma 2.12 (b) and (d) to the inequalities $\overline{\varepsilon} \preceq \sigma^k(\varepsilon) \preceq \varepsilon$, we get for all $k \geq 0$

$$\sigma(\overline{\mu(\varepsilon)}) = \sigma(\mu(\overline{\varepsilon})) \preceq \sigma(\mu(\sigma^k(\varepsilon))) \preceq \sigma(\mu(\varepsilon)).$$

Hence, from Lemma 2.12 (a),

$$\sigma(\overline{\mu(\varepsilon)}) \preceq \sigma^{2k+1}(\mu(\varepsilon)) \preceq \sigma(\mu(\varepsilon)). \tag{2.4}$$

Since $(\mu(\varepsilon))_1 = 1$, we have $\sigma(\mu(\varepsilon)) \preceq \mu(\varepsilon)$ and $\overline{\mu(\varepsilon)} \preceq \sigma(\overline{\mu(\varepsilon)})$. Hence the above inequalities yield

$$\overline{\mu(\varepsilon)} \preceq \sigma^{2k+1}(\mu(\varepsilon)) \preceq \mu(\varepsilon).$$

It remains to prove that for every $k \geq 0$

$$\overline{\mu(\varepsilon)} \preceq \sigma^{2k}(\mu(\varepsilon)) \preceq \mu(\varepsilon).$$

If $(\sigma^{2k}(\mu(\varepsilon)))_1 = 0$, then $\sigma^{2k}(\mu(\varepsilon)) \prec \mu(\varepsilon)$ as $\mu(\varepsilon)_1 = 1$. On the other hand, (2.4) implies that $\sigma(\overline{\mu(\varepsilon)}) \preceq \sigma(\sigma^{2k}(\mu(\varepsilon)))$. Hence $\overline{\mu(\varepsilon)} \preceq \sigma^{2k}(\mu(\varepsilon))$, since $(\overline{\mu(\varepsilon)})_1 = (\sigma^{2k}(\mu(\varepsilon)))_1$ $(= 0)$.

If $(\sigma^{2k}(\mu(\varepsilon)))_1 = 1$, then $\overline{\mu(\varepsilon)} \prec \sigma^{2k}(\mu(\varepsilon))$ as $\overline{\mu(\varepsilon)}_1 = 0$. On the other hand, (2.4) implies that $\sigma(\sigma^{2k}(\mu(\varepsilon))) \preceq \sigma(\mu(\varepsilon))$. Hence $\sigma^{2k}(\mu(\varepsilon)) \preceq \mu(\varepsilon)$, since $(\sigma^{2k}(\mu(\varepsilon)))_1 = (\mu(\varepsilon))_1$ $(= 1)$.

Now suppose that $\mu(\varepsilon)$ belongs to Γ, and that $\varepsilon_1 = 1$. We clearly have $\overline{\varepsilon} \prec \varepsilon$. It thus suffices to prove that, for any $k \geq 1$, both inequalities $\sigma^k(\varepsilon) \preceq \varepsilon$ and $\overline{\varepsilon} \preceq \sigma^k(\varepsilon)$ hold.

- Let us prove the first inequality. Let $k \geq 1$. If $\varepsilon_{k+1} = 0$, we have $\sigma^k(\varepsilon) = \varepsilon_{k+1}\varepsilon_{k+2}\cdots \prec \varepsilon$. If $\varepsilon_{k+1} = 1$, then either $\varepsilon_j = 1$ for all $j \in [1, k+1]$ and $\sigma^k(\varepsilon) \preceq \varepsilon$ since $\sigma^k(\varepsilon)$ begins with less 1's than ε, or there exists $\ell \in [2, k+1]$ such that $\varepsilon_j = 1$ for all $j \in [\ell, k+1]$ and $\varepsilon_{\ell-1} = 0$. But then

$$1\,\varepsilon_\ell\,\overline{\varepsilon_\ell}\,\varepsilon_{\ell+1}\,\overline{\varepsilon_{\ell+1}}\cdots = \overline{\varepsilon_{\ell-1}}\,\varepsilon_\ell\,\overline{\varepsilon_\ell}\,\varepsilon_{\ell+1}\,\overline{\varepsilon_{\ell+1}}\cdots = \sigma^{2\ell-2}(\mu(\varepsilon)) \preceq \mu(\varepsilon) = 1\,\varepsilon_1\,\overline{\varepsilon_1}\,\varepsilon_2\,\overline{\varepsilon_2}\cdots$$

  hence

$$\varepsilon_\ell\,\overline{\varepsilon_\ell}\,\varepsilon_{\ell+1}\,\overline{\varepsilon_{\ell+1}}\cdots \preceq \varepsilon_1\,\overline{\varepsilon_1}\,\varepsilon_2\,\overline{\varepsilon_2}\cdots$$

  which easily implies

$$\sigma^{\ell-1}(\varepsilon) = \varepsilon_\ell\,\varepsilon_{\ell+1}\cdots \preceq \varepsilon_1\,\varepsilon_2\cdots = \varepsilon.$$

  This in turn implies $\sigma^k(\varepsilon) \preceq \varepsilon$, since the sequence $\sigma^k(\varepsilon)$, beginning with less 1's than $\sigma^{\ell-1}(\varepsilon)$, is smaller than $\sigma^{\ell-1}(\varepsilon)$.
- Let us prove the second inequality. Let $k \geq 1$. If $\varepsilon_{k+1} = 1$, we have $\sigma^k(\varepsilon) = \varepsilon_{k+1}\varepsilon_{k+2}\cdots \succ \overline{\varepsilon}$. If $\varepsilon_{k+1} = 0$, then there exists $\ell \in [2, k+1]$ such that $\varepsilon_j = 0$ for all $j \in [\ell, k+1]$ and $\varepsilon_{\ell-1} = 1$ (remember that $\varepsilon_1 = 1$). But then

$$0\,\varepsilon_\ell\,\overline{\varepsilon_\ell}\,\varepsilon_{\ell+1}\,\overline{\varepsilon_{\ell+1}}\cdots = \overline{\varepsilon_{\ell-1}}\,\varepsilon_\ell\,\overline{\varepsilon_\ell}\,\varepsilon_{\ell+1}\,\overline{\varepsilon_{\ell+1}}\cdots = \sigma^{2\ell-2}(\mu(\varepsilon)) \succeq \overline{\mu(\varepsilon)} = 0\,\overline{\varepsilon_1}\,\varepsilon_1\,\overline{\varepsilon_2}\,\varepsilon_2\cdots$$

  hence

$$\varepsilon_\ell\,\overline{\varepsilon_\ell}\,\varepsilon_{\ell+1}\,\overline{\varepsilon_{\ell+1}}\cdots \succeq \overline{\varepsilon_1}\,\varepsilon_1\,\overline{\varepsilon_2}\,\varepsilon_2\cdots$$

which easily implies

$$\sigma^{\ell-1}(\varepsilon) = \varepsilon_\ell\, \varepsilon_{\ell+1} \cdots \succeq \overline{\varepsilon_1}\, \overline{\varepsilon_2} \cdots = \overline{\varepsilon}.$$

This in turn implies $\sigma^k(\varepsilon) \succeq \overline{\varepsilon}$, since the sequence $\sigma^k(\varepsilon)$, beginning with less 0's than $\sigma^{\ell-1}(\varepsilon)$, is larger than $\sigma^{\ell-1}(\varepsilon)$.

