

## A NOTE ON UNIVOQUE SELF-STURMIAN NUMBERS \*

JEAN-PAUL ALLOUCHE<sup>1</sup>

**Abstract.** We compare two sets of (infinite) binary sequences whose suffixes satisfy extremal conditions: one occurs when studying iterations of unimodal continuous maps from the unit interval into itself, but it also characterizes univoque real numbers; the other is a disguised version of the set of characteristic Sturmian sequences. As a corollary to our study we obtain that a real number  $\beta$  in  $(1, 2)$  is univoque and self-Sturmian if and only if the  $\beta$ -expansion of 1 is of the form  $1v$ , where  $v$  is a characteristic Sturmian sequence beginning itself in 1.

**Résumé.** Nous comparons deux ensembles de suites infinies dont les suffixes satisfont à des conditions extrémales : le premier de ces ensembles est lié à l'itération des fonctions continues unimodales de l'intervalle unité dans lui-même, mais il caractérise aussi les nombres réels univoques ; le second est un déguisement de l'ensemble des suites sturmiennes caractéristiques. Un corollaire de notre étude est qu'un nombre  $\beta$  dans l'intervalle ouvert  $]1, 2[$  est univoque et auto-sturmien si et seulement si le  $\beta$ -développement de 1 est de la forme  $1v$ , où  $v$  est une suite sturmienne caractéristique commençant elle-même par 1.

**1991 Mathematics Subject Classification.** 11A63, 68R15.

### 1. INTRODUCTION

The kneading sequences of a unimodal continuous map  $f$  from  $[0, 1]$  into itself, with  $f(1) = 0$  and  $\sup f = 1$ , are classically studied by first looking at the combinatorial properties of the kneading sequence of 1. Cosnard proved that, using a simple bijection on binary sequences (namely mapping the sequence  $(x_n)_{n \geq 0}$  to

---

*Keywords and phrases:* Sturmian sequences, univoque numbers, self-Sturmian numbers.

\* supported by MENESR, ACI NIM 154 Numération.

<sup>1</sup> CNRS, LRI, UMR 8623, Université Paris Sud, Bâtiment 490. F-91405 Orsay Cedex (France)  
email: [allouche@lri.fr](mailto:allouche@lri.fr)

$(y_n)_{n \geq 0}$ , where  $y_n := \sum_{0 \leq j \leq n} x_j \pmod{2}$ , the set of kneading sequences of 1 for all maps  $f$  as above, maps to the set  $\Gamma$  defined by

$$\Gamma := \{u = (u_n)_{n \geq 0} \in \{0, 1\}^{\mathbb{N}}, \forall k \geq 0, \bar{u} \leq S^k u \leq u\}$$

where  $\bar{u} = (\bar{u}_n)_{n \geq 0}$  is the sequence defined by  $\bar{u}_n := 1 - u_n$ , where  $S^k$  is the  $k$ th iterate of the shift (i.e.,  $S^k((u_n)_{n \geq 0}) := (u_{n+k})_{n \geq 0}$ ), and where  $\leq$  is the lexicographical order on sequences induced by  $0 < 1$ . See [2, 10], where the relevant set is actually  $\Gamma \setminus \{(10)^\infty\}$ . See also [1] for a detailed combinatorial study of the set  $\Gamma$ .

A slight modification of the set  $\Gamma$  describes the expansions of 1 in the bases  $\beta$ , where  $\beta$  runs through the univoque numbers belonging to  $(1, 2)$ . Recall that a number  $\beta$  is called *univoque* if 1 admits only one expansion in base  $\beta$ . More precisely, the set of expansions of 1 in all the univoque bases  $\beta \in (1, 2)$  is the set

$$\Gamma_1 := \{u = (u_n)_{n \geq 0} \in \{0, 1\}^{\mathbb{N}}, \forall k \geq 0, \bar{u} < S^k u < u\}$$

(see [11, Remark 1, page 379], see also [4] and the bibliography therein).

**Remark 1.1.** Note that a binary sequence belongs to  $\Gamma_1$  if and only if it belongs to  $\Gamma$  and is not purely periodic.

Other sequences can be defined by extremal properties of their suffixes: characteristic Sturmian sequences and Sturmian sequences. More precisely the following results can be found in several papers (see in particular [7, 8, 14–16]; see also the survey [6] and the discussion therein).

