

# Summation of Series Defined by Counting Blocks of Digits

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## Abstract

We discuss the summation of certain series defined by counting blocks of digits in the  $B$ -ary expansion of an integer. For example, if  $s_2(n)$  denotes the sum of the base-2 digits of  $n$ , we show that  $\sum_{n \geq 1} s_2(n)/(2n(2n+1)) = (\gamma + \log \frac{4}{\pi})/2$ . We recover this previous result of Sondow and provide several generalizations.

*MSC:* 11A63, 11Y60.

## 1 Introduction

A classical series with rational terms, known as Vacca's series [17] or in an equivalent integral form as Catalan's integral [7] (see also [6] and [16]), evaluates to Euler's constant  $\gamma$ :

$$\gamma = \sum_{n \geq 1} \frac{(-1)^n}{n} \left\lfloor \frac{\log n}{\log 2} \right\rfloor = \int_0^1 \frac{1}{1+x} \sum_{n \geq 1} x^{2^n - 1} dx.$$

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In a recent paper [15] Sondow gave the following two formulas:

$$\gamma^\pm = \sum_{n \geq 1} \frac{N_1(n) \pm N_0(n)}{2n(2n+1)}$$

where  $\gamma^+ = \gamma$  is the Euler constant,  $\gamma^- = \log \frac{4}{\pi}$  is the “alternating Euler constant” [14], and  $N_1(n)$  (resp.  $N_0(n)$ ) is the number of 1’s (resp. 0’s) in the binary expansion of the integer  $n$ . The series for  $\gamma^+ = \gamma$  is equivalent to Vacca’s. The formulas for  $\gamma^\pm$  show in particular that

$$\sum_{n \geq 1} \frac{s_2(n)}{2n(2n+1)} = \frac{\gamma + \log \frac{4}{\pi}}{2}$$

where  $s_2(n)$  is the sum of the binary digits of the integer  $n$ .

This last formula reminds us of one of the problems posed at the 1981 Putnam competition [9]: Determine whether or not

$$\exp \left( \sum_{n \geq 1} \frac{s_2(n)}{n(n+1)} \right)$$

is a rational number. In fact,  $\sum \frac{s_2(n)}{n(n+1)} = 2 \log 2$ . A generalization was proven by Shallit [13], where the base 2 is replaced by any integer base  $B \geq 2$ . A more general result, where the sum of digits is replaced by the function  $N_{w,B}(n)$ , which counts the number of occurrences of the block  $w$  in the  $B$ -ary expansion of the integer  $n$ , was given by Allouche and Shallit [2].

The purpose of the present paper is to show that the result of [15] cited above can be deduced from a general lemma in [2]. Furthermore, we sum the series

$$\sum_{n \geq 1} \frac{N_{w,2}(n)}{2n(2n+1)} \quad \text{and} \quad \sum_{n \geq 1} \frac{N_{w,2}(n)}{2n(2n+1)(2n+2)},$$

thus generalizing Corollary 1 in [15] and a series for Euler’s constant in [5, 1, 11] (dated February 1967, August 1967, February 1968), respectively. Finally, we indicate some generalizations of our results, including an extension to base  $B > 2$ , and a method for giving alternate proofs without using the general lemma from [2].

## 2 A general lemma

The first lemma in this section is taken from [2]; for completeness we recall the proof. We also give two classical results presented as lemmas, together with a new result (Lemma 4).

We start with some definitions. Let  $B \geq 2$  be an integer. Let  $w$  be a word on the alphabet of digits  $\{0, 1, \dots, B-1\}$  (that is,  $w$  is a finite block of digits). We denote by  $N_{w,B}(n)$  the number of (possibly overlapping) occurrences of  $w$  in the  $B$ -ary expansion of an integer  $n > 0$ , and we set  $N_{w,B}(0) = 0$ .

Given  $w$  as above, we denote by  $|w|$  the length of the word  $w$  (i.e., if  $w = d_1 d_2 \cdots d_k$ , then  $|w| = k$ ). Denote by  $w^j$  the concatenation of  $j$  copies of the word  $w$ .