To finish with the proof of Lemma 2.13 it suffices to prove (c): namely (d) is a consequence of (c) and of Lemma 2.12 (c). So, let ε be a sequence in Γ with $\varepsilon \preceq 1(10)^\infty$. Since $1(10)^\infty = \mu(1^\infty)$, we may suppose that $\varepsilon \prec 1(10)^\infty$. We thus have

$$0(01)^\infty \prec \overline{\varepsilon} \preceq \sigma^k(\varepsilon) \preceq \varepsilon \prec 1(10)^\infty$$

for all $k \geq 0$. This implies in particular that ε cannot contain three consecutive 1's nor three consecutive 0's. The sequence ε must begin with 1. If $\varepsilon = 10\cdots$, then $\varepsilon = (10)^\infty$ from Lemma 2.13 (a), which is excluded. Thus $\varepsilon = 11\cdots$, hence $\varepsilon = 110\cdots$. From the inequality $\varepsilon \prec 1(10)^\infty$, there is a maximal integer $i_1 \geq 1$ such that $\varepsilon = 1(10)^{i_1}\cdots$. Hence $\varepsilon = 1(10)^{i_1}01\cdots$ (recall that ε does not contain three consecutive 0's). Then there exists a maximal integer $j_1 \geq 1$ such that $\varepsilon = 1(10)^{i_1}(01)^{j_1}\cdots$. But $0(01)^{j_1}\cdots = \sigma^{2i_1}\varepsilon \succeq \overline{\varepsilon} = 0(01)^{i_1}(10)^{j_1}\cdots$. This implies $j_1 \leq i_1$ and the next term in $\sigma^{2i_1}\varepsilon$ must be 1, i.e., $\sigma^{2i_1}\varepsilon = 0(01)^{j_1}1\cdots$. Finally $\varepsilon = 1(10)^{i_1}(01)^{j_1}1\cdots$, with $0 \leq j_1 \leq i_1$. A similar reasoning can be applied to $\sigma^{2i_1+2j_1}\varepsilon = 11\cdots$, yielding $\sigma^{2i_1+2j_1}\varepsilon = 1(10)^{i_2}(01)^{j_2}1\cdots$ with $1 \leq i_2 \leq i_1$ and $0 \leq j_2 \leq i_1$. Thus, iterating, we get

$$\varepsilon = 1(10)^{i_1}(01)^{j_1}(10)^{i_2}(01)^{j_2}(10)^{i_3}(01)^{j_3}\cdots$$

with $1 \leq i_k \leq i_1$ and $0 \leq j_k \leq i_1$. This implies

$$\varepsilon = \mu(1^{i_1}0^{j_1}1^{i_2}0^{j_2}1^{i_3}0^{j_3}\cdots).$$

The sequence $1^{i_1}0^{j_1}1^{i_2}0^{j_2}1^{i_3}0^{j_3}\cdots$ belongs to Γ from Lemma 2.13 (b). ☐

Remark 2.14. In Lemma 2.13 Part (a) is [2, Lemme 2 b, p. 27]. The "only if" part of (b) is on Page 26-06 in [1]. The proof of Part (c) is inspired by the proof of the partly more general "Lemme 1" on page 47 of [2].

We are now ready to present and prove the following construction of the sequences (a_k).

Proposition 2.15. *Denote by a_k the smallest periodic sequence belonging to Γ whose smallest period is equal to $k \geq 1$. Let $k = 2^n(2m+1)$, then*

$$a_k = \begin{cases} 1^\infty & \text{if } m = n = 0 \text{ (i.e., } k = 1), \\ \mu^n(0^\infty) & \text{if } m = 0 \text{ and } n \geq 1, \\ \mu^n((1(10)^m)^\infty) & \text{if } m \geq 1. \end{cases}$$

Proof.

- The case $m = n = 0$ is trivial.
- Let us address the case $m = 0$. For $n = 1$, we have $k = 2$, and it is easy to see that $a_2 = (10)^\infty = \mu(0^\infty)$. Suppose that $a_{2^n} = \mu^n(0^\infty)$ for some $n \geq 1$. Then $\mu(a_{2^n})$ has smallest period 2^{n+1}, hence $a_{2^{n+1}} \preceq \mu(a_{2^n}) = \mu^{n+1}(0^\infty)$. Now $0^\infty \prec L \prec 1(10)^\infty$, hence $a_{2^{n+1}} \preceq \mu^{n+1}(0^\infty) \prec \mu^{n+1}(L) = L \prec 1(10)^\infty$. This implies, from Lemma 2.13(c), the

existence of a sequence z in Σ such that $a_{2^{n+1}} = \mu(z)$. We have that z is periodic, and its smallest period is 2^n. Furthermore $\mu(z) \preceq \mu^{n+1}(0^\infty)$, hence $z \preceq \mu^n(0^\infty)$. This forces $z = \mu^n(0^\infty)$, thus $a_{2^{n+1}} = \mu(z) = \mu^{n+1}(0^\infty)$.

