Infinite products with strongly B-multiplicative exponents

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To Professor Kátai on the occasion of his 70th birthday

Abstract

Let $N_{1,B}(n)$ denote the number of ones in the B-ary expansion of an integer n. Woods introduced the infinite product $P := \prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{N_{1,2}(n)}}$ and Robbins proved that $P = 1/\sqrt{2}$. Related products were studied by several authors. We show that a trick for proving that $P^2 = 1/2$ (knowing that P converges) can be extended to evaluating new products with (generalized) strongly B-multiplicative exponents. A simple example is

$$\prod_{n\geq 0}\left(\frac{Bn+1}{Bn+2}\right)^{(-1)^{N_{1,B}(n)}} = \frac{1}{\sqrt{B}}.$$

MSC: 11A63, 11Y60.

1 Introduction

In 1985 the following infinite product, for which no closed expression is known, appeared in [8, p. 193 and p. 209]:

$$R := \prod_{n\geq 1}\left(\frac{(4n+1)(4n+2)}{4n(4n+3)}\right)^{\varepsilon(n)}$$

where $(\varepsilon(n))_{n\geq 0}$ is the ± 1 Prouhet-Thue-Morse sequence, defined by

$$\varepsilon(n) = (-1)^{N_{1,2}(n)}$$

with $N_{1,2}(n)$ being the number of ones in the binary expansion of n. (For more on the Prouhet-Thue-Morse sequence, see for example [5].)

On the one hand, it is not difficult to see that $R = \frac{3}{2Q}$, where

$$Q := \prod_{n \geq 1} \left(\frac{2n}{2n+1} \right)^{\varepsilon(n)}.$$

Namely, splitting the simpler product into even and odd indices and using the relations $\varepsilon(2n) = \varepsilon(n)$ and $\varepsilon(2n+1) = -\varepsilon(n)$, we get

$$Q = \left(\prod_{n \geq 1} \left(\frac{4n}{4n+1} \right)^{\varepsilon(n)} \right) \left(\prod_{n \geq 0} \left(\frac{4n+2}{4n+3} \right)^{-\varepsilon(n)} \right) = \frac{3}{2} \prod_{n \geq 1} \left(\frac{4n(4n+3)}{(4n+1)(4n+2)} \right)^{\varepsilon(n)} = \frac{3}{2R}.$$

(Note that, whereas the logarithm of R is an absolutely convergent series, the logarithm of Q – and similarly the logarithm of the product P below – is a conditionally convergent series, as can be seen by partial summation, using the fact that the sums $\sum_{0 \leq k \leq n} \varepsilon(k)$ only take the values $+1$, 0 and -1, hence are bounded.)

On the other hand, the product Q reminds us of the Woods-Robbins product [18, 12]

$$P := \prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{\varepsilon(n)} = \frac{1}{\sqrt{2}}$$

(generalized for example in [13, 1, 2, 3, 4, 14]).

In 1987 during a stay at the University of Chicago, the first author, convinced that the computation of the infinite product Q should not resist the even-odd splitting techniques he was using with J. Shallit, discovered the following trick. First write QP as

$$QP = \left(\frac{1}{2} \right)^{\varepsilon(0)} \prod_{n \geq 1} \left(\frac{2n}{2n+1} \cdot \frac{2n+1}{2n+2} \right)^{\varepsilon(n)} = \frac{1}{2} \prod_{n \geq 1} \left(\frac{n}{n+1} \right)^{\varepsilon(n)}.$$

Now split the indices as we did above, obtaining

$$\prod_{n \geq 1} \left(\frac{n}{n+1} \right)^{\varepsilon(n)} = \left(\prod_{n \geq 1} \left(\frac{2n}{2n+1} \right)^{\varepsilon(n)} \right) \left(\prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{-\varepsilon(n)} \right) = QP^{-1}.$$

This gives $QP = \frac{1}{2} QP^{-1}$: as the hope of computing Q fades, the trick at least yields an easy way to compute $P = 1/\sqrt{2}$. By extending this trick to B-ary expansions, the second author [14] found the generalization of $P = 1/\sqrt{2}$ given in Corollary 5 of Section 4.2.

It happens that the sequence $(\varepsilon(n))_{n \geq 0}$ is strongly 2-multiplicative (see Definition 1 in the next section). The purpose of this paper is to extend the trick to products with more general exponents. For example, we prove the following.