Given  $w$  and  $B$  as above, we denote by  $v_B(w)$  the value of  $w$  when  $w$  is interpreted as the base  $B$ -expansion (possibly with leading 0's) of an integer.

**Remark 1** The occurrences of a given word in the  $B$ -ary expansion of the integer  $n$  may overlap. For example,  $N_{11,2}(7) = 2$ .

If the word  $w$  begins with 0, but  $v_B(w) \neq 0$ , then in computing  $N_{w,B}(n)$  we assume that the  $B$ -ary expansion of  $n$  starts with an arbitrarily long prefix of 0's. If  $v_B(w) = 0$  we use the usual  $B$ -ary expansion of  $n$  without leading zeros. For example,  $N_{011,2}(3) = 1$  (write 3 in base 2 as  $0 \cdots 011$ ) and  $N_{0,2}(2) = 1$ .

**Lemma 1 ([2])** Fix an integer  $B \geq 2$ , and let  $w$  be a non-empty word on the alphabet  $\{0, 1, \dots, B-1\}$ . If  $f : \mathbb{N} \rightarrow \mathbb{C}$  is a function with the property that  $\sum_{n \geq 1} |f(n)| \log n < \infty$ , then

$$\sum_{n \geq 1} N_{w,B}(n) \left( f(n) - \sum_{0 \leq j < B} f(Bn + j) \right) = \sum f(B^{|w|}n + v_B(w)),$$

where the last summation is over  $n \geq 1$  if  $w = 0^j$  for some  $j \geq 1$ , and over  $n \geq 0$  otherwise.

*Proof.* (See [2].) As  $N_{w,B}(n) \leq \lfloor \frac{\log n}{\log B} \rfloor + 1$ , all series  $\sum N_{w,B}(un + v)f(un + v)$ , where  $u$  and  $v$  are nonnegative integers, are absolutely convergent. Let  $\ell$  be the last digit of  $w$ , and let  $g := B^{|w|-1}$ . Then

$$\sum_{n \geq 0} N_{w,B}(n)f(Bn + \ell) = \sum_{0 \leq k < g} \sum_{n \geq 0} N_{w,B}(gn + k)f(Bgn + Bk + \ell)$$

and

$$\sum_{n \geq 0} N_{w,B}(Bn + \ell)f(Bn + \ell) = \sum_{0 \leq k < g} \sum_{n \geq 0} N_{w,B}(Bgn + Bk + \ell)f(Bgn + Bk + \ell).$$

Now, if either  $n \neq 0$  or  $v_B(w) \neq 0$ , then for  $k = 0, 1, \dots, g-1$  we have

$$N_{w,B}(Bgn + Bk + \ell) - N_{w,B}(gn + k) = \begin{cases} 1, & \text{if } k = \lfloor \frac{v_B(w)}{B} \rfloor; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, if  $n = 0$  and  $v_B(w) = 0$  (hence  $\ell = 0$ ), then the difference equals 0 for every  $k \in \{0, 1, \dots, g-1\}$ . Hence

$$\begin{aligned} \sum_{n \geq 0} N_{w,B}(Bn + \ell)f(Bn + \ell) - \sum_{n \geq 0} N_{w,B}(n)f(Bn + \ell) &= \sum f \left( Bgn + B \lfloor \frac{v_B(w)}{B} \rfloor + \ell \right) \\ &= \sum f(B^{|w|}n + v_B(w)) \quad (*), \end{aligned}$$

the last two summations being over  $n \geq 0$  if  $w$  is not of the form  $0^j$ , and over  $n \geq 1$  if  $w = 0^j$  for some  $j \geq 1$ . We then write

$$\begin{aligned} \sum_{n \geq 0} N_{w,B}(n) f(n) &= \sum_{0 \leq j < B} \sum_{n \geq 0} N_{w,B}(Bn + j) f(Bn + j) \\ &= \sum_{j \in [0, B) \setminus \{\ell\}} \sum_{n \geq 0} N_{w,B}(Bn + j) f(Bn + j) + \sum_{n \geq 0} N_{w,B}(Bn + \ell) f(Bn + \ell) \end{aligned}$$

which together with (\*) gives

$$\sum_{n \geq 0} N_{w,B}(n) \left( f(n) - \sum_{0 \leq j < B} f(Bn + j) \right) = \sum f(B^{|w|}n + v_B(w)).$$