- Let us prove the result for $m \geq 1$ by induction on $n \geq 0$. Take first $n = 0$. Let ε be a sequence in Γ with smallest period $(2m+1)$ for some $m \geq 1$. From Lemma 2.13 (d), we must have $\varepsilon \succ 1(10)^\infty$. Hence the prefix of ε of length $2m+1$ must be larger than or equal to the prefix of $1(10)^\infty$ of length $2m+1$, i.e., $1(10)^m$. The sequence ε being periodic with smallest period $(2m+1)$, this implies that $\varepsilon \succeq (1(10)^m)^\infty$. But this last sequence clearly belongs to Γ, which implies that it is the smallest sequence in Γ that has smallest period $2m+1$.
- Now, suppose the result is true for some $n \geq 0$. Let ε be the smallest sequence in Γ whose smallest period is $2^{n+1}(2m+1)$. Since the sequence $\mu^{n+1}((1(10)^m)^\infty)$ is in Γ (use Lemma 2.13 (b)) and has smallest period $2^{n+1}(2m+1)$ (use Lemma 2.12 (c); note that we *need* $m \neq 0$), we have $\varepsilon \preceq \mu^{n+1}((1(10)^m)^\infty)$. Since this clearly implies $\varepsilon \preceq 1(10)^\infty$, Lemma 2.13 (c) gives the existence of a sequence ε' in Γ such that $\varepsilon = \mu(\varepsilon')$. The inequality $\mu(\varepsilon') = \varepsilon \preceq \mu^{n+1}((1(10)^m)^\infty)$ implies, using Lemma 2.12 (d), that $\varepsilon' \preceq \mu^n((1(10)^m)^\infty)$. But ε' belongs to Γ and has smallest period $2^n(2m+1)$ (use Lemma 2.12 (c)). Hence, from the induction hypothesis, $\varepsilon' = \mu^n((1(10)^m)^\infty)$. Thus, $\varepsilon = \mu(\varepsilon') = \mu^{n+1}((1(10)^m)^\infty)$.

□

We conclude the section with an explicit formula for the a_k via fragments of the Thue-Morse sequence.

Proposition 2.16. *The sequences* (a_k) *are related to the Thue-Morse sequence as follows. Let* $k = 2^n(2m+1)$. *Then*

$$a_k = \begin{cases} \mathfrak{m}_1^\infty & \text{if } m = n = 0 \text{ (i.e., } k = 1\text{)}, \\ (\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^n-1}\overline{\mathfrak{m}_{2^n}})^\infty & \text{if } m = 0 \text{ and } n \geq 1, \\ (\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{3\cdot 2^n}(\mathfrak{m}_1 \cdots \overline{\mathfrak{m}_{2^{n+1}}})^{m-1})^\infty & \text{if } m \geq 1. \end{cases}$$

Proof. We may assume that $(m,n) \neq (0,0)$. We then note that (see the proof of Lemma 2.12 (e)), for any sequence ε in Σ and for any $n \geq 0$, we have $0\mu^n(\varepsilon) = \varphi^n(0\varepsilon)$. Hence to prove the proposition, it suffices to show that, for any $n \geq 0$,

$$\begin{array}{rcl} \varphi^n(0^\infty) & = & 0(\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^n-1}\overline{\mathfrak{m}_{2^n}})^\infty, \\ \varphi^n(0(1(10)^m)^\infty) & = & 0(\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{3\cdot 2^n}(\mathfrak{m}_1 \cdots \overline{\mathfrak{m}_{2^{n+1}}})^{m-1})^\infty \ \ \forall m \geq 1. \end{array}$$

Remembering that $\mathfrak{m}_n$ is nothing but the parity of the sum of the binary digits of n, we clearly have $\mathfrak{m}_{2^n} = 1$, $0(\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^n-1}\overline{\mathfrak{m}_{2^n}})^\infty = (\mathfrak{m}_0\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^n-1})^\infty$, and

$$\begin{array}{rcl} \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{3\cdot 2^n} & = & \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^{n+1}}\mathfrak{m}_{2^{n+1}+1} \cdots \mathfrak{m}_{2^{n+1}+2^n} \\ & = & \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cdots \mathfrak{m}_{2^{n+1}-1}\overline{\mathfrak{m}_0\mathfrak{m}_1 \cdots \mathfrak{m}_{2^n-1}}\, 0 \end{array}$$

Denoting by W_n the word (of length 2^n) $W_n := \mathfrak{m}_0\mathfrak{m}_1 \cdots \mathfrak{m}_{2^n-1}$, what we have to prove boils down to

$$\begin{array}{rcl} \varphi^n(0^\infty) & = & W_n^\infty \\ \varphi^n(0(1(10)^m)^\infty) & = & (W_{n+1}\,\overline{W_n}\,W_{n+1}^{m-1})^\infty \ \ \forall m \geq 1. \end{array}$$

Using the easily proven relation $W_{n+1} = \varphi(W_n)$ we have $W_n = \varphi^n(0)$, $\overline{W_n} = \varphi^n(1)$, and $W_{n+1} = W_n\overline{W_n}$, thus

$$\varphi^n(0^\infty) = (\varphi^n(0))^\infty = W_n^\infty$$

and

$$\begin{aligned}\varphi^n(0(1(10)^m)^\infty) &= \varphi^n(0)\varphi^n[(1(10)^m)^\infty] = \varphi^n(0)[\varphi^n(1)(\varphi^n(10))^m]^\infty \\ &= \varphi^n(0)[\varphi^n(1)(\varphi^n(1)\varphi^n(0))^m]^\infty \\ &= W_n[\overline{W_n}\ (\overline{W_n}W_n)^m]^\infty = W_n[\overline{W_n}\ (\overline{W_n}W_n)^{m-1}\overline{W_n}W_n]^\infty \\ &= (W_n\overline{W_n}\ (\overline{W_n}W_n)^{m-1}\overline{W_n})^\infty = (W_n\overline{W_n}\ \overline{W_n}(W_n\overline{W_n})^{m-1})^\infty \\ &= (W_{n+1}\overline{W_n}W_{n+1}^{m-1})^\infty.\end{aligned}$$

□

For the table of the first 8 values of β_n see Table 2.1 below.

β_n	$d(\beta_n)$	minimal polynomial	numerical value	below β_{KL}?
$n = 2$	11	$x^2 - x - 1$	1.61803	yes
$n = 3$	111	$x^3 - x^2 - x - 1$	1.83929	no
$n = 4$	1101	$x^3 - 2x^2 + x - 1$	1.75488	yes
$n = 5$	11011	$x^5 - x^4 - x^3 - x - 1$	1.81240	no
$n = 6$	110101	$x^6 - x^5 - x^4 - x^2 - 1$	1.78854	no
$n = 7$	1101011	$x^6 - 2x^5 + x^4 - x^3 - 1$	1.80509	no
$n = 8$	11010011	$x^5 - 2x^4 + x^2 - 1$	1.78460	yes

TABLE 2.1. The table of β_n for small values of n

3. IMPOSSIBILITY OF CONTINUOUS EXTENSION OF F_β

Recall that the map F_β acts on a nowhere dense subset of I_β. It would be tempting to try to explain the Sharkovskiĭ order in our model via some extension of F_β to the whole interval and then applying the classical Sharkovskiĭ theorem to that extended map. In this section we show that this is in fact impossible.

Theorem 3.1. *Let, as above,* $\beta_4 \approx 1.75488$ *denote the unique root of* $x^3 = 2x^2 - x + 1$ *lying in* $(1, 2)$. *Assume we have* $\beta \in (\beta_4, 2)$*; then any continuous map* $S_\beta : I_\beta \to I_\beta$ *such that* $S_\beta|_{X_\beta} = F_\beta$ *has a* k*-cycle for all* $k \in \mathbb{N}$ *provided* S_β *is monotonic on* $[0, 1/\beta]$.