Let $B > 1$ be an integer. For $k = 1, \ldots, B-1$ define $N_{k,B}(n)$ to be the number of occurrences of the digit k in the B-ary expansion of the integer n. Also, let

$$s_B(n) := \sum_{0 < k < B} k N_{k,B}(n)$$

be the sum of the B-ary digits of n, and let $q > 1$ be an integer. Then

$$\prod_{n\geq 0}\left(\frac{Bn+k}{Bn+k+1}\right)^{(-1)^{N_{k,B}(n)}} = \frac{1}{\sqrt{B}},$$

$$\prod_{n\geq 0}\prod_{\substack{0<k<B\\ k\not\equiv 0 \bmod q}}\left(\frac{Bn+k}{Bn+k+1}\right)^{\sin\frac{\pi k}{q}\sin\frac{\pi(2s_B(n)+k)}{q}} = \frac{1}{\sqrt{B}},$$

and

$$\prod_{n\geq 0}\prod_{\substack{0<k<B\\ k\not\equiv 0 \bmod q}}\left(\frac{Bn+k}{Bn+k+1}\right)^{\sin\frac{\pi k}{q}\cos\frac{\pi(2s_B(n)+k)}{q}} = 1.$$

Note that the use of the trick is not necessarily the only way to compute products of this type: real analysis is used for computing P in [12] and to compute products more general than P in [13]; the core of [1] is the use of Dirichlet series, while [2] deals with complex power series and the second part of [3] with real integrals. It may even happen that, in some cases, the use of the trick gives less general results than other methods. For example, in Remark 5 we show that Corollary 5 or [14] can also be obtained as an easy consequence of [2, Theorem 1].

2 Strongly B-multiplicative sequences

We recall the classical definition of a strongly B-multiplicative sequence. (For this and for the definitions of B-multiplicative, B-additive, and strongly B-additive, see [6, 9, 7, 11, 10].)

Definition 1. Let $B \geq 2$ be an integer. A sequence of complex numbers $(u(n))_{n\geq 0}$ is *strongly B-multiplicative* if $u(0) = 1$ and, for all $n \geq 0$ and all $k \in \{0, 1, \ldots, B-1\}$,

$$u(Bn+k) = u(n)u(k).$$

Example 1. If z is any complex number, then the sequence u defined by $u(0) := 1$ and $u(n) := z^{s_B(n)}$ for $n \geq 1$ is strongly B-multiplicative.

Remark 1. If we do not impose the condition $u(0) = 1$ in Definition 1, then either $u(0) = 1$ holds, or the sequence $(u(n))_{n\geq 0}$ must be identically 0. To see this, note that the relation $u(Bn+k) = u(n)u(k)$ implies, with $n = k = 0$, that $u(0) = u(0)^2$. Hence $u(0) = 1$ or $u(0) = 0$. If $u(0) = 0$, then taking $n = 0$ in the relation gives $u(k) = 0$ for all $k \in \{0, 1, \ldots, B-1\}$, which by (1) implies $u(n) = 0$ for all $n \geq 0$.

Proposition 1. *If the sequence $(u(n))_{n\geq 0}$ is strongly B-multiplicative, and if the B-ary expansion of $n \geq 1$ is $n = \sum_j e_j(n)B^j$, then $u(n) = \prod_j u(e_j(n))$. In particular, the only strongly B-multiplicative sequence with $u(1) = u(2) = \cdots = u(B-1) = \theta$, where $\theta = 0$ or 1, is the sequence $1, \theta, \theta, \theta, \ldots$.*

Proof. Use induction on the number of base B digits of n. ∎

We now generalize the notion of a strongly B-multiplicative sequence different from $1, 0, 0, 0, \ldots$.

Definition 2. Let $B \geq 2$ be an integer. A sequence of complex numbers $(u(n))_{n\geq 0}$ *satisfies Hypothesis* $\mathcal{H}_B$ if there exist an integer $n_0 \geq B$ and complex numbers $v(0), v(1), \ldots, v(B-1)$ such that $u(n_0) \neq 0$ and, for all $n \geq 1$ and all $k = 0, 1, \ldots, B-1$,

$$u(Bn + k) = u(n)v(k).$$

Proposition 2.

(1) *If a sequence* $(u(n))_{n\geq 0}$ *satisfies Hypothesis* $\mathcal{H}_B$*, then the values* $v(0), v(1), \ldots, v(B-1)$ *are uniquely determined.*

(2) *A sequence* $(u(n))_{n\geq 0}$ *has* $u(0) = 1$ *and satisfies Hypothesis* $\mathcal{H}_B$ *with* $u(Bn + k) = u(n)v(k)$ *not only for* $n \geq 1$ *but also for* $n = 0$*, if and only if the sequence is strongly* B*-multiplicative and not equal to* $1, 0, 0, 0, \ldots$*. In that case,* $v(k) = u(k)$ *for* $k = 0, 1, \ldots, B-1$.

Proof. If the sequence $(u(n))_{n\geq 0}$ satisfies Hypothesis $\mathcal{H}_B$, then $v(k) = u(Bn_0 + k)/u(n_0)$ for $k = 0, 1, \ldots, B-1$. This implies (1).

