

Euler, Pisot, Prouhet-Thue-Morse, Wallis and the duplication of sines

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Abstract

We all know Euler's product $\prod(1 + X^{2^n}) = (1 - X)^{-1}$ and its companion $\prod(1 - X^{2^n}) = \sum \pm X^j$, where the sequence of signs is the so-called Prouhet-Thue-Morse automatic sequence. Discussing generalizations of these two formulae, we are led respectively

- (1) to Wallis' famous infinite product for π ,
- (2) to a characterization of Pisot numbers,
- (3) to multigrade equalities and the Prouhet-Tarry-Escott problem,
- (4) to the product $\prod_{0 \leq j \leq n} \sin(2^j x)$ and its sequence of signs as x runs through the intervals $(j\pi/2^n, (j+1)\pi/2^n)$, $j \in [0, 2^n - 1]$,
- (5) and finally to the Gelfond and Newman-Slater product and its generalization $\prod \sin r^j x$, which plays a rôle in several papers when $r = 2$.

Keywords: Euler's identity, product of sines, product of cosines, Wallis' formula, PV numbers, multigrade equalities, Thue-Morse sequence.

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1 Introduction

If the product obtained by iterating the duplication of cosines has a well-known simple closed form, namely

$$\cos x \cos 2x \cos 4x \cdots \cos 2^n x = \frac{\sin 2^{n+1}x}{2^{n+1} \sin x},$$

there is no such formula for the analogous product of sines. Nevertheless the product:

$$P_n(x) := \sin x \sin 2x \sin 4x \cdots \sin 2^n x$$

occurs several times in the literature (see for example [8, 13, 14, 5, 6, 7]), where in particular the two quantities $\|P_n\|_\infty := \sup_{x \in \mathbb{R}} |P_n(x)|$ and $\|P_n\|_1 := \int_0^\pi |P_n(x)| dx$ enter the picture.

We will give a general identity, involving the sum of binary digits of an integer, from which formulas for the product $P_n(x)$ and related quantities will be deduced. Exploiting these identities will lead us from Pisot numbers to Euler and Wallis formulas, from the Prouhet-Thue-Morse sequence to asymptotic formulas in analytic number theory, from multigrade equalities to the natural introduction of morphisms of the free monoid on two letters.

2 Two propositions

2.1 An easy identity

The following result is not difficult to prove.

Proposition 1 *Let a be a complex number. Denote by $s(k)$ the sum of the binary digits of the nonnegative integer k . Then the following identity holds in $\mathbb{C}[[X]]$.*

$$\prod_{0 \leq k < n} (1 + aX^{2^k}) = \sum_{0 \leq j < 2^n} a^{s(j)} X^j. \quad (1)$$

Proof. This is a direct consequence of the uniqueness of the base 2 expansion of the integers in $[0, 2^n)$. Alternatively this can be proved by induction on n . \square

Corollary 1 *The following well-known identities hold:*

$$\begin{aligned} \prod_{0 \leq k < n} (1 + X^{2^k}) &= \frac{1 - X^{2^n}}{1 - X} \\ \prod_{k \geq 0} (1 + X^{2^k}) &= \frac{1}{1 - X} \\ \prod_{0 \leq k < n} (1 - X^{2^k}) &= \sum_{0 \leq j < 2^n} (-1)^{s(j)} X^j \\ \prod_{k \geq 0} (1 - X^{2^k}) &= \sum_{j \geq 0} (-1)^{s(j)} X^j. \end{aligned}$$

Proof. Take $a = 1$ and $a = -1$ in Identity 1 gives the first and third identities. Letting n tend to infinity in the first and third identities gives the second and fourth identities. \square

Remark 1

– The second identity in Corollary 1 above goes back to L. Euler (1707-1783). For an unusual occurrence of the Euler formula, the reader might want to look at [12].