Since  $N_{w,B}(0) = 0$ , the proof is complete.  $\square$

Now let  $\Gamma$  be the usual gamma function, let  $\Psi := \Gamma'/\Gamma$  be the logarithmic derivative of the gamma function, let  $\zeta(s)$  be the Riemann zeta function, let  $\zeta(s, x) := \sum_{n \geq 0} (n + x)^{-s}$  be the Hurwitz zeta function, and let  $\gamma$  denote Euler's constant.

**Lemma 2** *If  $a$  and  $b$  are positive real numbers, then*

$$\sum_{n \geq 1} \left( \frac{1}{an} - \frac{1}{an + b} \right) = \frac{1}{b} + \frac{\gamma + \Psi(b/a)}{a}.$$

*Proof.* We write

$$\begin{aligned} \sum_{n \geq 1} \left( \frac{1}{an} - \frac{1}{an + b} \right) &= \lim_{s \rightarrow 1+} \sum_{n \geq 1} \left( \frac{1}{(an)^s} - \frac{1}{(an + b)^s} \right) = \frac{1}{a} \lim_{s \rightarrow 1+} \sum_{n \geq 1} \left( \frac{1}{n^s} - \frac{1}{(n + \frac{b}{a})^s} \right) \\ &= \frac{1}{a} \lim_{s \rightarrow 1+} \left( \zeta(s) - \zeta\left(s, \frac{b}{a}\right) + \left(\frac{a}{b}\right)^s \right) \\ &= \frac{1}{b} + \frac{1}{a} \lim_{s \rightarrow 1+} \left( \left( \zeta(s) - \frac{1}{s-1} \right) - \left( \zeta\left(s, \frac{b}{a}\right) - \frac{1}{s-1} \right) \right) \\ &= \frac{1}{b} + \frac{1}{a} \left( \gamma + \frac{\Gamma'(b/a)}{\Gamma(b/a)} \right) = \frac{1}{b} + \frac{\gamma + \Psi(b/a)}{a} \end{aligned}$$

(see for example [18, p. 271]).  $\square$

**Lemma 3** *For  $x > 0$  we have*

$$\sum_{r \geq 1} \left( \frac{x}{r} - \log \left( 1 + \frac{x}{r} \right) \right) = \log x + \gamma x + \log \Gamma(x).$$

*Proof.* Take the logarithm of the Weierstraß product for  $1/\Gamma(x)$  (see, for example, [8, Section 1.1] or [18, Section 12.1]).  $\square$

The next lemma in this section is the last step before proving our theorems.

**Lemma 4** *Let  $a$  and  $b$  be positive real numbers. Then*

$$\sum_{n \geq 1} \left( \frac{1}{an} - \log \frac{an+1}{an} \right) = \log \Gamma \left( \frac{1}{a} \right) + \frac{\gamma}{a} - \log a$$

and

$$\sum_{n \geq 0} \left( \frac{1}{an+b} - \log \frac{an+b+1}{an+b} \right) = \log \Gamma \left( \frac{b+1}{a} \right) - \log \Gamma \left( \frac{b}{a} \right) - \frac{\Psi(b/a)}{a}.$$

*Proof.* The proof is straightforward. The first formula follows directly from Lemma 3. To prove the second, write the  $n$ th term of the series for  $n \geq 1$  as the following sum of  $n$ th terms of three absolutely convergent series:

$$\frac{1}{an+b} - \frac{1}{an} - \frac{b}{an} + \log \left( 1 + \frac{b}{an} \right) + \frac{b+1}{an} - \log \left( 1 + \frac{b+1}{an} \right);$$

then use Lemmas 2 and 3.  $\square$

### 3 Two theorems

In this section we give two theorems that are consequences of Lemma 1, and that generalize results in [15] and [5, 1, 11].

