Restricted towers of Hanoi and morphisms

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Abstract. The classical towers of Hanoi have been generalized in several ways. In particular the second named author has studied the 3-peg Hanoi towers with all possible restrictions on the permitted moves between pegs. We prove that all these Hanoi puzzles give rise to infinite morphic sequences of moves, whose appropriate truncations describe the transfer of any given number of disks. Furthermore two of these infinite sequences are actually automatic sequences.

1 Introduction

The towers of Hanoi have been introduced by the famous French number-theoretist Lucas, see [17]. Actually Lucas even invented in 1883 under the pseudonym Claus (anagram of Lucas) a legend about these towers, see e.g., `http://www.cs.wm.edu/~pkstoc/toh.html`

We recall briefly that this puzzle consists of three pegs and N distinct disks numbered $1, 2, \ldots, N$. The disks are all placed on peg 1 in increasing order of size (smallest disk on top). A move consists of taking the topmost disk on some peg and placing it on the top of some other peg, with the requirement that no disk should be covered by a larger one. The purpose is to move all the disks from peg 1 to some other peg in a minimal number of moves.

While the original game has been extensively studied, several variations have been proposed: cyclic towers of Hanoi, more than three pegs, colored disks, ... The reader can read in particular the bibliographies of [5, 2, 14], and the 207-item bibliography by Stockmeyer available at `http://www.cs.wm.edu/~pkstoc/biblio.ps`

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Several papers on Hanoi towers deal with links between the Hanoi puzzle (or related algorithms) and other objects from mathematics or computer science, e.g., finite automata [5], morphisms of free monoids [4], Toeplitz sequences [5, 3], Pascal's triangle [15], Stern's diatomic sequence [16].

In [18] the second named author has studied optimal algorithms for the 3-peg Hanoi puzzle where some moves from a given peg to a given peg are forbidden: there are five non-isomorphic possibilities (the case where the pegs are aligned and only moves between adjacent pegs are permitted goes back to [19]; the case where the pegs lie on a circle and only clockwise moves are permitted goes back to [7]). We will prove in this paper that these five possibilities give rise to infinite sequences of moves that are *morphic* (a definition is given in the next section) and such that appropriate prefixes (truncations) of these infinite sequences yield (possibly up to a small number of final moves) a minimal sequence of moves for transferring any given number of disks. This generalizes the case of classical and cyclic Hanoi towers, see [4, 5].

2 Morphic sequences and automatic sequences

We give a short reminder about morphic and automatic sequences (see for example [6] for a more complete account).

Let Σ be an alphabet (finite set) and let Σ^* be the free monoid generated by Σ (i.e., the set of finite words on Σ equipped with the concatenation operation). Let $h : \Sigma^* \to \Sigma^*$ be a morphism (i.e., a homomorphism of monoid). The morphism h is said to be *prolongable* on a if there exists a letter $a \in \Sigma$ and a word $x \in \Sigma^*$ such that $h(a) = ax$, and, for all $k \geq 0$, $h^k(x) \neq \emptyset$. It is then clear that the following limit exists

$$h^\infty(a) := \lim_{k\to\infty} h^k(a) = ah(a)h^2(a)h^3(a)\cdots$$

A sequence $\mathbf{u} = (u_n)_{n\geq 0}$ with values in Σ is said to be an *iterative fixed point* of the morphism h if h is prolongable on u_0. It is then clear that

$$\mathbf{u} = h^\infty(u_0) := \lim_{k\to\infty} h^k(u_0) = u_0h(u_0)h^2(u_0)h^3(u_0)\cdots$$

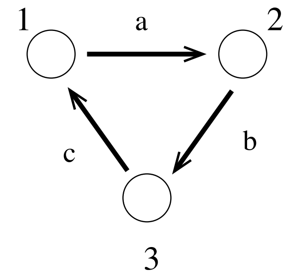

Fig. 1. The direct moves for the towers of Hanoi

Of course the sequence $\mathbf{u}$ is then a fixed point of (the extension by continuity of) h (to infinite sequences).