– The third and fourth identities in Corollary 1 above involve the sequence $((-1)^{s(j)})_{j \in \mathbb{N}}$, which is the celebrated Prouhet-Thue-Morse sequence. This sequence first appeared in 1851 in a paper by E. Prouhet (1817-1867) for multigrade equalities (see below); it was then studied at the beginning of the 20th century by A. Thue (1863-1922) for a question about repetitions in infinite sequences (which lead to the nowadays vivid combinatorics of words); in the 20's H.C.M. Morse (1892-1977) rediscovered it for a construction in differential geometry: this and more can be found, e.g., in [1] and the references therein. Also note that K. Mahler (1903-1988) studied this sequence, without calling it the Thue-Morse sequence: in [9] he proved that the correlation spectrum of this sequence contains a nonzero singular continuous component (Kakutani proved in the 60's that the spectrum is singular continuous); Mahler told the second author that [9] was the first paper he wrote in English – with the help of N. Wiener. Speaking of Mahler and the Thue-Morse sequence, it is worth mentioning that the transcendence results he proved in [10] directly imply the transcendence of the real number whose b -ary expansion is the Thue-Morse sequence.

2.2 A generalization

The following generalization of Identity 1 holds; it will be the core of our paper.

Theorem 1 *Let a be a complex number. If $j = \sum_q e_q(j)2^q$ with $e_q(j) \in \{0, 1\}$ is the binary expansion of the nonnegative integer j , and if $\lambda := (\lambda_j)_{j \in \mathbb{N}}$ is a sequence of complex numbers, let $u(j, \lambda) := \sum_q e_q(j)\lambda_q$. Then the following identity holds in $\mathbb{C}[[Z]]$.*

$$\prod_{0 \leq k < n} (1 + a \exp(\lambda_k Z)) = \sum_{0 \leq j < 2^n} a^{s(j)} \exp(u(j, \lambda)Z). \quad (2)$$

Proof. As for Identity 1 the proof relies on an induction on n . \square

Remark 2

– In Identity 2 the formal series $\exp Y$ is defined as usual by $\exp Y := \sum_{n \geq 0} Y^n/n!$. This identity could be written in the more visual – but maybe less conventional – form

$$\prod_{0 \leq k < n} (1 + aX^{\lambda_k}) = \sum_{0 \leq j < 2^n} a^{s(j)} X^{u(j, \lambda)}.$$

– Taking $\lambda_j := 2^j$ and $X := \exp Z$ in Identity 2 clearly implies Identity 1.

3 More product formulas

The identities above can be turned into trigonometric identities including in particular a formula for the duplication of sines and cosines.

Proposition 2 *Let a be a complex number. Let $\lambda := (\lambda_j)_{j \in \mathbb{N}}$ be a sequence of complex numbers. If the binary expansion of the nonnegative integer j is $j = \sum_q e_q(j)2^q$, with $e_q(j) = 0, 1$, define $\sigma_q(j)$ by $\sigma_q(j) := 2e_q(j) - 1$. Then, the following identity holds true in $\mathbb{C}[[Z]]$.*

$$\begin{aligned} a^n \prod_{0 \leq k < n} (\exp(\lambda_k Z) + a^{-1} \exp(-\lambda_k Z)) \\ = \sum_{0 \leq j < 2^n} a^{s(j)} \exp \left(\left(\sum_{0 \leq q < n} \sigma_q(j) \lambda_q \right) Z \right). \end{aligned} \quad (3)$$

Proof. Dividing the two sides of Identity 2 by $\exp(\frac{\lambda_0 + \lambda_1 + \dots + \lambda_{n-1}}{2} Z)$ yields

$$\begin{aligned} \prod_{0 \leq k < n} \left(\exp \left(-\frac{\lambda_k}{2} Z \right) + a \exp \left(\frac{\lambda_k}{2} Z \right) \right) \\ = \sum_{0 \leq j < 2^n} a^{s(j)} \exp \left(\left(u(j, \lambda) - \frac{\lambda_0 + \dots + \lambda_{n-1}}{2} \right) Z \right). \end{aligned}$$

Since

$$u(j, \lambda) - \frac{\lambda_0 + \cdots + \lambda_{n-1}}{2} = \frac{1}{2} \sum_{0 \leq q < n} (2e_q(j) - 1)\lambda_q = \frac{1}{2} \sum_{0 \leq q < n} \sigma_q(j)\lambda_q$$

we get Identity 3 after factoring a out and replacing Z by $2Z$. \square

Identity 3 implies several product identities.