To prove the "only if" part of (2), take $n = 0$ in the relation $u(Bn + k) = u(n)v(k)$, yielding $u(k) = u(0)v(k) = v(k)$ for $k = 0, 1, \ldots, B-1$. Hence $u(Bn+k) = u(n)u(k)$ for all $n \geq 0$ and $k = 0, 1, \ldots, B-1$. Thus $(u(n))_{n\geq 0}$ is strongly B-multiplicative. Since $u(n_0) \neq 0$ for some $n_0 \geq B$, the sequence is not $1, 0, 0, 0, \ldots$

Conversely, suppose that $(u(n))_{n\geq 0}$ is strongly B-multiplicative and is not $1, 0, 0, 0, \ldots$ Then there exists an integer $\ell_0 \geq 1$ such that $u(\ell_0) \neq 0$. Hence $n_0 := B\ell_0 \geq B$ and $u(n_0) = u(B\ell_0) = u(\ell_0)u(0) = u(\ell_0) \neq 0$. Defining $v(k) := u(k)$ for $k = 0, 1, \ldots, B-1$, we see that $(u(n))_{n\geq 0}$ satisfies Hypothesis $\mathcal{H}_B$, and the proposition follows. ∎

Example 2. We construct a sequence which satisfies Hypothesis $\mathcal{H}_B$ but is not strongly B-multiplicative. Let z be a complex number, with $z \notin \{0, 1\}$, and define $u(n) := z^{N_{0,B}(n)}$, where $N_{0,B}(n)$ counts the number of zeros in the B-ary expansion of n for $n > 0$, and $N_{0,B}(0) := 0$ (which corresponds to representing zero by the empty sum, that is, the empty word). Note that for all $n \geq 1$ the relation $N_{0,B}(Bn) = N_{0,B}(n) + 1$ holds, and for all $k \in \{1, 2, \ldots, B-1\}$ and all $n \geq 0$ the relation $N_{0,B}(Bn+k) = N_{0,B}(n) = N_{0,B}(n) + N_{0,B}(k)$ holds. Hence the nonzero sequence $(u(n))_{n\geq 0}$ satisfies Hypothesis $\mathcal{H}_B$, with $v(0) := z$ and $v(k) := 1 = u(k)$ for $k = 1, 2, \ldots, B-1$. But the sequence is not strongly B-multiplicative: $u(B \times 1 + 0) = z \neq 1 = u(1)u(0)$.

Remark 2. The alternative definition $N_{0,B}(0) := 1$ (which would correspond to representing zero by the single digit 0 instead of by the empty word) would also not lead to a strongly B-multiplicative sequence u, since then $u(0) = z \neq 1$, which does not agree with Definition 1 (see also Remark 1). On the other hand, the new sequence would still satisfy Hypothesis $\mathcal{H}_B$, with the same values $v(k)$, as the same proof shows, since $u(0)$ does not appear in it.

3 Convergence of infinite products

Inspired by the Woods-Robbins product P, we want to study products given in the following lemma.

Lemma 1. *Let $B > 1$ be an integer. Let $(u(n))_{n\geq 0}$ be a sequence of complex numbers with $|u(n)| \leq 1$ for all $n \geq 0$. Suppose that it satisfies Hypothesis $\mathcal{H}_B$ with $|v(k)| \leq 1$ for all $k \in \{0, 1, \ldots, B-1\}$, and that $|\sum_{0\leq k<B} v(k)| < B$. Then for each $k \in \{0, 1, \ldots, B-1\}$, the infinite product*

$$\prod_{n\geq \delta_k} \left(\frac{Bn+k}{Bn+k+1}\right)^{u(n)}$$

converges, where δ_k —a special case of the Kronecker delta— is defined by

$$\delta_k := \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0. \end{cases}$$

Proof. For $N = 1, 2, \ldots$, let

$$F(N) := \sum_{0\leq n<N} u(n).$$

Also define for $j = 1, 2, \ldots, B-1$

$$G(j) := \sum_{0\leq n<j} v(n)$$

and set $G(0) := 0$. Then, for each $b \in \{0, 1, \ldots, B-1\}$, and for every $N \geq 1$,

$$\begin{aligned} F(BN+b) &= \sum_{0\leq n<BN} u(n) + \sum_{BN\leq n<BN+b} u(n) \\ &= \sum_{0\leq n<N} \sum_{0\leq \ell<B} u(Bn+\ell) + \sum_{0\leq \ell<b} u(BN+\ell) \\ &= \sum_{0\leq \ell<B} u(\ell) + \sum_{1\leq n<N} \sum_{0\leq \ell<B} u(n)v(\ell) + u(N) \sum_{0\leq \ell<b} v(\ell). \end{aligned}$$

Hence, using $|u(N)| \leq 1$ and $|G(b)| \leq B-1 < B$,

$$\begin{aligned} |F(BN+b)| &= |F(B) + (F(N) - u(0))G(B) + u(N)G(b)| \\ &< |F(B) - u(0)G(B)| + |F(N)||G(B)| + B. \end{aligned}$$

This gives the case $d = 1$ of the following inequality, which holds for $d \geq 1$ and $e_t \in \{0, 1, \ldots, B-1\}$, and which is proved by induction on d using $|F(e_t)| \leq B$:

$$\left| F\left(\sum_{0\leq t\leq d} e_t B^t\right)\right| < |F(B) - u(0)G(B)| \left(1 + \sum_{1\leq t\leq d-1} |G(B)|^t\right) + B\left(1 + \sum_{1\leq t\leq d} |G(B)|^t\right).$$

Hence

$$\left|F\left(\sum_{0\le t\le d} e_t B^t\right)\right| < \begin{cases} B(3d+1) & \text{if } |G(B)| \le 1,\\ 3B\dfrac{|G(B)|^{d+1}-1}{|G(B)|-1} & \text{if } |G(B)| > 1.\end{cases}$$