**Theorem 1** *Let  $w$  be a non-empty word on the alphabet  $\{0, 1\}$ , and let  $\Psi$  denote the logarithmic derivative of the Gamma function.*

(a) *If  $v_2(w) = 0$ , then*

$$\sum_{n \geq 1} \frac{N_{w,2}(n)}{2n(2n+1)} = \log \Gamma \left( \frac{1}{2^{|w|}} \right) + \frac{\gamma}{2^{|w|}} - |w| \log 2.$$

(b) *If  $v_2(w) \neq 0$ , then*

$$\sum_{n \geq 1} \frac{N_{w,2}(n)}{2n(2n+1)} = \log \Gamma \left( \frac{v_2(w)+1}{2^{|w|}} \right) - \log \Gamma \left( \frac{v_2(w)}{2^{|w|}} \right) - \frac{1}{2^{|w|}} \Psi \left( \frac{v_2(w)}{2^{|w|}} \right).$$

*Proof.* Let

$$A_n := \frac{1}{n} - \log \frac{n+1}{n}$$

for  $n \geq 1$ . Noting that  $A_n - A_{2n} - A_{2n+1} = \frac{1}{2n(2n+1)}$ , the theorem follows from Lemma 1 with  $B = 2$ , and  $f(n) := A_n$  for  $n \geq 1$ , together with Lemma 4.  $\square$

**Example 1** Taking  $w = 0$  and  $w = 1$ , and recalling that  $\Gamma(1/2) = \sqrt{\pi}$  and  $\Psi(1/2) = -\gamma - 2 \log 2$  by Gauß's theorem (see for example [8, p. 19] or [10, p. 94]), we get

$$\sum_{n \geq 1} \frac{N_{0,2}(n)}{2n(2n+1)} = \frac{1}{2} \log \pi + \frac{\gamma}{2} - \log 2$$

and

$$\sum_{n \geq 1} \frac{s_2(n)}{2n(2n+1)} = \sum_{n \geq 1} \frac{N_{1,2}(n)}{2n(2n+1)} = -\frac{1}{2} \log \pi + \frac{\gamma}{2} + \log 2.$$

These equalities imply the formulas in the Introduction:

$$\sum_{n \geq 1} \frac{N_{1,2}(n) \pm N_{0,2}(n)}{2n(2n+1)} = \gamma^\pm$$

where (following the notations of [15])  $\gamma^+ := \gamma$  and  $\gamma^- := \log \frac{4}{\pi}$ , which is Corollary 1 of [15].

**Remark 2** The formulas in Theorem 1 are analogous to those in [2, p. 25]. The analogy becomes more striking if one uses Gauß's theorem to write all expressions of the form  $\Psi(x)$ , with  $x$  a rational number in  $(0, 1]$ , using only trigonometric functions, logarithms, and Euler's constant.

**Theorem 2** Let  $w$  be a non-empty word on the alphabet  $\{0, 1\}$ .

(a) If  $v_2(w) = 0$ , then

$$\sum_{n \geq 1} \frac{N_{w,2}(n)}{2n(2n+1)(2n+2)} = \log \Gamma \left( \frac{1}{2^{|w|}} \right) + \frac{\gamma}{2^{|w|+1}} - |w| \log 2 - \frac{1}{2^{|w|+1}} \Psi \left( \frac{1}{2^{|w|}} \right) - \frac{1}{2}.$$

(b) If  $v_2(w) \neq 0$ , then

$$\begin{aligned} \sum_{n \geq 1} \frac{N_{w,2}(n)}{2n(2n+1)(2n+2)} &= \log \Gamma \left( \frac{v_2(w)+1}{2^{|w|}} \right) - \log \Gamma \left( \frac{v_2(w)}{2^{|w|}} \right) \\ &\quad - \frac{1}{2^{|w|+1}} \left( \Psi \left( \frac{v_2(w)}{2^{|w|}} \right) + \Psi \left( \frac{v_2(w)+1}{2^{|w|}} \right) \right). \end{aligned}$$

*Proof.* Noting that  $\frac{1}{2n(2n+1)} - \frac{1}{4} \cdot \frac{1}{n(n+1)} = \frac{1}{2n(2n+1)(2n+2)}$ , it suffices to use Theorem 1 and the following result, deduced from [2, top of p. 26] in the case  $B = 2$ .