Definition 1. *A sequence* $\mathbf{u} = (u_n)_{n \geq 0}$ *on the alphabet* Σ *is said to be* pure morphic *if it is an iterative fixed point of some prolongable morphism on* Σ.

A sequence $\mathbf{v} = (v_n)_{n \geq 0}$ *on the alphabet* Δ *is said to be* morphic *if it is a coding of a pure morphic sequence on some alphabet* Σ, *i.e., if there exist an alphabet* Σ, *a morphism from* Σ^* *to itself, a map* τ *from* Σ *to* Δ, *and an infinite sequence* $\mathbf{u} = (u_n)_{n \geq 0}$ *with values in* Σ *such that* $\mathbf{u}$ *is an iterative fixed point of* h *and for each* $n \geq 0$, $v_n = \tau(u_n)$. *If, furthermore, the morphism* h *has* constant length ℓ *(i.e., for each letter* $a \in \Sigma$, *the length, i.e., the number of letters, of* $h(a)$ *is equal to* ℓ*) the sequence* $\mathbf{v}$ *is said to be* ℓ-automatic.

The following result shows how to solve the Hanoi puzzle in the classical and cyclic cases by using morphic or automatic sequences. Let a, resp. b, c, be the move that takes the topmost disk from peg 1 and put it on top of peg 2, resp. from peg 2 to peg 3 and from peg 3 to peg 1 (see Figure 1), and let $\overline{a}, \overline{b}, \overline{c}$ be the corresponding reverse moves (e.g., $\overline{a}$ takes the topmost disk from peg 2 and put it on top of peg 1).

Theorem 1 **([4, 5]).**
(i) There exists a sequence $\mathbf{u} = (u_n)_{n \geq 0}$ *on the alphabet* $\{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$ *that is* 2*-automatic, and such that its prefixes of length* $2^N - 1$ *describe a minimal set of moves in the* classical *Hanoi puzzle to transfer* N *disks from peg* 1 *to peg* 2 *if* N *is odd and from peg* 1 *to peg* 3 *if* N

is even. The sequence $\mathbf{u}$ *is the iterative fixed point of the morphism of length* 2 *defined on the alphabet* $\{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$ *by*

$$a \to a\overline{c}, \ \ b \to c\overline{b}, \ \ c \to b\overline{a}$$
$$\overline{a} \to ac, \ \ \overline{b} \to cb, \ \ \overline{c} \to ba$$

(ii) There exists an infinite sequence on the alphabet $\{a, b, c\}$ *that is the common limit of the finite minimal sequences of moves given by Atkinson for the* cyclic *towers of Hanoi that permit to transfer* N *disks from peg* 1 *to peg* 2 *or from peg* 1 *to peg* 3*. Furthermore this sequence is morphic: it is the image under the map* $\varphi : \{f, g, h, u, v, w\} \to \{a, b, c\}$ *where* $\varphi(f) = \varphi(w) := a$*,* $\varphi(g) = \varphi(u) := c$*,* $\varphi(h) = \varphi(v) := b$ *of the iterative fixed point of the morphism* s *defined on* $\{f, g, h, u, v, w\}$ *by*

$$s(f) = fvf, \quad s(g) = gwg, \quad s(h) = huh,$$
$$s(u) = fg, \quad s(v) = gh, \quad s(w) = hf$$

3 3-peg towers of Hanoi with forbidden moves

3.1 Known results

All the possible restrictions of moves under which the three-peg Hanoi puzzle can be solved are given in [18]. Up to "isomorphism" there are only five cases: these cases are given in Figure 2:

- the *complete* puzzle uses all possible moves in $\{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$;
- the *three-in-a-row* puzzle ("lazy" puzzle) only uses the moves in $\{a, b, \overline{a}, \overline{b}\}$;
- the *cyclic* puzzle only uses the moves in $\{a, b, c\}$;
- the *complete−−* puzzle only uses the moves in $\{a, b, c, \overline{a}, \overline{b}\}$;
- the *cyclic++* puzzle only uses the moves in $\{a, b, c, \overline{a}\}$.