Corollary 2 *Let $\lambda := (\lambda_j)_{j \in \mathbb{N}}$ be a sequence of complex numbers. If the binary expansion of the nonnegative integer j is $j = \sum_q e_q(j)2^q$, with $e_q(j) = 0, 1$, define $\sigma_q(j)$ by $\sigma_q(j) := 2e_q(j) - 1$. Then,*

$$\prod_{0 \leq k < n} \cosh \lambda_k = 2^{-n} \sum_{0 \leq j < 2^n} \cosh \left(\sum_{0 \leq q < n} \sigma_q(j)\lambda_q \right) \quad (4)$$

$$\prod_{0 \leq k < n} \cos \lambda_k = 2^{-n} \sum_{0 \leq j < 2^n} \cos \left(\sum_{0 \leq q < n} \sigma_q(j)\lambda_q \right) \quad (5)$$

$$\prod_{0 \leq k < 2m} \sinh \lambda_k = 2^{-2m} \sum_{0 \leq j < 2^{2m}} (-1)^{s(j)} \cosh \left(\sum_{0 \leq q < 2m} \sigma_q(j)\lambda_q \right) \quad (6)$$

$$\prod_{0 \leq k < 2m} \sin \lambda_k = (-1)^m 2^{-2m} \sum_{0 \leq j < 2^{2m}} (-1)^{s(j)} \cos \left(\sum_{0 \leq q < 2m} \sigma_q(j)\lambda_q \right) \quad (7)$$

Furthermore

$$\prod_{0 \leq k < n} \cos(2^k x) = 2^{-n} \sum_{0 \leq j < 2^n} \cos((2j+1)x - 2^n x) \quad (8)$$

$$\prod_{0 \leq k < 2m} \sin(2^k x) = (-1)^m 2^{-2m} \sum_{0 \leq j < 2^{2m}} (-1)^{s(j)} \cos((2j+1)x - 2^{2m} x) \quad (9)$$

Hence

$$\prod_{0 \leq k < n} \cos(2^k x) = 2^{1-n} \sum_{0 \leq j < 2^{n-1}} \cos((2j+1)x) \quad \left(= \frac{\sin 2^n x}{2^n \sin x} \right) \quad (10)$$

$$\prod_{0 \leq k < 2m} \sin(2^k x) = (-1)^{m+1} 2^{1-2m} \sum_{0 \leq j < 2^{2m-1}} (-1)^{s(j)} \cos((2j+1)x) \quad (11)$$

Proof. Taking $a = 1$ in Identity 3, we see that the left-hand expression is invariant under $Z \rightarrow -Z$; hence the right-hand expression is also invariant under $Z \rightarrow -Z$; hence adding Identity 3 with Z to Identity 3 with $-Z$ yields Identity 4. Changing then λ_k into $i\lambda_k$ yields Identity 5.

Taking $a = -1$ in Identity 3, we see that changing Z into $-Z$ multiplies out the left-hand expression by $(-1)^n$. Taking $n = 2m$ gives

$$\prod_{0 \leq k < n} (\exp(\lambda_k Z) - \exp(-\lambda_k Z)) = \sum_{0 \leq j < 2^n} (-1)^{s(j)} \exp \left(\left(\sum_{0 \leq q < n} \sigma_q(j) \lambda_q \right) Z \right)$$

whose left-hand member (hence right-hand member as well) is invariant under $Z \rightarrow -Z$. Arguing as above, dividing by 2^{2m} , and taking $Z = 1$ gives Identity 6. Replacing λ_k by $i\lambda_k$ yields Identity 7.