This implies that for some constant $C = C(B,v)$, and for every N large enough,

$$|F(N)| < \begin{cases} C\log N & \text{if } |G(B)| \le 1,\\ C|G(B)|^{\frac{\log N}{\log B}} = CN^{\frac{\log|G(B)|}{\log B}} & \text{if } |G(B)| > 1.\end{cases}$$

Since $|G(B)| < B$ by hypothesis, we can define $\alpha \in (0,1)$ by

$$\alpha := \begin{cases} \frac{1}{2} & \text{if } |G(B)| \le 1,\\ \frac{\log|G(B)|}{\log B} & \text{if } |G(B)| > 1.\end{cases}$$

Hence for every N large enough $|F(N)| < CN^{\alpha}$. It follows, using summation by parts, that the series $\sum_n u(n)\log\frac{Bn+k}{Bn+k+1}$ converges, hence the lemma. ∎

Remark 3.

(1) Here and in what follows, expressions of the form a^z, where a is a positive real number and z a complex number, are defined by $a^z := e^{z\log a}$, and $\log a$ is real.

(2) For more precise estimates of summatory functions of (strongly) B-multiplicative sequences, see for example [7, 10]. (In [10] strongly B-multiplicative sequences are called completely B-multiplicative.)

4 Evaluation of infinite products

This section is devoted to computing some infinite products with exponents that satisfy Hypothesis $\mathcal{H}_B$, including some whose exponents are strongly B-multiplicative.

4.1 General results

Theorem 1. *Let $B > 1$ be an integer. Let $(u(n))_{n\ge 0}$ be a sequence of complex numbers with $|u(n)| \le 1$ for all $n \ge 0$. Suppose that u satisfies Hypothesis $\mathcal{H}_B$, with complex numbers $v(0), v(1), \dots, v(B-1)$ such that $|v(k)| \le 1$ for $k \in \{0, 1, \dots B-1\}$ and $|\sum_{0\le k<B} v(k)| < B$. Then the following relation between nonempty products holds:*

$$\prod_{\substack{0\le k<B\\ v(k)\ne 1}} \prod_{n\ge\delta_k} \left(\frac{Bn+k}{Bn+k+1}\right)^{u(n)(1-v(k))} = \frac{1}{B^{u(0)}} \prod_{0<k<B} \left(\frac{k}{k+1}\right)^{u(k)-u(0)v(k)}.$$

Proof. The condition $|\sum_{0\le k<B} v(k)| < B$ prevents v from being identically equal to 1 on $\{0, 1, \dots, B-1\}$, so the left side of the equation is not empty. Since $B > 1$, so is the right.

We first show that

$$\prod_{0\le k<B} \prod_{n\ge\delta_k} \left(\frac{Bn+k}{Bn+k+1}\right)^{u(n)} = \frac{1}{B^{u(0)}} \prod_{n\ge 1} \left(\frac{n}{n+1}\right)^{u(n)} \qquad (*)$$

(note that, by Lemma 1, all the products converge). To see this, write the left side as

$$\left(\frac{1}{2}\frac{2}{3}\cdots\frac{B-1}{B}\right)^{u(0)}\prod_{n\geq 1}\left(\frac{Bn}{Bn+1}\frac{Bn+1}{Bn+2}\cdots\frac{Bn+B-1}{Bn+B}\right)^{u(n)}$$

and use telescopic cancellation. Now, splitting the product on the right side of $(*)$ according to the values of n modulo B gives

$$\begin{aligned}\prod_{n\geq 1}\left(\frac{n}{n+1}\right)^{u(n)} &= \prod_{0\leq k<B}\prod_{n\geq\delta_k}\left(\frac{Bn+k}{Bn+k+1}\right)^{u(Bn+k)}\\ &= \prod_{0<k<B}\left(\frac{k}{k+1}\right)^{u(k)}\prod_{0\leq k<B}\prod_{n\geq 1}\left(\frac{Bn+k}{Bn+k+1}\right)^{u(n)v(k)}\\ &= \prod_{0<k<B}\left(\frac{k}{k+1}\right)^{u(k)-u(0)v(k)}\prod_{0\leq k<B}\prod_{n\geq\delta_k}\left(\frac{Bn+k}{Bn+k+1}\right)^{u(n)v(k)}.\end{aligned}$$

Using $(*)$ and the fact that convergent infinite products are nonzero, the theorem follows. ■

Example 3. As in Example 2, the sequence u defined by $u(n) = z^{N_{0,B}(n)}$, with $z \notin \{0, 1\}$, satisfies Hypothesis $\mathcal{H}_B$, and $\sum_{0\leq k<B} v(k) = z + B - 1$. If furthermore $|z| \leq 1$, then

$$\prod_{n\geq 1}\left(\frac{Bn}{Bn+1}\right)^{(1-z)z^{N_{0,B}(n)}} = B.$$

Corollary 1. *Fix an integer $B > 1$. If $(u(n))_{n\geq 0}$ is strongly B-multiplicative, satisfies $|u(n)| \leq 1$ for all $n \geq 0$, and is not equal to either of the sequences $1, 0, 0, 0, \ldots$ or $1, 1, 1, \ldots$, then*

$$\prod_{n\geq 0}\prod_{\substack{0<k<B\\ u(k)\neq 1}}\left(\frac{Bn+k}{Bn+k+1}\right)^{u(n)(1-u(k))} = \frac{1}{B}.$$