*A binary sequence  $u = (u_n)_{n \geq 0}$  is characteristic Sturmian if and only if it is not periodic and belongs to the set  $\Xi$  defined by*

$$\Xi := \{u = (u_n)_{n \geq 0} \in \{0, 1\}^{\mathbb{N}}, \forall k \geq 0, 0u \leq S^k u \leq 1u\}.$$

*A binary sequence  $u = (u_n)_{n \geq 0}$  is Sturmian if and only if it is not periodic and there exists a binary sequence  $v = (v_n)_{n \geq 0}$  such that  $u$  belongs to  $\Xi_v$ , where*

$$\Xi_v := \{u = (u_n)_{n \geq 0} \in \{0, 1\}^{\mathbb{N}}, \forall k \geq 0, 0v \leq S^k u \leq 1v\}.$$

*The sequence  $v$  has the property that  $1v = \sup_k S^k u$  and  $0v = \inf_k S^k u$ . This is the characteristic Sturmian sequence having the same slope as  $u$ .*

**Remark 1.2.** The reader can find the essentials on Sturmian sequences in [13, Chapter 2]. A hint for the proof of the two assertions above is that a sequence is Sturmian if and only if it is not periodic and for any binary (finite) word  $w$ , the words  $0w0$  and  $1w1$  cannot be simultaneously factors of the sequence. Furthermore a sequence  $u$  is characteristic Sturmian if and only if  $0u$  and  $1u$  are both Sturmian.

## 2. COMPARING THE SETS $\Gamma$ AND $\Xi$

The analogy between the definitions of  $\Gamma$  and  $\Xi$  suggests the natural question whether any sequence can belong to their intersection. The disappointing answer is the following proposition.

**Proposition 2.1.** *A sequence  $u \in \{0, 1\}^{\mathbb{N}}$  belongs to  $\Gamma \cap \Xi$  if and only if it is equal to  $1^\infty$  or there exists  $j \geq 1$  such that  $u = (1^j 0)^\infty$ .*

*Proof.* If the sequence  $u$  belongs to  $\Gamma$ , we have in particular  $u \geq \bar{u}$ . Hence  $u = 1w$  for some binary sequence  $w$ . If  $u$  is not equal to  $1^\infty$  (which clearly belongs to  $\Gamma \cap \Xi$ ), let us write  $u = 1^j 0z$  for some integer  $j \geq 1$  and some binary sequence  $z$ . Since  $u$  belongs to  $\Gamma$  we have  $S^{j+1}u \leq u$ , i.e.,  $z \leq u$ . Now  $u$  belongs to  $\Xi$ , thus  $S^j u \geq 0u$ , i.e.,  $0z \geq 0u$ , hence  $z \geq u$ . This gives  $z = u$ . Hence  $u = (1^j 0)^\infty$ , which in turn clearly belongs to  $\Gamma \cap \Xi$ .  $\square$

The next question is whether a Sturmian sequence can belong to  $\Gamma$ . The answer is more interesting.

**Proposition 2.2.** *A (binary) Sturmian sequence  $u$  belongs to  $\Gamma$  if and only if there exists a characteristic Sturmian sequence  $v$  such that  $v$  begins in 1 and  $u = 1v$ .*

*Proof.* Let us first suppose that the Sturmian sequence  $u$  belongs to  $\Gamma$ . As above, since  $u$  belongs to  $\Gamma$ ,  $u$  begins in 1. Hence  $u = 1w$  for some binary sequence  $w$ . The inequalities  $S^k u \leq u$  for all  $k \geq 0$  imply that  $\sup_k S^k u = u$  (the inequality  $\geq$  is trivial since  $S^0 u = u$ ). This can be written  $\sup_k S^k u = 1w$ . On the other hand  $u$  is Sturmian, hence there exists a characteristic Sturmian sequence  $v$  such that  $u$  belongs to  $\Xi_v$ . We also know that  $v$  is such that  $1v = \sup_{k \geq 0} S^k u$ . Hence  $v = w$ . Now  $\inf_k S^k u = 0v = 0w$ . But  $S^k u \geq \bar{u}$  for all  $k \geq 0$ , since  $u$  belongs to  $\Gamma$ . Hence  $0v = 0w \geq \bar{u} = 0\bar{w}$ , thus  $w \geq \bar{w}$ , hence  $w$  begins in 1.

If, conversely,  $u = 1v$  where  $v$  is a characteristic Sturmian sequence (which actually implies that  $u$  is Sturmian) beginning in 1, we first note that  $0v \leq S^k v \leq 1v$  for all  $k \geq 0$ . Hence, immediately,  $S^k u \leq 1v = u$  for all  $k \geq 0$  (using that  $S^{k+1} u = S^k v$  and that  $S^0 u = u = 1v$ ). And also  $S^k u \geq 0v \geq 0\bar{v} = \bar{u}$  (using furthermore that  $v \geq \bar{v}$  since  $v$  begins in 1, and that  $S^0 u = u \geq \bar{u}$  since  $u$  begins in 1).  $\square$

**Remark 2.3.** We see in particular that a Sturmian sequence belonging to  $\Gamma$  must begin in 11. This is not surprising since the only sequence belonging to  $\Gamma$  that begins in 10 is  $(10)^\infty$ . This claim is a particular case of a lemma in [1]: *if a sequence  $t$  belonging to  $\Gamma$  begins with  $m\bar{m}$ , where  $m$  is a (finite) nonempty binary word, then  $t = (m\bar{m})^\infty$ .*

## 3. UNIVOQUE SELF-STURMIAN NUMBERS

Several papers were devoted to univoque numbers having an extra property. For example:

- the smallest univoque number in  $(1, 2)$  is determined in [12]; it is related to the celebrated Thue-Morse sequence and was proven transcendental in [3];
- univoque Pisot numbers belonging to  $(1, 2)$  are studied in [5].