In [18] A. Sapir gives minimal recursive algorithms to solve these five Hanoi problems. Furthermore he shows that the number of moves for N disks has order of magnitude λ^N, where λ is respectively equal to 2, 3, $1 + \sqrt{3}$, $(1 + \sqrt{17})/2$, and the largest root of the polynomial $x^3 - x^2 - 4x + 2$ in each of the five cases above. We will show that these algorithms give more: they permit to construct infinite morphic sequences of moves from which the moves for N disks can be easily deduced for any N.

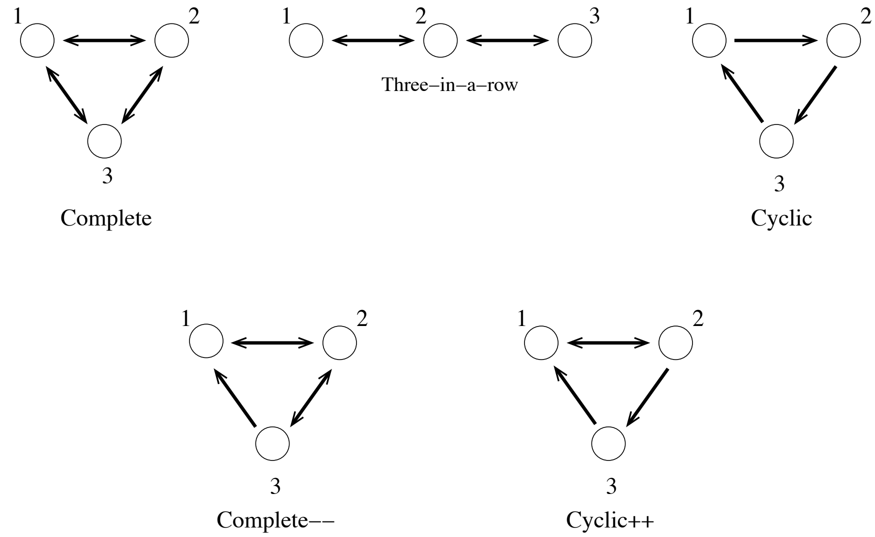

Fig. 2. The five 3-peg Hanoi puzzles

3.2 Morphic sequences for the restricted 3-peg Hanoi puzzle

We prove the following theorem.

Theorem 2. *1. The five* 3*-peg Hanoi puzzles with restricted moves give rise to infinite sequences, obtained as limits of the sequence of moves for N disks when N goes to infinity. Let us call these sequences the "complete Hanoi sequence", the "three-in-a-row Hanoi sequence", the "cyclic Hanoi sequence", the "complete−− Hanoi sequence", and the "cyclic++ Hanoi sequence".*

2. The five restricted Hanoi sequences are morphic. Furthermore the complete Hanoi sequence is 2*-automatic and the three-in-a-row Hanoi sequence is* 3*-automatic. The morphisms and codings can be given explicitly:*

– *the* complete Hanoi sequence *is the iterative fixed point of the morphism*

$$a \to a\overline{c},\ \ b \to c\overline{b},\ \ c \to b\overline{a}$$
$$\overline{a} \to ac,\ \ \overline{b} \to cb,\ \ \overline{c} \to ba$$

on the alphabet $\{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$*;*

– *the* three-in-a-row Hanoi sequence *is the iterative fixed point of the morphism*

$$a \to a\ b\ a,\quad \overline{a} \to a\ b\ \overline{a},\quad b \to \overline{b}\ \overline{a}\ b,\quad \overline{b} \to \overline{b}\ \overline{a}\ \overline{b}.$$

– *the* cyclic Hanoi sequence *can be obtained as the coding under the map* $\varphi : \{f, g, h, u, v, w\} \to \{a, b, c\}$ *where* $\varphi(f) = \varphi(w) := a$, $\varphi(g) = \varphi(u) := c$, $\varphi(h) = \varphi(v) := b$ *of the iterative fixed point of the morphism* s *defined on* $\{f, g, h, u, v, w\}$ *by*

$$\begin{array}{lll} s(f) = fvf, & s(g) = gwg, & s(h) = huh, \\ s(u) = fg, & s(v) = gh, & s(w) = hf \end{array};$$