Proof. Using Theorem 1 and Proposition 2 part (2) it suffices to prove that $|\sum_{0\leq k<B} u_k| < B$. Since $|u_n| \leq 1$ for all $n \geq 0$, we have $|\sum_{0\leq k<B} u_k| \leq B$. From the equality case of the triangle inequality, it thus suffices to prove that the numbers $u_0, u_1, \ldots, u_{B-1}$ are not all equal to a same complex number z with $|z| = 1$. If they were, then, since $u_0 = 1$, we would have $u_0 = u_1 = \ldots = u_{B-1} = 1$. Hence $(u(n))_{n\geq 0} = 1, 1, 1, \ldots$ from Proposition 1, a contradiction. ■

Addendum. *Theorem 1 and Corollary 1 can be strengthened, as follows.*

(1) *If B, u, and v satisfy the hypotheses of Theorem 1, then*

$$\sum_{\substack{0\leq k<B\\ v(k)\neq 1}}(1-v(k))\sum_{n\geq\delta_k} u(n)\log\frac{Bn+k}{Bn+k+1} = -u(0)\log B + \sum_{0<k<B}(u(k)-u(0)v(k))\log\frac{k}{k+1}.$$

(2) *If B and u satisfy the hypotheses of Corollary 1, then*

$$\sum_{n\geq 0}\sum_{\substack{0<k<B\\ u(k)\neq 1}} u(n)(1-u(k))\log\frac{Bn+k}{Bn+k+1} = -\log B.$$

Proof. Write the proofs of Theorem 1 and Corollary 1 additively instead of multiplicatively. ■

Remark 4. The Addendum cannot be proved by just taking logarithms in the formulas in Theorem 1 and Corollary 1. To illustrate the problem, note that while

$$\prod_{n\geq 0} e^{\frac{(-1)^n 8i}{2n+1}} = 1$$

(because the product converges to $e^{2\pi i}$), the log equation is false:

$$\sum_{n\geq 0}\frac{(-1)^n 8i}{2n+1} = 2\pi i \neq 0 = \log 1.$$

Example 4. With the same u and z as in Example 3, Addendum (1) yields

$$\sum_{n\geq 1} z^{N_{0,B}(n)}\log\frac{Bn}{Bn+1} = \frac{\log B}{z-1}.$$

Hence

$$\prod_{n\geq 1}\left(\frac{Bn}{Bn+1}\right)^{z^{N_{0,B}(n)}} = B^{\frac{1}{z-1}}.$$

(Note the similarity between this product and the one in Example 3. Neither implies the other, but of course the preceding log equation implies both.)

If we modify the sequence u as in Remark 2, we get the same two formulas, because the value $N_{0,B}(0)$ does not appear in them.

Corollary 2. *Fix integers B, q, p with $B > 1$, $q > p > 0$, and $B \equiv 1 \bmod q$. Then*

$$\prod_{n\geq 0}\prod_{\substack{0<k<B\\ k\not\equiv 0 \bmod q}}\left(\frac{Bn+k}{Bn+k+1}\right)^{\sin\frac{\pi kp}{q}\sin\frac{\pi(2n+k)p}{q}} = \frac{1}{\sqrt{B}}$$

and

$$\prod_{n\geq 0}\prod_{\substack{0<k<B\\ k\not\equiv 0 \bmod q}}\left(\frac{Bn+k}{Bn+k+1}\right)^{\sin\frac{\pi kp}{q}\cos\frac{\pi(2n+k)p}{q}} = 1.$$

Proof. Let $\omega := e^{2\pi i p/q}$. Since $B \equiv 1 \bmod q$, we may take $u(n) := \omega^n$ in Addendum (2), yielding the formula

$$\sum_{n\geq 0} \sum_{\substack{0<k<B \\ k\not\equiv 0 \bmod q}} \omega^n(1-\omega^k) \log \frac{Bn+k}{Bn+k+1} = -\log B.$$

Writing $\omega^n(1-\omega^k) = -2i\omega^{n+\frac{k}{2}} \sin \frac{\pi kp}{q}$, and multiplying the real and imaginary parts of the formula by $1/2$, the result follows. ■

Example 5. Take $B = 5$, $p = 1$, and $q = 4$. Squaring the products, we get

Define $\sigma(n)$ to be $+1$ if n is a square modulo 4, and -1 otherwise, that is,

$$\sigma(n) := \begin{cases} +1 & \text{if } n \equiv 0 \text{ or } 1 \bmod 4, \\ -1 & \text{if } n \equiv 2 \text{ or } 3 \bmod 4. \end{cases}$$