Remark 3.2. In particular, this means that any map of the form

$$S_\beta(x) = \begin{cases} \beta x, & 0 \le x < \frac{1}{\beta}, \\ G(x), & \frac{1}{\beta} \le x \le \frac{1}{\beta(\beta-1)} \\ \beta x - 1, & \frac{1}{\beta(\beta-1)} < x \le \frac{1}{\beta-1}, \end{cases}$$

where G is continuous, and $G\big(\frac{1}{\beta}\big) = 1$, $G\big(\frac{1}{\beta(\beta-1)}\big) = \frac{2-\beta}{\beta-1}$, has cycles of any length provided $\beta > \beta_4$ – see Figs below. On the other hand, as we know, if $\beta < \beta_8$, then F_β itself has only

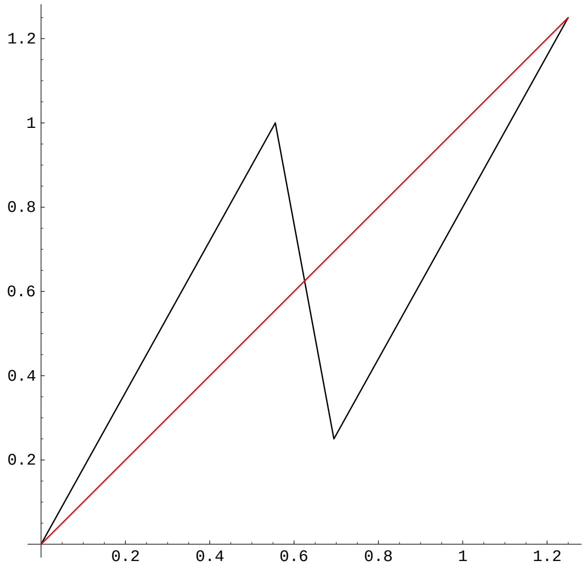

FIGURE 1. A continuous extension of F_β for $\beta = 1.8$.

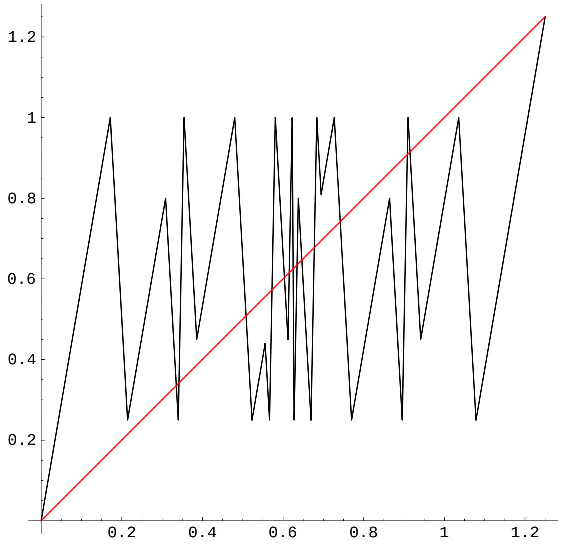

FIGURE 2. …and its third iterate – observe all those "parasite" fixed points!

cycles of length 2 and 4. This means that there is no immediate connection between the classical Sharkovskiĭ theorem and our Theorem 1.3. (See Section 4 for a less immediate connection.)

Proof. Since $\beta > \beta_4$, we have the following 4-cycle in Σ_β:

$$x_1 \sim (0011)^\infty,\ x_2 \sim (0110)^\infty$$
$$x_3 \sim (1100)^\infty,\ x_4 \sim (1001)^\infty$$

Consequently, we have a 4-cycle for F_β which we denote by $\{x_1, x_2, x_3, x_4\}$ as well. Notice that x_1 and x_2 lie in the left hand side interval, while x_3 and x_4 belong to the right hand side one.

Suppose such a map S_β exists; then we have $S_\beta^3(x_1) = x_4 > x_1,\ S_\beta^3(x_2) = x_1 < x_2$. By the mean value theorem, the map S_β^3 has a fixed point x_* between x_1 and x_2. Since $F_\beta(x_1) > x_1$ and $F_\beta(x_2) > x_2$ and our assumption on the monotonicity of S_β between x_1 and x_2, we conclude that $S_\beta(x_*) > x_*$, whence x_* is a period 3 point for S_β. By the classical Sharkovskiĭ theorem, this implies that S_β has cycles of all possible lengths for $\beta > \beta_4$. ☐

4. ANOTHER PROOF OF OUR MAIN THEOREM USING SHARKOVSKIĬ'S CLASSICAL THEOREM FOR TRAPEZOIDAL MAPS

In the previous section we established the fact that there is no natural extension of F_β to the whole interval which preserves the delicate structure of the Sharkovskiĭ ordering on the periodic orbits of F_β. In this section we modify F_β by flipping the right branch, which leads to the family of *trapezoidal maps* (see, e.g., [16]) and links our result to the classical Sharkovskiĭ theorem.

More precisely, define the map $T_\beta : I_\beta \longrightarrow I_\beta$ as follows:

$$T_\beta(x) = \begin{cases} \beta x, & 0 \le x < \frac{1}{\beta} \\ 1, & \frac{1}{\beta} \le x \le \frac{1}{\beta(\beta-1)} \\ \frac{\beta}{\beta-1} - \beta x, & \frac{1}{\beta(\beta-1)} < x \le \frac{1}{\beta-1}. \end{cases}$$

Following the standard notation, we denote the corresponding intervals by L, C and R respectively – see Fig. 3. (Here $C = \left[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\right]$.) We will write the itineraries of points under T_β using this notation.

Our goal is to present another proof of the Sharkovskiĭ theorem for the family $(\Sigma_\beta, \sigma_\beta)$ using the classical Sharkovskiĭ theorem for T_β.

Define the map $h : \Sigma \to \{L, R\}^{\mathbb{N}}$ as follows (from here on $*$ denotes an arbitrary – but fixed – tail):

- $h(0*) = Lh(*)$;
- $h(1^a 0^b 1*) = RL^{a-1}RL^{b-1}h(1*)$ for $a, b \ge 1$;
- $h(1^a 0^\infty) = RL^{a-1}RL^\infty$;
- $h(1^\infty) = RL^\infty$.

It is clear that h is well defined and is one-to-one, with $h^{-1}(RL^aRL^b) = 1^{a+1}0^{b+1}$ for $a, b \ge 0$ (on blocks). We claim that h in fact maps the orbits of σ_β into the orbits of T_β which do not fall into C.