(a) If  $v_2(w) = 0$ , then

$$\sum_{n \geq 1} \frac{N_{w,2}(n)}{n(n+1)} = \frac{1}{2^{|w|-1}} \left( \Psi \left( \frac{1}{2^{|w|}} \right) + \gamma + 2^{|w|} \right).$$

(b) If  $v_2(w) \neq 0$ , then

$$\sum_{n \geq 1} \frac{N_{w,2}(n)}{n(n+1)} = \frac{1}{2^{|w|-1}} \left( \Psi \left( \frac{v_2(w)+1}{2^{|w|}} \right) - \Psi \left( \frac{v_2(w)}{2^{|w|}} \right) \right).$$

□

**Example 2** Taking  $w = 0$  and  $w = 1$ , we get

$$\sum_{n \geq 1} \frac{N_{0,2}(n)}{2n(2n+1)(2n+2)} = \frac{1}{2} \log \pi + \frac{\gamma}{2} - \frac{1}{2} \log 2 - \frac{1}{2}$$

and

$$\sum_{n \geq 1} \frac{s_2(n)}{2n(2n+1)(2n+2)} = \sum_{n \geq 1} \frac{N_{1,2}(n)}{2n(2n+1)(2n+2)} = -\frac{1}{2} \log \pi + \frac{\gamma}{2} + \frac{1}{2} \log 2.$$

Hence

$$\sum_{n \geq 1} \frac{N_{1,2}(n) \pm N_{0,2}(n)}{2n(2n+1)(2n+2)} = \delta^\pm$$

where  $\delta^+ := \gamma - \frac{1}{2}$  and  $\delta^- := \frac{1}{2} - \log \frac{\pi}{2}$ , which are respectively a formula given in [5, 1, 11] and a seemingly new companion formula.

**Remark 3** As mentioned, all expressions of the form  $\Psi(x)$ , with  $x$  a rational number in  $(0, 1]$ , can be written using only trigonometric functions, logarithms, and Euler's constant.

## 4 Generalizations

Several extensions or generalizations of our results are possible. We give some of them in this section.

### 4.1 Variation on $A_n$

Instead of applying Lemma 1 with  $f(n) = A_n = \frac{1}{n} - \log \frac{n+1}{n}$  for  $n \geq 1$ , we could replace  $A_n$  with

$$A_n^{(k)} := \frac{1}{n+k} - \log \frac{n+1}{n}$$

for  $n \geq 1$ , where  $k$  is a nonnegative integer. Defining the (rational) function  $Q^{(k)}$  by

$$Q^{(k)}(n) := A_n^{(k)} - A_{2n}^{(k)} - A_{2n+1}^{(k)}$$

and noting that summing  $\sum_{n \geq 1} A_{an+b}^{(k)}$  boils down to summing  $\sum_{n \geq 1} \left( \frac{1}{an+b+k} - \frac{1}{an+b} \right)$ , which as in the proof of Lemma 2 involves the Hurwitz zeta function, we obtain explicit formulas for the sum of the series  $\sum_{n \geq 1} N_{w,2}(n) Q^{(k)}(n)$ .

### 4.2 Extension to base $B > 2$

Lemma 1 has been used above only for base  $B = 2$ . There are applications to other bases in [2]. We also note that the relation among the  $A_n$ 's,

$$A_n = \frac{1}{2n(2n+1)} + A_{2n} + A_{2n+1}$$

for  $n \geq 1$ , can be generalized to base  $B$ . Namely,

$$A_n = Q(n, B) + R(n, B)$$

where

$$Q(n, B) := \frac{1}{Bn(Bn+1)} + \frac{2}{Bn(Bn+2)} + \cdots + \frac{B-1}{Bn(Bn+B-1)}$$

and

$$R(n, B) := A_{Bn} + A_{Bn+1} + \cdots + A_{Bn+B-1}.$$

This allows us to use Lemmas 1 and 4 to sum, for example, the series

$$\sum_{n \geq 1} N_{w,3}(n) \frac{9n+4}{3n(3n+1)(3n+2)},$$

since

$$Q(n, 3) = A_n - A_{3n} - A_{3n+1} - A_{3n+2} = \frac{9n+4}{3n(3n+1)(3n+2)}.$$

### 4.3 Weighted $A_n$ 's

In this section we consider a weighted form of the  $A_n$ 's. First we need to study a relation between sequences of real numbers.