Then

$$\prod_{n\geq 0} \left(\frac{5n+1}{5n+2}\right)^{\sigma(n)} \left(\frac{5n+2}{5n+3}\right)^{\sigma(n)+\sigma(n+1)} \left(\frac{5n+3}{5n+4}\right)^{\sigma(n+1)} = \frac{1}{5}$$

and

$$\prod_{n\geq 0} \left(\frac{5n+1}{5n+2}\right)^{\sigma(n-1)} \left(\frac{5n+2}{5n+3}\right)^{\sigma(n-1)+\sigma(n)} \left(\frac{5n+3}{5n+4}\right)^{\sigma(n)} = 1.$$

4.2 The sum-of-digits function $s_B(n)$

Other products can also be obtained from Corollary 1. We give three corollaries, each of which generalizes the Woods-Robbins formula $P = 1/\sqrt{2}$ in the Introduction. Recall that $s_B(n)$ denotes the sum of the B-ary digits of the integer n.

Corollary 3. *Fix an integer $B > 1$ and a complex number z with $|z| \leq 1$. If $z \notin \{0, 1\}$, then*

$$\prod_{n\geq 0} \prod_{\substack{0<k<B \\ z^k\neq 1}} \left(\frac{Bn+k}{Bn+k+1}\right)^{z^{s_B(n)}(1-z^k)} = \frac{1}{B}.$$

Proof. Take $u(n) := z^{s_B(n)}$ in Corollary 1 and note that $s_B(k) = k$ when $0 < k < B$. ■

Example 6. Take $B = 2$ and $z = 1/2$. Squaring the product, we obtain

$$\prod_{n\geq 0} \left(\frac{2n+1}{2n+2}\right)^{(1/2)^{s_2(n)}} = \frac{1}{4}.$$

Corollary 4. *Let B, p, q be integers with $B > 1$ and $q > p > 0$. Then*

$$\prod_{n\geq 0} \prod_{\substack{0<k<B \\ k\not\equiv 0 \bmod q}} \left(\frac{Bn+k}{Bn+k+1}\right)^{\sin \frac{\pi kp}{q} \sin \frac{\pi(2s_B(n)+k)p}{q}} = \frac{1}{\sqrt{B}}$$

and

$$\prod_{n\geq 0}\prod_{\substack{0<k<B\\ k\not\equiv 0 \bmod q}}\left(\frac{Bn+k}{Bn+k+1}\right)^{\sin\frac{\pi kp}{q}\cos\frac{\pi(2s_B(n)+k)p}{q}}=1.$$

Proof. Use the proof of Corollary 2, but replace $B\equiv 1 \bmod q$ with $s_B(Bn+k)=s_B(n)+k$ when $0\leq k<B$, and replace ω^n with $\omega^{s_B(n)}$. ■

Example 7. Take $B=2$, $q=4$, and $p=1$. Squaring the products and defining $\sigma(n)$ as in Example 5, we get

$$\prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{\sigma(s_2(n))}=\frac{1}{2}\quad\text{and}\quad\prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{\sigma(s_2(n)+1)}=1.$$

In the same spirit, we recover a result from [3, p. 369-370].

Example 8. Taking $B=q=3$ and $p=1$ in Corollary 4, we obtain two infinite products. Raising the second to the power $-2/\sqrt{3}$ and multiplying by the square of the first, we get

Define $\theta(n)$ *by*

$$\theta(n):=\begin{cases}1 & \text{if } n\equiv 0 \text{ or } 1 \bmod 3,\\ -2 & \text{if } n\equiv 2 \bmod 3.\end{cases}$$

Then

$$\prod_{n\geq 0}(3n+1)^{\theta(s_3(n))}(3n+2)^{\theta(s_3(n)+1)}(3n+3)^{\theta(s_3(n)+2)}=\frac{1}{3}.$$

Corollary 5 ([14]). *Let $B>1$ be an integer. Then*

$$\prod_{n\geq 0}\prod_{\substack{0<k<B\\ k\ \text{odd}}}\left(\frac{Bn+k}{Bn+k+1}\right)^{(-1)^{s_B(n)}}=\frac{1}{\sqrt{B}}.$$

Proof. Take $z=-1$ in Corollary 3 (or take $q=2$ and $p=1$ in Corollary 4). ■

Example 9. With $B=2$, since $s_2(n)=N_{1,2}(n)$, we recover the Woods-Robbins formula $P=1/\sqrt{2}$. Taking $B=6$ gives

$$\prod_{n\geq 0}\left(\frac{(6n+1)(6n+3)(6n+5)}{(6n+2)(6n+4)(6n+6)}\right)^{(-1)^{s_6(n)}}=\frac{1}{\sqrt{6}}.$$

Remark 5. Corollary 5 can also be obtained from [2, Theorem 1], as follows. Taking x equal to -1 and j equal to 0 in that theorem gives

$$\sum_{n\geq 0}(-1)^{s_B(n)}\log\frac{n+1}{B\lfloor n/B\rfloor+B}=-\frac{1}{2}\log B$$

where $\lfloor x \rfloor$ is the integer part of x. But the series is equal to

$$\begin{aligned}\sum_{m\geq 0}\sum_{0\leq k<B}(-1)^{s_B(Bm+k)}\log\frac{Bm+k+1}{Bm+B} &= \sum_{m\geq 0}(-1)^{s_B(m)}\sum_{0\leq k<B}(-1)^k\log\frac{Bm+k+1}{Bm+B}\\ &= \sum_{m\geq 0}(-1)^{s_B(m)}\sum_{\substack{k \text{ odd}\\ 0<k<B}}\log\frac{Bm+k}{Bm+k+1}\end{aligned}$$

where the last equality follows by looking separately at the cases B even and B odd.