More precisely, put $\rho_\beta : \{L, R\} \to I_\beta$, where $\rho_\beta(\xi) = x$ such that ξ is the itinerary of x under T_β.

Lemma 4.1. *For* $\varepsilon = 1^\ell 0*$ *with* $\ell \ge 0$ *we have*

$$F_\beta^{\ell+1}(x) = T_\beta^{\ell+1}(x),$$

where $x = \rho_\beta h(\varepsilon)$.

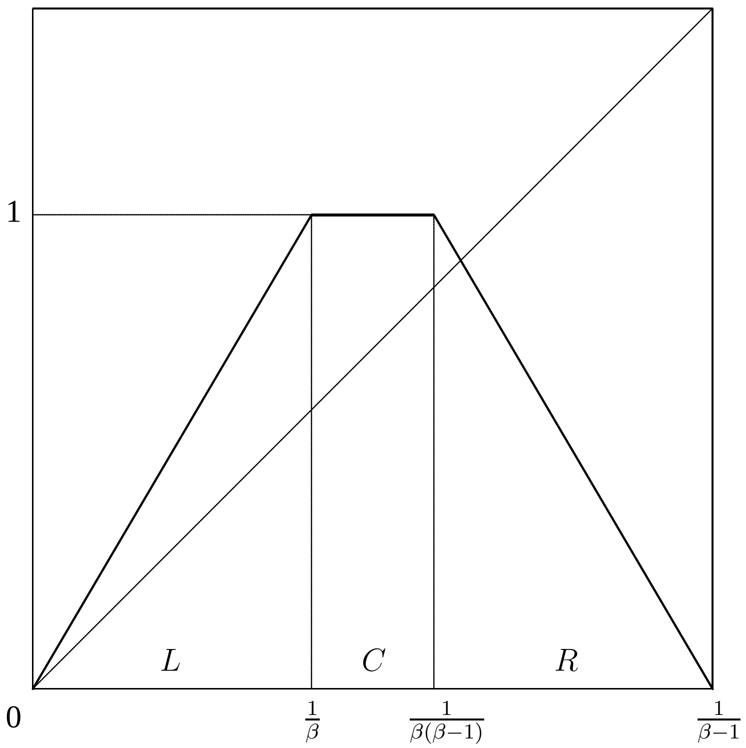

FIGURE 3. The trapezoidal map T_β for $\beta = 1.7$

Proof. Assume first that $\ell \geq 1$; we have $h(\varepsilon) = RL^{\ell-1}Rh(*)$, whence by the definition of T_β,

$$\begin{aligned} T_\beta(x) &= \frac{\beta}{\beta-1} - \beta x, \\ T_\beta^\ell(x) &= \frac{\beta^\ell}{\beta-1} - \beta^\ell x, \\ T_\beta^{\ell+1}(x) &= \frac{\beta}{\beta-1} - \frac{\beta^{\ell+1}}{\beta-1} + \beta^{\ell+1}x \\ &= \beta^{\ell+1}x - \beta^\ell - \cdots - \beta^2 - \beta. \end{aligned}$$

It is easy to verify that the greedy β-expansion of x begins with $1^\ell 0$, whence by the definition of F_β (see Section 1), $F_\beta^{\ell+1}(x) = \beta^{\ell+1}x - \beta^\ell - \cdots - \beta^2 - \beta = T_\beta^{\ell+1}(x)$.

If $\ell = 0$, then $F_\beta(x) = T_\beta(x) = \beta x$. □

Corollary 4.2. *For* $\varepsilon = 1^{a_1}0^{b_1}\dots 1^{a_s}0*$ *with* $a_j \geq 1, b_j \geq 1$*, we have*

$$F_\beta^{\ell+1}(x) = T_\beta^{\ell+1}(x),$$

where $x = \rho_\beta h(\varepsilon)$ *and* $\ell = \sum_{j=1}^{s-1}(a_j + b_j) + a_s$*.*

This result allows us to link the periodic orbits of F_β to those of T_β. Notice that since h acts blockwise and does not alter the length of a block, a p-periodic orbit of F_β maps into a q-periodic orbit of F_β, where q divides p. In fact, we will show that there are only two possibilities: either $q = p$ or $q = p/2$ – see below.

Example 4.3. If $\varepsilon = (1100)^\infty$, then $h(\varepsilon) = (RL)^\infty$. A more complicated example: $\varepsilon = (1101011\ 0010100)^\infty$ and $h(\varepsilon) = (RLRRRRL)^\infty$. Notice that in both cases $\varepsilon = (v\overline{v})^\infty$ for some v. We will see later that this is always the case when h cuts a period in half.

Now we are ready to present an alternative proof of the main theorem of this paper. Since the case of powers of 2 is considered in [13], we assume $m \rhd k$ in the Sharkovskiĭ ordering and m is not a power of 2. Suppose F_β has an m-cycle; then T_β has an m-cycle or an $m/2$-cycle. In either case, by the classical Sharkovskiĭ theorem, T_β has a k-cycle. We need to make sure however that such a cycle does not involve C. Let us call a cycle with this property an *L-R cycle* and prove a version of the Sharkovskiĭ theorem[1].

Proposition 4.4. *If* T_β *has an L-R* m*-cycle, then it has an L-R* k*-cycle for any* $k \lhd m$*.*

Proof. Let $\{x_1, \dots, x_m\}$ be the cycle in question. Without loss of generality assume that x_1 is the point of this cycle closest to C. If $x_1 < 1/\beta$, we put

$$\widetilde{T}_\beta(x) = \begin{cases} \beta x, & 0 \leq x < x_1, \\ \beta x_1, & x_1 \leq x \leq \frac{1}{\beta-1} - x_1, \\ \frac{\beta}{\beta-1} - \beta x, & \frac{1}{\beta-1} - x_1 < x \leq \frac{1}{\beta-1}. \end{cases}$$

If $x_1 > \frac{1}{\beta(\beta-1)}$, put

$$\widetilde{T}_\beta(x) = \begin{cases} \beta x, & 0 \leq x < \frac{1}{\beta-1} - x_1, \\ \frac{\beta}{\beta-1} - \beta x_1, & \frac{1}{\beta-1} - x_1 \leq x \leq x_1, \\ \frac{\beta}{\beta-1} - \beta x, & x_1 < x \leq \frac{1}{\beta-1}. \end{cases}$$

In other words, $\widetilde{T}_\beta$ is a trapezoidal map whose graph is made out of the graph of T_β by "sawing off" its top at the level $y = \beta x_1$ or $y = \frac{\beta}{\beta-1} - \beta x_1$ respectively. Notice that $\{x_1, \dots, x_m\}$ is still an m-cycle for $\widetilde{T}_\beta$, whence, by the classical Sharkovskiĭ theorem, $\widetilde{T}_\beta$ has a k-cycle $\{y_1, \dots, y_k\}$. It suffices to observe that $y_j \notin C$ for all j, because otherwise $\widetilde{T}_\beta(y_j) = \widetilde{T}_\beta(x_1)$, and consequently, $\widetilde{T}_\beta^r(y_j) \notin C$ for any $r \geq 1$.