– *the* complete−− Hanoi sequence *can be obtained as the coding under the map* $\varphi : \{x, s, z, d, e, f, a, b, c, \overline{a}, \overline{b}\} \to \{a, b, c, \overline{a}, \overline{b}\}$ *defined by*

$$\begin{array}{lll} \varphi(x) = a, & \varphi(s) = a, & \varphi(z) = \overline{a}, \\ \varphi(d) = b, & \varphi(e) = c, & \varphi(f) = \overline{b} \\ \varphi(a) = a, & \varphi(\overline{a}) = \overline{a}, & \\ \varphi(b) = b, & \varphi(\overline{b}) = \overline{b}, & \\ \varphi(c) = c & & \end{array}$$

of the iterative fixed point of the morphism θ *defined by*

$$\begin{array}{lll} \theta(x) = s\ b\ a\ f, & \theta(s) = s\ b\ a\ e\ b\ s, & \theta(z) = d\ \overline{a}\ e, \\ \theta(d) = z\ b\ s\ b, & \theta(e) = f\ c\ z, & \theta(f) = e\ \overline{b}\ x, \end{array}$$
$$\theta(a) = a,\quad \theta(b) = b,\quad \theta(\overline{a}) = \overline{a},\quad \theta(\overline{b}) = \overline{b},\quad \theta(c) = c.$$

– *the* cyclic++ Hanoi sequence *can be obtained as the coding under the map* $\psi : \{x, r, z, t, u, s, a, b, c, \overline{a}\} \to \{a, b, c, \overline{a}\}$ *defined by*

$$\begin{array}{lll} \psi(x) = a, & \psi(r) = a, & \psi(z) = \overline{a}, \\ \psi(t) = b, & \psi(u) = c, & \psi(s) = c \\ \psi(a) = a, & \psi(\overline{a}) = \overline{a}, & \\ \psi(b) = b, & \psi(\overline{b}) = \overline{b}, & \\ \psi(c) = c & & \end{array}$$

of the iterative fixed point of the morphism η *defined by*

$$\begin{array}{lll} \eta(x) = r\ b\ a\ s\ a, & \eta(r) = r\ b\ a\ u\ b\ r, & \eta(z) = t\ \overline{a}\ u, \\ \eta(t) = z\ b\ r\ b, & \eta(u) = s\ a\ c\ z\ c, & \eta(s) = s\ a\ c\ t\ a\ s, \end{array}$$
$$\eta(a) = a,\quad \eta(b) = b,\quad \eta(\overline{a}) = \overline{a},\quad \eta(\overline{b}) = \overline{b},\quad \eta(c) = c.$$

Proof (sketch).

Our proof will follow the lines of [5, 4, 1]. We define families of words that describe the restricted Hanoi algorithms. These words converge to the infinite Hanoi sequences. Since they are *locally catenative* (roughly speaking there exists a fixed δ such that the nth word depends on the words indexed by $n-1$, $n-2$, ..., $n-\delta$) we can construct morphisms and codings proving that the sequences are morphic. Note that a general result of Shallit (see [20]) proves that locally catenative sequences satisfying mild properties are morphic.

Proofs for complete and cyclic Hanoi sequences can be found in [4, 5] so that it suffices to address the three remaining cases. We are sketching the proofs for the three remaining cases inspired by [5].