4.3 The counting function $N_{j,B}(n)$

We can also compute some infinite products associated with counting the number of occurrences of one or several given digits in the base B expansion of an integer.

Definition 3. If B is an integer ≥ 2 and if j is in $\{0, 1, \dots, B-1\}$, let $N_{j,B}(n)$ be the number of occurrences of the digit j in the B-ary expansion of n when $n > 0$, and set $N_{j,B}(0) := 0$.

Corollary 6. *Let B, q, p be integers with $B > 1$ and $q > p > 0$. Let J be a nonempty, proper subset of $\{0, 1, \dots, B-1\}$. Define $N_{J,B}(n) := \sum_{j\in J} N_{j,B}(n)$. Then the following equalities hold:*

$$\prod_{k\in J}\prod_{n\geq\delta_k}\left(\frac{Bn+k}{Bn+k+1}\right)^{\sin\frac{\pi(2N_{J,B}(n)+1)p}{q}} = B^{-\frac{1}{2\sin\frac{\pi p}{q}}}$$

and

$$\prod_{k\in J}\prod_{n\geq\delta_k}\left(\frac{Bn+k}{Bn+k+1}\right)^{\cos\frac{\pi(2N_{J,B}(n)+1)p}{q}} = 1.$$

Proof. Let $\omega := e^{2\pi ip/q}$. We denote $u_{q,j,B}(n) := \omega^{N_{j,B}(n)}$ and $u_{q,J,B}(n) := \prod_{j\in J} u_{q,j,B}(n) = \omega^{N_{J,B}(n)}$. Note that, for every j in $\{1, 2, \dots, B-1\}$, the sequence $(u_{q,j,B}(n))_{n\geq 0}$ is strongly B-multiplicative and nonzero, hence satisfies Hypothesis $\mathcal{H}_B$. The sequence $(u_{q,0,B}(n))_{n\geq 0}$ also satisfies Hypothesis $\mathcal{H}_B$, as is seen by taking $z = \omega$ in Example 2. Therefore the sequence $(u_{q,J,B}(n))_{n\geq 0}$ satisfies Hypothesis $\mathcal{H}_B$, with, for $k = 0, 1, \dots, B-1$, the value $v(k) := \omega$ if $k \in J$ and $v(k) := 1$ otherwise.

Now $|u_{q,J,B}(n)| = 1$ for $n \geq 0$, and $|v(k)| = 1$ for $k = 0, 1, \dots, B-1$. Furthermore, $|\sum_{0\leq k<B} v(k)| < B$, since v is not constant on $\{0, 1, \dots, B-1\}$. Thus we may apply Addendum (1) with $u(n) := u_{q,J,B}(n)$, obtaining

$$(1-\omega)\sum_{k\in J}\sum_{n\geq\delta_k}\omega^{N_{J,B}(n)}\log\frac{Bn+k}{Bn+k+1} = -\log B.$$

Writing $(1-\omega)\omega^{N_{J,B}(n)} = -2i\omega^{N_{J,B}(n)+\frac{1}{2}}\sin\frac{\pi p}{q}$, and taking the real and imaginary parts of the summation, the result follows. ∎

Example 10. Taking $q = 2$ and $p = 1$ in the first formula gives

$$\prod_{k \in J} \prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{(-1)^{N_{J,B}(n)}} = \frac{1}{\sqrt{B}}.$$

An application is an alternate proof of Corollary 5: take J to be the set of odd numbers in $\{1, 2, \ldots, B-1\}$; since $s_B(n) = \sum_{0<k<B} k N_{k,B}(n)$, it follows that $(-1)^{\sum_{j \in J} N_{j,B}(n)} = (-1)^{s_B(n)}$.

Remark 6. Corollary 6 requires that J be a proper subset of $\{0, 1, \ldots, B-1\}$. Suppose instead that $J = \{0, 1, \ldots, B-1\}$. Then $N_{J,B}(n)$ is the number of B-ary digits of n if $n > 0$ (that is, $N_{J,B}(n) = \lfloor \frac{\log n}{\log B} \rfloor + 1$), and $N_{J,B}(0) = 0$. In that case, Corollary 6 does not apply, and the products may diverge. For example, when $B = q = 2$ and $p = 1$ the logarithm of the first product is equal to the series

$$-\log 2 + \sum_{n \geq 1} (-1)^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \log \frac{n+1}{n},$$

which does not converge. However, note its resemblance with Vacca's (convergent) series for Euler's constant [16]

$$\gamma = \sum_{n \geq 1} \left\lfloor \frac{\log n}{\log 2} \right\rfloor \frac{(-1)^n}{n}.$$