Hence, by our construction, $\{y_1, \dots, y_k\}$ is a sought L-R cycle for T_β. □

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[1] The authors are grateful to Sebastian van Strien for the idea of the proof.

Thus, T_β has an L-R k-cycle, whence by applying h^{-1}, we conclude that F_β (or σ_β) has a k-cycle or an ℓk-cycle for some $\ell \geq 2$. It suffices to discard the latter case.

Let $\varepsilon = u^\infty$, with

$$u = 1^{a_1}0^{b_1}\dots 1^{a_{s-1}}0^{b_{s-1}}1^{a_s} \ \mid\ 0^{b_s}1^{a_{s+1}}0^{b_{s+1}}\dots 1^{a_r}0^{b_r},$$

where | separates the two halves of u. Then

$$h(u) = RL^{a_1-1}RL^{b_1-1}\dots RL^{a_s-1} \ \mid\ RL^{b_s-1}RL^{a_{s+1}-1}RL^{b_{s+1}-1}\dots RL^{a_r-1}RL^{b_r-1}. \tag{4.1}$$

From (4.1) it is clear that if a word u is not a power of another word itself and $h(u)$ is a power, then it must be a square, i.e., $\ell = 2$.

Suppose $h(u) = ww$ for some w. Then

$$RL^{a_1-1}RL^{b_1-1}\dots RL^{a_s-1} = RL^{b_s-1}RL^{a_{s+1}-1}RL^{b_{s+1}-1}\dots RL^{a_r-1}RL^{b_r-1},$$

i.e., $a_1 = b_s, b_1 = a_{s+1}, \dots, a_s = b_r$. In other words, $u = v\overline{v}$ for $v = 1^{a_1}0^{b_1}\dots 1^{a_{s-1}}0^{b_{s-1}}1^{a_s}$.

Thus, if a sequence ε is of smallest period $2k$ and $h(\varepsilon)$ has smallest period k, then $\varepsilon = (v\overline{v})^\infty$ for some v. Our goal is to show that for such an ε one can find $\varepsilon' \in \Sigma_\beta$ of smallest period k, which will conclude the proof.

Analogously to the proof of Proposition 2.8, we consider all the shifts of ε and $\overline{\varepsilon}$ (which are all in Σ_β) and choose the maximal one. Hence without loss of generality, we may again assume $\varepsilon \in \Gamma$, where Γ is given by (2.2).

Proposition 4.5. *Assume $\varepsilon \in \Gamma$ is a sequence of smallest period $2k$, of the form $\varepsilon = (v\overline{v})^\infty$ for some v, where $|v| = k > 1$. Then there exists a sequence $\varepsilon' \in \Gamma$ such that ε' has smallest period k and $\varepsilon' \prec \varepsilon$.*

Proof. This result can be deduced from the combination of two results of [2] (namely Proposition 2 on p. 34 applied to the sequence ε and Proposition 1 on p. 32), but we give a direct proof. Since the sequence ε begins with 1, and since $\sigma^{2k-1}\varepsilon \prec \varepsilon$, the word $\overline{v}$ must end in 0, hence the word v must end in 1. Let $v := w1$, hence $v\overline{v} = w1\overline{w}0$.

The sequence $\varepsilon' := (w0)^\infty$ has period k. This is the smallest period of the sequence ε': if this were not the case, we would have $w0 = (z0)^\ell$ for some word z and some integer $\ell \geq 2$. Thus $w = (z0)^{\ell-1}z$. This would imply $\varepsilon = (w1\overline{w}0)^\infty = ((z0)^{\ell-1}z1(\overline{z}1)^{\ell-1}\overline{z}0)^\infty$. But then $\sigma^{2\ell-2}\varepsilon$ begins with $z1$ and the condition $\sigma^{2\ell-2}\varepsilon \preceq \varepsilon$ would not be satisfied.

We clearly have $\varepsilon' \prec \varepsilon$. To prove that ε' belongs to Γ, it clearly suffices to prove that if the word w is written as $w := xy$ (thus $(w0)^\infty = (xy0)^\infty$), with the condition $0 < |x| < |w| = k-1$, then

$$\overline{x}\,\overline{y}\,1 \prec y\,0\,x \prec x\,y\,0, \tag{4.2}$$

thus yielding $\overline{\varepsilon'} \prec \sigma^{|x|}\varepsilon' \prec \varepsilon'$ or $\overline{x}\,\overline{y}\,1 = y\,0\,x \prec x\,y\,0$, thus yielding $\overline{\varepsilon'} = \sigma^{|x|}\varepsilon' \prec \varepsilon'$.

Let us prove (4.2). Since $\varepsilon = (w\,1\,\overline{w}\,0)^\infty = (x\,y\,1\,\overline{x}\,\overline{y}\,0)^\infty$ belongs to Γ, we have $\sigma^{|x|}(\varepsilon) \preceq \varepsilon$, hence $y\,1\,\overline{x} \preceq x\,y\,1$. Notice that $y\,1\,\overline{x}$ cannot be equal to $x\,y\,1$. Namely, if these two words were equal, this would imply $y\,1\,\overline{x}\,\overline{y}\,0\,x = x\,y\,1\,\overline{x}\,\overline{y}\,0$. This shows that the words x and $y\,1\,\overline{x}\,\overline{y}\,0$ would commute: from a theorem of Lyndon and Schützenberger [17] this would imply that there exist a word z and two positive integers a and b such that $x = z^a$ and $y\,1\,\overline{x}\,\overline{y}\,0 = z^b$.

Hence $w\ 1\ \overline{w}\ 0 = z^{a+b}$ and $2k$ would not be the smallest period of the sequence ε. We thus have $y\ 1\ \overline{x} \prec x\ y\ 1$, whence

$$y\ 1\ \overline{x} \preceq x\ y\ 0. \tag{4.3}$$

Obviously, $y\ 0\ x \prec y\ 1\ \overline{x}$, whence $y\ 0\ x \prec x\ y\ 0$, which proves the RHS inequality in (4.2).