– Consider the algorithm given in [18] for the three-in-a-row Hanoi puzzle. Let X_N be the sequence of moves on the alphabet $\{a, b, \overline{a}, \overline{b}\}$ that takes N disks from peg 1 to peg 2, and Y_N the sequence of moves that takes N disks from peg 1 to peg 3, in the minimal algorithm given in [18]. Then, defining the map g by $g(a) := \overline{b}$, $g(b) := \overline{a}$, $g(\overline{a}) := b$, $g(\overline{b}) := a$, a straightforward consequence of the algorithm in [18] is that

$$X_{N+1} = Y_N a g(X_N) \ \text{ and } Y_{N+1} = Y_N a g(Y_N) b Y_N$$

with $X_1 = a$, $Y_1 = ab$. We first note that the two sequences of words $(X_N)_{N\geq 1}$ and $(Y_N)_{N\geq 1}$ converge to a same infinite sequence (since Y_{N+1} begins with Y_N and X_{N+1} begins with Y_N). Let us denote by Y_∞ this common limit.

Let us define $E_N := Y_N a$, $F_N := Y_N \overline{a}$, $G_N := g(E_N)$, and $H_N := g(F_N)$. We see that $E_1 = a\,b\,a$, $F_1 = a\,b\,\overline{a}$, $G_1 = \overline{b}\,\overline{a}\,\overline{b}$, and $H_1 = \overline{b}\,\overline{a}\,b$. Furthermore

$$\begin{aligned} E_{N+1} &= E_N\ H_N\ E_N \\ F_{N+1} &= E_N\ H_N\ F_N \\ G_{N+1} &= G_N\ F_N\ G_N \\ H_{N+1} &= G_N\ F_N\ H_N \end{aligned}$$

and the sequence of words $(E_N)_{N\geq 1}$ clearly converges to Y_∞. Now let σ the morphism defined on the alphabet $\{a, b, \overline{a}, \overline{b}\}$ by

$$\begin{aligned}\sigma(a) &= a\ b\ a\\ \sigma(\overline{a}) &= a\ b\ \overline{a}\\ \sigma(b) &= \overline{b}\ \overline{a}\ b\\ \sigma(\overline{b}) &= \overline{b}\ \overline{a}\ \overline{b}.\end{aligned}$$

we easily see by induction on N that the following four relations simultaneously hold

$$\sigma(E_N) = E_{N+1},\ \ \sigma(F_N) = F_{N+1},\ \ \sigma(G_N) = G_{N+1},\ \ \sigma(H_N) = H_{N+1}.$$

Hence, for all $N \geq 1$, we have $\sigma^N(a) = E_N$. This implies that $Y_\infty = \sigma^\infty(a)$.

– Case of the complete−− Hanoi puzzle. The permitted moves are the elements of $\{a, b, c, \overline{a}, \overline{b}\}$. Let us define the following words on this alphabet:

X_N is the word given by the algorithm in [18] to transfer N disks from peg 1 to peg 2,

Y_N is the word given by the algorithm in [18] to transfer N disks from peg 1 to peg 3,

Z_N is the word given by the algorithm in [18] to transfer N disks from peg 2 to peg 1,

D_N is the word given by the algorithm in [18] to transfer N disks from peg 2 to peg 3,

E_N is the word given by the algorithm in [18] to transfer N disks from peg 3 to peg 1,

F_N is the word given by the algorithm in [18] to transfer N disks from peg 3 to peg 2.

A rephrasing of the algorithm given in [18] yields that:

$$\begin{aligned}X_{N+1} &= Y_N a F_N\\ Y_{N+1} &= Y_N a E_N b Y_N\\ Z_{N+1} &= D_N \overline{a} E_N\\ D_{N+1} &= Z_N b Y_N\\ E_{N+1} &= F_N c Z_N\\ F_{N+1} &= E_N \overline{b} X_N\end{aligned}$$

together with $X_1 = a$, $Y_1 = ab$, $Z_1 = \overline{a}$, $D_1 = b$, $E_1 = c$, and $F_1 = \overline{b}$. It is not hard to see that Y_N ends in b for any $N \geq 1$. We define S_N by $Y_N := S_N b$; hence $S_1 = a$. The relations above can then be written as

$$\begin{aligned} X_{N+1} &= S_N ba F_N \\ S_{N+1} &= S_N ba E_N b S_N \\ Z_{N+1} &= D_N \overline{a} E_N \\ D_{N+1} &= Z_N b S_N b \\ E_{N+1} &= F_N c Z_N \\ F_{N+1} &= E_N \overline{b} X_N \end{aligned}$$