**Lemma 5** *Let  $(r_n)_{n \geq 1}$  and  $(R_i)_{i \geq 1}$  be sequences of real numbers. Set  $r_0 := 0$  and  $R_0 := 0$ . Then the following two properties are equivalent:*

(1) for  $i \geq 1$

$$R_i = \sum_{k \geq 0} r_{\lfloor \frac{i}{2^k} \rfloor} = r_i + r_{\lfloor \frac{i}{2} \rfloor} + r_{\lfloor \frac{i}{4} \rfloor} + \cdots$$

(note that this is actually a finite sum);

(2) for  $n \geq 1$

$$r_n = R_n - R_{\lfloor \frac{n}{2} \rfloor}.$$

*Proof.* The implication (1)  $\Rightarrow$  (2) is easily seen by considering the cases  $n$  even and  $n$  odd. Likewise, for (2)  $\Rightarrow$  (1) take  $i$  even and  $i$  odd.  $\square$

**Remark 4** See [4, Theorem 9] for more about this relation.

**Theorem 3** *Assume that  $r_1, r_2, \dots$  and  $R_1, R_2, \dots$  are real numbers related as in Lemma 5. Then the series  $\sum |r_n|n^{-2}$  converges if and only if the series  $\sum |R_i|i^{-2}$  converges, and in this case we have*

$$S := \sum_{n \geq 1} r_n \left( \frac{1}{n} - \log \frac{n+1}{n} \right) = \sum_{i \geq 1} \frac{R_i}{2i(2i+1)}$$

*Proof.* First note that if the series  $\sum |R_i| i^{-2}$  converges, then so does the series  $\sum |r_n| n^{-2}$ : use the expression for  $r_n$  in terms of the  $R_i$ 's in Lemma 5. Now suppose that the series  $\sum |r_n| n^{-2}$  converges. As before, let  $A_n := \frac{1}{n} - \log \frac{n+1}{n}$ . Then  $0 < A_n < \frac{1}{n} - \frac{1}{n+1}$ . This implies that the series  $S := \sum r_n \left( \frac{1}{n} - \log \frac{n+1}{n} \right)$  is absolutely convergent. Now

$$A_n = \frac{1}{2n(2n+1)} + A_{2n} + A_{2n+1}$$

implies

$$A_n = \frac{1}{2n(2n+1)} + \frac{1}{4n(4n+1)} + \frac{1}{(4n+2)(4n+3)} + A_{4n} + A_{4n+1} + A_{4n+2} + A_{4n+3}.$$

Hence, repeating  $K$  times,

$$A_n = \sum_{1 \leq k \leq K} \sum_{0 \leq m < 2^{k-1}} \frac{1}{(2^k n + 2m)(2^k n + 2m + 1)} + \sum_{0 \leq q < 2^K} A_{2^K n + q}.$$

Using the bounds  $0 < A_n < \frac{1}{n} - \frac{1}{n+1}$  and telescoping, the last sum is less than  $2^{-K}$ . Letting  $K$  tend to infinity, we obtain the (rapidly convergent) series

$$A_n = \sum_{k \geq 1} \sum_{0 \leq m < 2^{k-1}} \frac{1}{(2^k n + 2m)(2^k n + 2m + 1)}.$$