Let us prove that $\overline{x}\ \overline{y}\ 1 \prec y\ 0\ x$, or, equivalently, that

$$\overline{y}\ 1\ \overline{x} \prec x\ y\ 0. \tag{4.4}$$

We can write $xy = YX$, with $|X| = |x|$ and $|Y| = |y|$. Thus, $\varepsilon = (Y\ X\ 1\ \overline{x}\ \overline{y}\ 0)^\infty$. Since $\sigma^{2|x|+1+|y|}\varepsilon \preceq \varepsilon$, we have $\overline{y} \preceq Y$. Now, if $\overline{y} \prec Y$, then $\overline{y}\ 1\ \overline{x} \prec Y\ X\ 0 = x\ y\ 0$, which is the sought inequality (4.4), and we are done.

If $\overline{y} = Y$, then $\varepsilon = (Y\ X\ 1\ \overline{x}\ Y\ 0)^\infty$. We claim that $X\ 1\ \overline{x}$ cannot begin with 0, because if this were the case, say $X\ 1\ \overline{x} := 0\ t$, then $\varepsilon = (Y\ 0\ t\ Y\ 0)^\infty$. The inequality $\sigma^{|y|+|t|+1}\varepsilon \preceq \varepsilon$ would imply that $Y\ 0\ Y\ 0\ t \preceq Y\ 0\ t\ Y\ 0$, hence $Y\ 0\ t \preceq t\ Y\ 0$. On the other hand $\sigma^{|y|+1}\varepsilon \preceq \varepsilon$ implies that $t\ Y\ 0 \preceq Y\ 0\ t$. We would thus have $t\ Y\ 0 = Y\ 0\ t$. In other words, $Y0$ and t commute, whence, as above, there exist a word z and two positive integers a and b, with $Y0 = z^a$ and $t = z^b$. Consequently, $\varepsilon = (z^{2a+b})^\infty$, which contradicts the minimality of the period of the sequence ε.

Therefore, $X\ 1\ \overline{x}$ must begin with 1, say, $X\ 1\ \overline{x} := 1\ t$. We have $\varepsilon = (Y\ 1\ t\ Y\ 0)^\infty$. Now $\overline{\varepsilon} \preceq \sigma^{|y|+1}\varepsilon$ implies that $\overline{Y}\ 0\ \overline{t} \preceq t\ Y\ 0 \prec t\ Y\ 1$. Hence, in view of $\overline{y} = Y$ and $\overline{X}\ 0\ x = 0\ \overline{t}$, we have

$$\begin{aligned} \overline{y}\ 1\ \overline{x}\ \overline{y}\ 0\ x &= Y\ 1\ \overline{Y}\ \overline{X}\ 0\ x = Y\ 1\ \overline{Y}\ 0\ \overline{t} \\ &\prec Y\ 1\ t\ Y\ 1 = Y\ X\ 1\ \overline{x}\ Y\ 1 \\ &= x\ y\ 1\ \overline{x}\ Y\ 1. \end{aligned} \tag{4.5}$$

This, in turn, implies $\overline{y}\ 1\ \overline{x} \preceq x\ y\ 1$. Notice that if we had $\overline{y}\ 1\ \overline{x} = x\ y\ 1$, then (4.5) would imply $\overline{y}\ 0\ x \prec \overline{x}\ Y\ 1 = \overline{x}\ \overline{y}\ 1$, i.e., by barring everything, $x\ y\ 0 \prec y\ 1\ \overline{x}$ – which clearly contradicts (4.3).

Hence $\overline{y}\ 1\ \overline{x} \prec x\ y\ 1$, and thus, by (4.5), $\overline{y}\ 1\ \overline{x} \preceq x\ y\ 0$. Either $\overline{y}\ 1\ \overline{x} \prec x\ y\ 0$, which is precisely the required LHS inequality in (4.2), and we are done, or $\overline{y}\ 1\ \overline{x} = x\ y\ 0$, i.e., $y\ 0\ x = \overline{x}\ \overline{y}\ 1$. This implies $\sigma^{|x|}\varepsilon' = (y\ 0\ x)^\infty = (\overline{x}\ \overline{y}\ 1)^\infty = \overline{\varepsilon'}$, which, together with the proven RHS of the inequality (4.2) yields $\overline{\varepsilon'} \in \Gamma$. ☐

Thus, we have constructed a periodic sequence $\varepsilon' \in \Gamma$ with smallest period k, which implies $\sigma^j\varepsilon' \preceq \varepsilon' \prec \varepsilon \prec d(\beta)$, and similarly, $\sigma^j\overline{\varepsilon'} \prec d(\beta)$. Hence $\varepsilon' \in \Sigma_\beta$, and this concludes the second proof of Theorem 1.3 in the case when m is not a power of 2.

5. FINAL REMARKS AND AN OPEN PROBLEM

Remark 5.1. The case $m = 2^n$ is a bit more delicate: here h can map a periodic sequence with smallest period m into a periodic sequence with smallest period $m/2$, for instance, $h((1100)^\infty) = (RL)^\infty$. A direct inspection shows that a 2-cycle for T_β does appear at $\beta = \beta_2$, but it is $(RC)^\infty$, not $(RL)^\infty$. The latter cycle in fact appears at $\beta = \beta_4$, where the former one disappears.

We conjecture that the first 2^n-cycle to arise for T_β appears at $\beta = \beta_{2^n}$ but always involves C. The proof is left to the interested reader.

Remark 5.2. H. Bruin is his Ph. D. Thesis [7] proved various results on the Sharkovskiĭ theorem for unimodal maps.

Remark 5.3. Let, as above, $\tau_\beta : [0,1) \to [0,1)$ denote the (greedy) β-transformation, i.e., $\tau_\beta x = \beta x \bmod 1$. Our remark consists in a simple observation that there is no Sharkovskiĭ theorem for the family $([0,1), \tau_\beta)_{\beta \in (1,2)}$. Indeed, the set of admissible sequences here is

$$\widetilde{\Sigma}_\beta = \{\varepsilon \in \Sigma : \sigma^j \varepsilon \prec d(\beta), \quad \forall j \geq 0\} \tag{5.1}$$

(see [19]), and it is obvious that the smallest periodic sequence ε with smallest period n such that $\varepsilon_1 = 1$ and $\sigma^j \varepsilon \preceq \varepsilon$ for all $j \geq 0$, is (10^{n-1}). Hence the analogue of U_n is $\widetilde{U}_n = (q_n, 2)$, where q_n is the appropriate root of $x^n = x^{n-1} + 1$, i.e., $d'(q_n) = (10^{n-1})^\infty$.

Hence $\widetilde{U}_n \subset \widetilde{U}_k$ iff $n < k$, which is not a particularly interesting result. Comparing (2.1) and (5.1), we see that the extra condition $\sigma^j \overline{\varepsilon} \prec d(\beta)$ makes all the difference.