Corollary 7. *Let B, q, p be integers with $B > 1$ and $q > p > 0$. Then for $k = 0, 1, \ldots, B-1$ the following equalities hold:*

$$\prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{\sin \frac{\pi(2N_{k,B}(n)+1)p}{q}} = B^{-\frac{1}{2 \sin \frac{\pi p}{q}}}$$

and

$$\prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{\cos \frac{\pi(2N_{k,B}(n)+1)p}{q}} = 1.$$

Proof. Take $J := \{k\}$ in Corollary 6. (The case $k = 0$ and $p = 1$ is Example 4 with $z = e^{2\pi i/q}$.) ■

Example 11. Taking $q = 2$ and $p = 1$ in the first formula (or taking $J = \{k\}$ in Example 10) yields

$$\prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{(-1)^{N_{k,B}(n)}} = \frac{1}{\sqrt{B}}.$$

In particular, if $B = 2$ the choice $k = 1$ gives the Woods-Robbins formula $P = 1/\sqrt{2}$, and $k = 0$ gives

$$\prod_{n \geq 1} \left(\frac{2n}{2n+1} \right)^{(-1)^{N_{0,2}(n)}} = \frac{1}{\sqrt{2}}.$$

Remark 7. For base $B = 2$, the formulas in Example 11 are special cases of results in [4], where $N_{j,2}(n)$ is generalized to counting the number of occurrences of a given *word* in the binary expansion of n. On the other hand, the value of the product Q in the Introduction,

$$Q = \prod_{n\geq 1} \left(\frac{2n}{2n+1}\right)^{(-1)^{N_{1,2}(n)}},$$

remains a mystery.

Example 12. Take $B = q = 3$ and $p = 1$. Raising the first product to the power $2/\sqrt{3}$ and squaring the second, we obtain

Define $\eta(n)$ by

$$\eta(n) := \begin{cases} +1 & \text{if } n \equiv 0 \bmod 3, \\ 0 & \text{if } n \equiv 1 \bmod 3, \\ -1 & \text{if } n \equiv 2 \bmod 3, \end{cases}$$

and define $\theta(n)$ as in Example 8. Then for $k = 0, 1,$ and 2

$$\prod_{n\geq\delta_k} \left(\frac{3n+k}{3n+k+1}\right)^{\eta(N_{k,3}(n))} = \frac{1}{3^{2/3}} \quad \textit{and} \quad \prod_{n\geq\delta_k} \left(\frac{3n+k}{3n+k+1}\right)^{\theta(N_{k,3}(n)+1)} = 1.$$

4.4 The Gamma function

It can happen that the exponent in some of our products is a periodic function of n. For example, this is obviously the case in Corollary 2. To take another example, it is not hard to see that if B odd, then $(-1)^{s_B(n)} = (-1)^n$. Hence Corollary 5 gives

$$\prod_{n\geq 0} \prod_{\substack{0<k<B \\ k \text{ odd}}} \left(\frac{Bn+k}{Bn+k+1}\right)^{(-1)^n} = \frac{1}{\sqrt{B}} \quad (B \text{ odd}). \tag{**}$$

(This formula can also be obtained from Corollary 2 with $q = 2$ and $p = 1$.) For instance

$$P_{1,3} := \prod_{n\geq 0} \left(\frac{3n+1}{3n+2}\right)^{(-1)^n} = \frac{1}{\sqrt{3}}.$$

The product $P_{1,3}$ can also be computed using the following corollary of the Weierstrass product for the Gamma function [17, Section 12.13].

If d is a positive integer and $a_1 + a_2 + \cdots + a_d = b_1 + b_2 + \cdots + b_d$, where the a_j and b_j are complex numbers and no b_j is zero or a negative integer, then

$$\prod_{n\geq 0} \frac{(n+a_1)\cdots(n+a_d)}{(n+b_1)\cdots(n+b_d)} = \frac{\Gamma(b_1)\cdots\Gamma(b_d)}{\Gamma(a_1)\cdots\Gamma(a_d)}.$$

Combining this with the relation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ gives $P_{1,3} = 1/\sqrt{3}$.

The computation can be generalized, using Gauss' multiplication theorem for the Gamma function, to give another proof of Corollary 5 for B odd. Likewise, an analog of the odd-B case of Corollary 5 can be proved for even k:

$$\prod_{n\geq 1}\prod_{\substack{0\leq k<B\\ k\text{ even}}}\left(\frac{Bn+k}{Bn+k+1}\right)^{(-1)^n} = \frac{\pi\sqrt{B}}{2^B}\binom{B-1}{(B-1)/2} \quad (B\text{ odd}).$$

Multiplying this by the formula

$$\prod_{n\geq 1}\prod_{\substack{0<k<B\\ k\text{ odd}}}\left(\frac{Bn+k}{Bn+k+1}\right)^{(-1)^n} = \frac{2^{B-1}}{\sqrt{B}}\binom{B-1}{(B-1)/2}^{-1} \quad (B\text{ odd}),$$

which is $(**)$ rewritten, yields Wallis' product for π. (For an evaluation of the preceding two products when $B=2$, see [15, Example 7].)

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