The notion of self-Sturmian numbers was introduced in [9]. These are the real numbers  $\beta$  such that the greedy  $\beta$ -expansion of 1 is a Sturmian sequence on some two-digit alphabet. It is tempting to ask which univoque numbers are self-Sturmian. We restrict the study to the numbers in  $(1, 2)$  for simplicity.

**Proposition 3.1.** *The real self-Sturmian numbers in  $(1, 2)$  that are univoque are exactly the real numbers  $\beta$  such that  $1 = \sum_{n \geq 1} \frac{u_n}{\beta^n}$ , where  $u = (u_n)_{n \geq 0}$  is a binary sequence of the form  $u = 1v$ , with  $v$  a characteristic Sturmian sequence beginning in 1.*

*Proof.* This is a rephrasing of Proposition 2.2. □

**Remark 3.2.** - The equality  $1 = \sum_{n \geq 1} \frac{u_n}{\beta^n}$ , where  $u = (u_n)_{n \geq 0}$  is a binary sequence, uniquely determines the real number  $\beta$  in  $(1, 2)$ .

- Self-Sturmian numbers correspond to Sturmian sequences of the form  $u = 1v$ , where  $v$  is any characteristic Sturmian sequence (see [9, Remark p. 399]). All self-Sturmian numbers are transcendental [9].

The author wants to thank Amy Glen for her comments on a previous version of this note.

## REFERENCES

- [1] J.-P. Allouche, *Théorie des nombres et automates*, Thèse d'État, Université Bordeaux I, 1983.
- [2] J.-P. Allouche, M. Cosnard, Itérations de fonctions unimodales et suites engendrées par automates, *C. R. Acad. Sci. Paris, Série I* **296** (1983) 159–162.
- [3] J.-P. Allouche, M. Cosnard, The Komornik-Loreti constant is transcendental, *Amer. Math. Monthly* **107** (2000) 448–449.
- [4] J.-P. Allouche, M. Cosnard, Non-integer bases, iteration of continuous real maps, and an arithmetic self-similar set, *Acta Math. Hung.* **91** (2001) 325–332.
- [5] J.-P. Allouche, C. Frougny, K. G. Hare, On univoque Pisot numbers, *Math. Comp.* **76** (2007) 1639–1660.
- [6] J.-P. Allouche, A. Glen, Extremal properties of (epi)sturmian sequences and distribution modulo 1, Preprint (2007).
- [7] Y. Bugeaud, A. Dubickas, Fractional parts of powers and Sturmian words, *C. R. Math. Acad. Sci. Paris* **341** (2005) 69–74.
- [8] S. Bullett, P. Sentenac, Ordered orbits of the shift, square roots, and the devil's staircase, *Math. Proc. Cambridge Philos. Soc.* **115** (1994) 451–481.
- [9] D. P. Chi, D. Kwon, Sturmian words,  $\beta$ -shifts, and transcendence, *Theoret. Comput. Sci.* **321** (2004) 395–404.
- [10] M. Cosnard, Étude de la classification topologique des fonctions unimodales, *Ann. Inst. Fourier* **35** (1985) 59–77.
- [11] P. Erdős, I. Joó, V. Komornik, Characterization of the unique expansions  $1 = \sum q^{-n_i}$  and related problems, *Bull. Soc. Math. France* **118** (1990) 377–390.

- [12] V. Komornik, P. Loreti, Unique developments in non-integer bases, *Amer. Math. Monthly* **105** (1998) 636–639.
- [13] M. Lothaire, *Algebraic Combinatorics On Words*, *Encyclopedia of Mathematics and its Applications*, vol. 90, Cambridge University Press, 2002.
- [14] G. Pirillo, Inequalities characterizing standard Sturmian words, *Pure Math. Appl.* **14** (2003) 141–144.
- [15] P. Veerman, Symbolic dynamics and rotation numbers, *Physica A* **134** (1986) 543–576.
- [16] P. Veerman, Symbolic dynamics of order-preserving orbits, *Physica D* **29** (1987) 191–201.

Communicated by (The editor will be set by the publisher).