Substituting into the sum defining  $S$  yields the double series

$$S = \sum_{n \geq 1} \sum_{k \geq 1} \sum_{0 \leq m < 2^{k-1}} \frac{r_n}{(2^k n + 2m)(2^k n + 2m + 1)},$$

which converges absolutely. Thus we may collect terms with the same denominator, and we arrive at the series

$$S = \sum_{i \geq 1} \frac{R'_i}{2i(2i+1)},$$

where

$$R'_i := \sum_{n \in \mathcal{E}_i} r_n$$

with  $\mathcal{E}_i := \{n \geq 1, \exists k \geq 1, \exists m \in [0, 2^{k-1}), 2^{k-1}n + m = i\}$ . On the one hand, this proves that the series  $\sum \frac{R'_i}{2i(2i+1)}$  is absolutely convergent (hence the series  $\sum |R'_i| i^{-2}$  is convergent). On the other hand,  $R'_i$  can also be written as

$$R'_i := \sum_{1 \leq k \leq \frac{\log i}{\log 2} + 1} r_{\lfloor \frac{i}{2^{k-1}} \rfloor} = \sum_{k \geq 0} r_{\lfloor \frac{i}{2^k} \rfloor}$$

(recall that we have set  $r_0 := 0$ ). Thus  $R'_i = R_i$  by the hypothesis, and the proof is complete.  $\square$

**Example 3** Theorem 3 yields in particular the series for  $\gamma$  and  $\log \frac{4}{\pi}$  in the Introduction, in Example 1, and in [15, Corollary 1]. Namely,

If  $r_1 = r_2 = \dots = 1$ , then the series defining  $S$  sums to  $\gamma$  from Lemma 4, and the formula defining  $R'_i = R_i$  reduces to  $R_i = \lfloor \frac{\log 2^i}{\log 2} \rfloor = N_{1,2}(i) + N_{0,2}(i)$ .

If  $r_n = (-1)^{n-1}$ , then  $S = \log \frac{4}{\pi}$  (see [14] or decompose  $S$  into  $\sum$  (odd terms)  $- \sum$  (even terms) and apply Lemma 4), and the formula defining  $R'_i = R_i$  implies  $R_i = N_{1,2}(i) - N_{0,2}(i)$ . To see this equality, first note that if it holds for  $i \geq 1$ , then using Lemma 5 and looking at the cases  $n$  odd and  $n$  even,

$$r_n = R_n - R_{\lfloor \frac{n}{2} \rfloor} = (-1)^{n-1}$$

for  $n \geq 1$  (compare [15, Lemma 2]). Now recall that properties (1) and (2) in Lemma 5 are equivalent.

**Remark 5** Example 3 shows that it is possible to deduce the formula

$$\sum_{n \geq 1} \frac{N_{1,2}(n) - N_{0,2}(n)}{2n(2n+1)} = \log \frac{4}{\pi}$$

from Theorem 3 and Lemma 4 without using Lemma 1: this yields a proof of the formula that is different from those in [15] and Example 1. Similar reasoning applies for any ultimately periodic sequence  $(r_n)_{n \geq 1}$ . In particular, it is not hard to see that the relations giving  $r_{2n}$  and  $r_{2n+1}$  in terms of the  $R_i$ 's imply that the sequence  $(r_n)_{n \geq 1}$  is periodic whenever  $R_i = N_{w,2}(i)$  for some fixed  $w$  and for every  $i \geq 1$ . Hence Theorem 1 can be deduced from Theorem 3 and Lemma 4 (along with the method for decomposing series employed in Example 3), without using Lemma 1. In the same vein, the generalization in Section 4.2 can be proved using a generalization of Theorem 3 to base  $B$  together with Lemma 4.

## 5 Future directions

Lemma 1 is the main tool for summing series in [2] and in the present paper. It might be possible to use the lemma to obtain the base  $B$  accelerated series for Euler's constant in [15, Theorem 2], and to sum more general series with  $N_{w,B}(n)$ . On the other hand, it might also be possible to extend the results of [2] and the present paper, and sum series where  $(N_{w,B}(n))_{n \geq 1}$  is replaced by a more general integer sequence  $(a_n)_{n \geq 1}$ , using the decomposition in [12] of a sequence  $(a_n)_{n \geq 1}$  into a (possibly infinite) linear combination of block-counting sequences  $(N_{w,B}(n))_{n \geq 1}$  (see also [3]). Of course, since this may replace a series with an infinite sum, for the method to work the new series must be summable in closed form.

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