Remark 5.4. Let $\prec_u$ denote the *unimodal order* on the itineraries of T_β, i.e., $L \prec_u C \prec_u R$ and $\varepsilon \prec_u \varepsilon'$ if $\varepsilon_i \equiv \varepsilon'_i$, $1 \leq i \leq k$ and either $\varepsilon_{k+1} \prec_u \varepsilon'_{k+1}$ with $\#\{i \in [1,k] : \varepsilon_i = R\}$ even or $\varepsilon_{k+1} \succ_u \varepsilon'_{k+1}$ with $\#\{i \in [1,k] : \varepsilon_i = R\}$ odd (see, e.g., [18]).

Proposition 5.5. *We have for* $\varepsilon, \varepsilon' \in \Sigma$,

$$\varepsilon \prec \varepsilon' \iff h(\varepsilon) \prec_u h(\varepsilon').$$

Proof. Essentially this claim can be found in [8], but for the reader's convenience we will give a sketch of the proof. Let

$$\varepsilon = 1^{a_1} 0^{b_1} \dots 1^{a_s} 0^{b_s} 0 \dots,$$
$$\varepsilon' = 1^{a_1} 0^{b_1} \dots 1^{a_s} 0^{b_s} 1 \dots.$$

Then

$$h(\varepsilon) = RL^{a_1-1} RL^{b_1-1} \dots RL^{a_s-1} RL^{b_s-1} L \dots$$
$$h(\varepsilon') = RL^{a_1-1} RL^{b_1-1} \dots RL^{a_s-1} RL^{b_s-1} R \dots,$$

whence $h(\varepsilon) \prec_u h(\varepsilon')$. The other cases are similar. □

Open problem. Let $\Omega = \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_m\}$ be points in $\mathbb{R}^d$ and let $S_\Omega(\beta)$ denote the set of "β-expansions", where the "digits" are taken from the set Ω. More precisely, put, for $\beta > 1$,

$$S_\Omega(\beta) = \left\{ (\beta - 1) \sum_{n=1}^{\infty} \beta^{-n} \boldsymbol{a}_n \mid \boldsymbol{a}_n \in \Omega \right\}.$$

Clearly, $S_\Omega(\beta)$ is a subset of the convex hull of Ω, and each $\boldsymbol{x} \in S_\Omega(\beta)$ has at least one *address* $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots) \in \Omega^{\mathbb{N}}$. Similarly to our setting, one can define the set of points which have a unique address and enquire about possible periods for such points.

In [22, Section 4] the third author studied the case $d = 2, m = 3$ (with noncollinear points $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$) and showed that the first period to appear is period 3. It is also easy to show that the last period to appear is period 2, so we have a reverse Sharkovskiĭ order here – at least at the endpoints. Other periods are much harder to deal with though, because of the holes in S_Ω, and it is not even clear whether U_n in this model is an interval for each n.

Obtaining a direct analogue of the Sharkovskiĭ theorem for the shift on the set of unique addresses would be intriguing.

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REFERENCES

[1] J.-P. Allouche, *Une propriété extrémale de la suite de Thue-Morse en rapport avec les cascades de Feigenbaum*, Sém. Théorie des Nombres Bordeaux (1981-1982), 26-01–26-17.

[2] J.-P. Allouche, Théorie des Nombres et Automates, Thèse d'État, Université Bordeaux I, (1983).

[3] J.-P. Allouche and M. Cosnard, *Itérations de fonctions unimodales et suites engendrées par automates*, C. R. Acad. Sci. Paris, Série I **296** (1983), 159–162.

[4] J.-P. Allouche and M. Cosnard, *The Komornik-Loreti constant is transcendental*, Amer. Math. Monthly **107** (2000), 448–449.

[5] J.-P. Allouche and M. Cosnard, *Non-integer bases, iteration of continuous real maps, and an arithmetic self-similar set*, Acta Math. Hung. **91** (2001), 325–332.

[6] J.-P. Allouche and J. Shallit, *The ubiquitous Prouhet-Thue-Morse sequence*, in C. Ding. T. Helleseth, and H. Niederreiter, eds., Sequences and Their Applications: Proceedings of SETA '98, Springer-Verlag, 1999, pp. 1–16.

[7] H. Bruin, *Invariant measures of interval maps*, Ph. D. Thesis, Delft, 1994.

[8] M. Cosnard, *Étude de la classification topologique des fonctions unimodales*, Ann. Inst. Fourier **35** (1985), 59–77.

[9] Z. Daroczy and I. Katai, *Univoque sequences*, Publ. Math. Debrecen **42** (1993), no. 3–4, 397–407.

[10] Z. Daróczy and I. Kátai, *On the structure of univoque numbers*, Publ. Math. Debrecen **46** (1995), 385–408.

[11] R. Devaney, An Introduction to Chaotic Dynamical Systems, Addison-Wesley Publishing Company, 1989.

[12] P. Erdős, I. Joó, and V. Komornik, *Characterization of the unique expansions* $1 = \sum_{i=1}^{\infty} q^{-n_i}$ *and related problems*, Bull. Soc. Math. Fr. **118** (1990), 377–390.

[13] P. Glendinning and N. Sidorov, *Unique representations of real numbers in non-integer bases*, Math. Res. Letters **8** (2001), 535–543.

[14] V. Komornik and P. Loreti, *Unique developments in non-integer bases*, Amer. Math. Monthly **105** (1998), 636–639.

[15] V. Komornik and M. de Vries, *Unique expansions of real numbers*, Preprint 2007. Available at `http://arxiv.org/abs/math/0609708v3`

[16] J. D. Louck and N. Metropolis, *Symbolic Dynamics of Trapezoidal Maps*, Mathematics and its Applications, 27. D. Reidel Publishing Co., Dordrecht, 1986.

[17] R.C. Lyndon and M. P. Schützenberger, *The equation* $a^M = b^N c^P$ *in a free group*, Michigan Math. J. **9** (1962), 289–298.

[18] J. Milnor and W. Thurston, *On iterated maps of the interval*. Dynamical systems (College Park, MD, 1986–87), 465–563, Lecture Notes in Math., 1342, Springer, Berlin, 1988.

[19] W. Parry, *On the* β*-expansions of real numbers*, Acta Math. Acad. Sci. Hung. **11** (1960), 401–416.

[20] O. M. Sharkovskiĭ, *Co-existence of cycles of a continuous mapping of a line onto itself.*, Ukranian Math. Z. **16** 1964, 61–71.

[21] N. Sidorov, *Almost every number has a continuum of* β*-expansions*, Amer. Math. Monthly **110** (2003), 838–842.

[22] N. Sidorov, *Combinatorics of linear iterated function systems with overlaps*, Nonlinearity **20** (2007) 1299–1312.

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