In particular there exist infinite sequences X_∞, Z_∞ and E_∞ on $\{a, b, c, \overline{a}, \overline{b}\}$ such that

$$\begin{aligned} &\lim_{N\to\infty} X_N = \lim_{N\to\infty} Y_N = \lim_{N\to\infty} S_N = X_\infty \\ &\lim_{N\to\infty} Z_N = \lim_{N\to\infty} D_N = Z_\infty \\ &\lim_{N\to\infty} E_N = \lim_{N\to\infty} F_N = E_\infty . \end{aligned}$$

Now, taking the morphism θ and the map φ given in the statement of Theorem 2, we get by an easy simultaneous induction that $\varphi(\theta^N(w)) = W_N$ where w is any of the letters x, s, z, d, e, f and W is the corresponding capital letter in X, S, Z, D, E, F. In particular $\varphi(\theta^\infty(s)) = X_\infty$.

– Case of the cyclic++ Hanoi puzzle. The permitted moves are the elements of $\{a, b, c, \overline{a}\}$. Let us define the following words on this alphabet:

X_N is the word given by the algorithm in [18] to transfer N disks from peg 1 to peg 2,

Y_N is the word given by the algorithm in [18] to transfer N disks from peg 1 to peg 3,

Z_N is the word given by the algorithm in [18] to transfer N disks from peg 2 to peg 1,

T_N is the word given by the algorithm in [18] to transfer N disks from peg 2 to peg 3,

U_N is the word given by the algorithm in [18] to transfer N disks from peg 3 to peg 1,

V_N is the word given by the algorithm in [18] to transfer N disks from peg 3 to peg 2.

A rephrasing of the algorithm given in [18] yields that:

$$\begin{aligned}
X_{N+1} &= Y_N a V_N \\
Y_{N+1} &= Y_N a U_N b Y_N \\
Z_{N+1} &= T_N \overline{a} U_N \\
T_{N+1} &= Z_N b Y_N \\
U_{N+1} &= V_N c Z_N \\
V_{N+1} &= V_N c T_N a V_N
\end{aligned}$$

together with $X_1 = a$, $Y_1 = ab$, $Z_1 = \overline{a}$, $T_1 = b$, $U_1 = c$, and $V_1 = ca$. It is not hard to see that Y_N ends in b for any $N \geq 1$. We define R_N by $Y_N := R_N b$; hence $R_1 = a$. We also see that V_N ends in a for any $N \geq 1$. We define S_N by $V_N := S_N a$; hence $S_1 = c$. The relations above can then be written as

$$\begin{aligned}
X_{N+1} &= R_N b a S_N a \\
R_{N+1} &= R_N b a U_N b R_N \\
Z_{N+1} &= T_N \overline{a} U_N \\
T_{N+1} &= Z_N b R_N b \\
U_{N+1} &= S_N a c Z_N \\
S_{N+1} &= S_N a c T_N a S_N
\end{aligned}$$

In particular there exist infinite sequences X_∞, Z_∞ and S_∞ with values in the alphabet $\{a, b, c, \overline{a}\}$ such that

$$\begin{aligned}
\lim_{N\to\infty} X_N &= \lim_{N\to\infty} R_N = X_\infty \\
\lim_{N\to\infty} Z_N &= \lim_{N\to\infty} T_N = Z_\infty \\
\lim_{N\to\infty} U_N &= \lim_{N\to\infty} S_N = S_\infty.
\end{aligned}$$

Now, taking the morphism η and the map ψ given in the statement of Theorem 2, we get by an easy simultaneous induction that $\psi(\eta^N(w)) = W_N$ where w is any of the letters x, r, z, t, u, s and W is the corresponding capital letter in X, R, Z, T, U, S. In particular $\psi(\eta^\infty(r)) = X_\infty$.