Then Identities 8 and 9 are deduced from Identities 5 and 7 respectively by taking $\lambda_k := 2^k x$ and by noting that $\sum_{0 \leq q < n} \sigma_q(j) 2^j = (2j + 1) - 2^n$.

Finally if we split the sum in Identity 8 as $\sum_{0 \leq j < 2^{n-1}} + \sum_{2^{n-1} \leq j < 2^n}$ and if we put $j = 2^{n-1} - t - 1$ in the first sum and $j = 2^{n-1} + t$ in the second sum, we get

$$\sum_{0 \leq j < 2^n} \cos((2j + 1)x - 2^n x) = 2 \sum_{0 \leq t < 2^{n-1}} \cos((2t + 1)x)$$

which gives Identity 10. Similarly Identity 9 implies Identity 11 after noting that for $t \in [0, 2^{2m-1})$ we have $(-1)^{s(2^{2m-1}-t-1)} = -(-1)^{s(t)}$ and $(-1)^{s(2^{2m-1}+t)} = -(-1)^{s(t)}$. \square

Remark 3

– In Identities 3 to 7 above, the quantities

$$\sum_{0 \leq j < 2^n} f(s(j)) F \left(\sum_{0 \leq q < n} \sigma_q(j) \lambda_q \right)$$

can be replaced by the (equal) expressions

$$\sum_{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1} \in \{-1, +1\}^n} f \left(\sum_{0 \leq q < n} \frac{1 + \varepsilon_q}{2} \right) F \left(\sum_{0 \leq q < n} \varepsilon_q \lambda_q \right).$$

– Identities analogous to Identities 6, 7, 9, and 11 above, where $2m$ is replaced by $2m+1$, can be obtained with similar arguments. For example

$$\prod_{0 \leq k < 2m+1} \sin \lambda_k = (-1)^{m+1} 2^{-2m-1} \sum_{0 \leq j < 2^{2m+1}} (-1)^{s(j)} \sin \left(\sum_{0 \leq q < 2m+1} \sigma_q(j) \lambda_q \right)$$

(this relation can be deduced directly from Identity 7 by multiplying by $\sin \lambda_{2m}$ and using $\sin \alpha \cos \beta = (\sin(\alpha + \beta) + \sin(\alpha - \beta))/2$).

4 Wallis formula

John Wallis (1616-1703), while calculating the value of π by finding the area under the quadrant of a circle (ten years before Newton discovered calculus), obtained the now called “Wallis formula”. It is worth mentioning that Sondow showed how to use Wallis formula to compute in an elementary way $\zeta(0)$ and $\zeta'(0)$, where ζ is the Riemann zeta function [19]. We show here how Wallis formula can be deduced from Identity 7.

Corollary 3 (Wallis formula) *If n tends to infinity, then*

$$\lim \frac{2^{4n}}{n \binom{2n}{n}^2} = \pi.$$

This can also be written

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \frac{4 \cdot 4}{3 \cdot 5} \frac{6 \cdot 6}{5 \cdot 7} \cdots$$

Proof. Applying Identity 7 with $\lambda_0 = \lambda_1 = \cdots = \lambda_{2m-1} := x$ we obtain

$$(\sin x)^{2m} = (-1)^m 2^{-2m} \sum_{0 \leq j < 2^{2m}} (-1)^{s(j)} \cos \left(x \sum_{0 \leq q < 2m} \sigma_q(j) \right).$$

Since $\sigma_q(j) = 2e_q(j) - 1$, we have $\sum_{0 \leq q < 2m} \sigma_q(j) = 2s(j) - 2m$, hence

$$(\sin x)^{2m} = (-1)^m 2^{-2m} \sum_{0 \leq j < 2^{2m}} (-1)^{s(j)} \cos(2x(s(j) - m)).$$

Integrating yields

$$\int_0^{\pi/2} (\sin x)^{2m} dx = (-1)^m 2^{-2m} \sum_{0 \leq j < 2^{2m}} (-1)^{s(j)} \int_0^{\pi/2} \cos(2x(s(j) - m)) dx.$$

