Note on an integral of Ramanujan

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Abstract

We answer a question of Berndt and Bowman, asking whether it is possible to deduce the value of the Ramanujan integral I from the value of the Ramanujan integral J, where

$$I := \int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r} \right) dx \quad (= \psi(q/r) - \psi(p) + \log r)$$

and

$$J := \int_0^\infty \frac{(1+ax)^{-p} - (1+bx)^{-q}}{x} dx \quad (= \psi(q) - \psi(p) + \log \frac{b}{a}).$$

We also show that the second integral can be deduced from a classical expression of the ψ function due to Dirichlet and from the classical equality

$$\int_0^\infty (e^{-ax} - e^{-bx}) \frac{dx}{x} = \log \frac{b}{a},$$

which is a simple consequence of Frullani-Cauchy's theorem.

1 Introduction

In the recent paper [2], B. C. Berndt and D. C. Bowman prove the following proposition for an integral that can be found in a short unpublished partial manuscript of Ramanujan that was photocopied with his lost notebook (see [6, pp. 274-275]).

Proposition 1 (Ramanujan) *Let p, q, r be positive real numbers. Then*

$$I := \int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r} \right) dx = \psi(q/r) - \psi(p) + \log r$$

where ψ is the function defined from the Gamma function Γ by $\psi(x) := \Gamma'(x)/\Gamma(x)$.

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Berndt and Bowman also give a special instance of a theorem of Ramanujan [1, p. 314] that generalizes Frullani's theorem, namely: for $a, b, p, q > 0$,

$$\int_0^\infty \frac{(1+ax)^{-p} - (1+bx)^{-q}}{x} dx = \psi(q) - \psi(p) + \log \frac{b}{a}. \tag{1}$$

The two authors then ask whether Proposition 1 can be deduced from Equation (1). We show here that this is indeed the case. We also show that Equation (1) can be deduced from a classical result on the ψ function due to Dirichlet (see [7, p. 247]).

2 Two propositions

Proposition 2 *Equation (1) directly implies Proposition 1.*

Proof. In the integral I, making the change of variable $x = 1/t$ yields

$$I = \int_1^\infty \left(\frac{t^{-p}}{t-1} - r\frac{t^{r-1-q}}{t^r - 1} \right) dt = \lim_{\varepsilon \to 0+} \left(\int_{1+\varepsilon}^\infty \frac{t^{-p}}{t-1} - \int_{1+\varepsilon}^\infty r \frac{t^{r-1-q}}{t^r-1} \right) dt.$$

Putting $t - 1 = z$ in the first integral and $t^r - 1 = z$ in the second one, we get

$$I = \lim_{\varepsilon \to 0+} \left(\int_\varepsilon^\infty \frac{(1+z)^{-p}}{z} - \int_{(1+\varepsilon)^r - 1}^\infty \frac{(1+z)^{-q/r}}{z} \right) dz.$$

Hence

$$I = \lim_{\varepsilon \to 0+} \left(\int_\varepsilon^\infty \left(\frac{(1+z)^{-p} - (1+z)^{-q/r}}{z} \right) dz + \int_\varepsilon^{(1+\varepsilon)^r - 1} \frac{(1+z)^{-q/r}}{z} dz \right)$$

i.e.,

$$I = \int_0^\infty \left(\frac{(1+z)^{-p} - (1+z)^{-q/r}}{z} \right) dz + \lim_{\varepsilon \to 0+} \int_\varepsilon^{(1+\varepsilon)^r - 1} \frac{(1+z)^{-q/r}}{z} dz. \tag{2}$$

But, using the mean value theorem for integrals (see for example [3, p. 108]), the remaining limit in (2) can be written, for some ξ in the interval $(\varepsilon, (1+\varepsilon)^r - 1)$, as

$$\lim_{\varepsilon \to 0+} (1+\xi)^{-q/r} \int_\varepsilon^{(1+\varepsilon)^r - 1} \frac{dz}{z} = \lim_{\varepsilon \to 0+} (1+\xi)^{-q/r} \log \frac{(1+\varepsilon)^r - 1}{\varepsilon} = \log r. \tag{3}$$

Using (3) in (2) and then using (1), we find that

$$I = \int_0^\infty \left(\frac{(1+z)^{-p} - (1+z)^{-q/r}}{z} \right) dz + \log r = \psi(q/r) - \psi(p) + \log r.$$

□

We now prove that Equation (1) can be deduced from a classical integral form of the ψ function due to Dirichlet (see [7, p. 247] and [4, p. 943]).

Proposition 3 *Equation (1) can be deduced from the formula*

$$\psi(z) = \int_0^\infty (e^{-t} - (1+t)^{-z}) \frac{dt}{t}.$$

Proof. Letting

$$J := \int_0^\infty \frac{(1+ax)^{-p} - (1+bx)^{-q}}{x} dx$$

we write

$$J = \int_0^\infty (e^{-bx} - (1+bx)^{-q}) \frac{dx}{x} - \int_0^\infty (e^{-ax} - (1+ax)^{-p}) \frac{dx}{x} + \int_0^\infty (e^{-ax} - e^{-bx}) \frac{dx}{x}.$$

Putting $bx = t$ in the first integral and $ax = t$ in the second integral yields

$$\begin{aligned} J &= \int_0^\infty (e^{-t} - (1+t)^{-q}) \frac{dt}{t} - \int_0^\infty (e^{-t} - (1+t)^{-p}) \frac{dt}{t} + \int_0^\infty (e^{-ax} - e^{-bx}) \frac{dx}{x} \\ &= \psi(q) - \psi(p) + \int_0^\infty (e^{-ax} - e^{-bx}) \frac{dx}{x}. \end{aligned}$$

By a simple application of the Cauchy-Frullani theorem (see for example [5, p. 57]), this last integral has the value $\log b/a$ (proving this formula directly is actually a well-known exercise).

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