4 Conclusion

Our Theorem 2 above somehow means that all restricted Hanoi puzzles belong to the same class of "regularity" formed by the morphic sequences. For example, computing the ith term of a morphic sequence can be done in time at most polynomial in $\log i$ (see [21] where more is proved). On the other hand automatic sequences form a (strict) subclass of morphic sequences and can be seen as "more regular". The computation of the ith term of an automatic sequence can be done in linear time in $\log i$ (this is a consequence of the possible computation by finite automata with output function, as proved in [10]). Note that a sequence can well be both morphic with a non-uniform morphism and d-automatic for some $d \geq 2$: for example taking the celebrated Thue-Morse sequence and counting the number of 1's between two consecutive zeros, one obtains the sequence

$$2\ 1\ 0\ 2\ 0\ 1\ 2\ 1\ 0\ 1\ 2\ 0\ 1\ 2\ 0\ \cdots$$

This sequence is both (see [8]) the iterative fixed point of the morphism

$$2 \to 210, \quad 1 \to 20, \quad 0 \to 1$$

and the image under the map $x \to x \bmod 3$ of the iterative fixed point of the 2-morphism

$$2 \to 21, \quad 1 \to 02, \quad 0 \to 04, \quad 4 \to 20.$$

What can be said about the infinite sequences associated with restricted Hanoi puzzles? We have seen that the complete (classical) and three-in-a-row Hanoi sequences are respectively 2-automatic and 3-automatic. We also know that the cyclic Hanoi sequence is not d-automatic for any d (see [1]). Proving that a given sequence is not d-automatic for any d is not an easy task and can be done by using various criteria (frequencies, repetitions, subsequences of certain types, etc.). We hope to finish up the proof that the only restricted Hanoi sequences that are d-automatic for some d are the complete and the three-in-a-row Hanoi sequences, by proving the non-automaticity of the cyclic++ and the complete−− Hanoi sequences. A hint is that the dominant eigenvalues of the transition matrices of the morphisms entering the picture are respectively $\lambda_1 := (1 + \sqrt{17})/2$ and λ_2 the maximal root of the polynomial $X^3 - X^2 - 4X + 2$. A conjecture of Hansel (see [11–13] for several results toward a proof) asserts that if the iterative fixed point of a morphism with dominant eigenvalue λ is also d-automatic and not ultimately periodic, then $(\log \lambda)/(\log d)$ is rational: this is a generalization of a theorem due to Cobham for automatic sequences [9]. Under this conjecture it would suffice to prove that $(\log \lambda_j)/(\log d)$ is irrational for $j = 1, 2$, and to prove that the images of the fixed points of the Hanoi morphisms with dominant eigenvalues λ_j under the corresponding codings are still non-automatic (in other words that these codings are not "trivial"). A small step is given in the proposition below.

Proposition 1. *Let* $\lambda_1 := (1 + \sqrt{17})/2$ *and* λ_2 *be the maximal root of the polynomial* $X^3 - X^2 - 4X + 2$. *Then, for any integer* $d \geq 1$, *the real numbers* $(\log \lambda_j)/(\log d)$ *with* $j = 1, 2$ *are irrational.*

Proof. It suffices to show that for any integer $n \geq 1$ the real numbers λ_j^n are not integers.

For λ_1 this is clear since for any $n \geq 1$ there exist integers $s_n > 0$ and $t_n > 0$ such that $2^n \lambda_1^n = (1 + \sqrt{17})^n = s_n + t_n\sqrt{17}$.

For λ_2: if α were an integer such that $\lambda_2^n = \alpha$, then the polynomial $X^3 - X^2 - 4X + 2$ would divide $X^n - \alpha$: there would exist a polynomial $Q(X)$ with integer coefficients such that

$$(X^3 - X^2 - 4X + 2)Q(X) = X^n - \alpha.$$

Reducing modulo 2 gives

$$X^2(X - 1)Q(X) \equiv X^n - \alpha \bmod 2$$

which would imply that α is both congruent to 0 and to 1 modulo 2, hence the desired contradiction.

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