The integrals in the right-hand quantity are equal to zero if $s(j) \neq m$ and to $\pi/2$ if $s(j) = m$. There are exactly $\binom{2m}{m}$ integers $j \in [0, 2^{2m})$ for which $s(j) = m$, hence

$$\int_0^{\pi/2} (\sin x)^{2m} dx = (-1)^m 2^{-2m} (-1)^m \binom{2m}{m} \frac{\pi}{2} = \frac{1}{2^{2m}} \binom{2m}{m} \frac{\pi}{2}.$$

Now, using the classical trick, define $I_n := \int_0^{\pi/2} (\sin x)^n dx$. Then

$$\int_0^{\pi/2} (\sin x)^{n+2} dx = \int_0^{\pi/2} (\sin x)^n dx - \int_0^{\pi/2} \cos x (\cos x (\sin x)^n) dx.$$

Integrating the last integral by parts gives

$$(n+2)I_{n+2} = (n+1)I_n \text{ hence } (n+2)I_{n+2}I_{n+1} = (n+1)I_{n+1}I_n.$$

This shows that the quantity $(n+1)I_{n+1}I_n$ does not depend on n , hence is equal to $I_1I_0 = \pi/2$. Furthermore the equality $(n+2)I_{n+2} = (n+1)I_n$ shows that I_{n+2}/I_n tends to 1 as n tends to infinity. But $I_{n+2} \leq I_{n+1} \leq I_n$; hence dividing by I_n shows that I_{n+1}/I_n tends to 1 as n tends to infinity. Hence

$$\frac{\pi}{2} = (n+1)I_{n+1}I_n \sim nI_n^2$$

which gives

$$\pi = \lim_{n \rightarrow \infty} 2nI_n^2 = \lim_{n \rightarrow \infty} 4nI_{2n}^2 = \lim_{n \rightarrow \infty} \frac{n\pi^2}{2^{4n}} \binom{2n}{n}^2.$$

Hence

$$\lim_{n \rightarrow \infty} \frac{2^{4n}}{n \binom{2n}{n}^2} = \pi. \quad \square$$

5 Pisot numbers

Pisot-Vijayaraghavan numbers were first studied by C. Pisot (1910-1988) and T. Vijayaraghavan (1902-1955): these are the algebraic integers > 1 such that all their other algebraic conjugates lie in the open disk $z < 1$. Their principal property is to behave almost like integers in questions of distribution modulo 1.

The product formulas above have applications for the distribution modulo 1 of quantities involving Pisot-Vijayaraghavan numbers. Before stating such an application (Corollary 4 below) we need a lemma.

Lemma 1 *Let $(\lambda_q)_{q \geq 0}$ be a sequence of reals. For any real x define $\|x\| := \min_{x \in \mathbb{Z}} |x - n|$. Then the infinite product $\prod_{q \geq 0} |\cos \pi \lambda_q|$ is equal to 0 if and only if either $\sum_{q \geq 0} \|\lambda_q\|^2 = +\infty$, or there exists $q \geq 0$ such that $\lambda_q \equiv \frac{1}{2} \pmod{1}$.*

Proof. Left to the reader. \square

Corollary 4 (Mendès France [11]) *Let $\lambda := (\lambda_j)_{j \in \mathbb{N}}$ be a sequence of real numbers. For every integer j with binary expansion $j = \sum_q e_q(j) 2^q$, where $e_q(j) \in \{0, 1\}$, define as previously $u(j, \lambda) := \sum_q e_q(j) \lambda_q$. Then the sequence $j \rightarrow u(j, \lambda)$ is uniformly distributed modulo 1 if and only if for each $\ell \in \mathbb{N} \setminus \{0\}$ either $\sum_q \|\ell \lambda_q\|^2 = +\infty$ or there exists $q = q(\ell) \geq 0$ such that $\ell \lambda_q \equiv \frac{1}{2} \pmod{1}$.*

As a consequence, supposing $\theta > 1$, the sequence $(x \sum_q e_q(k) \theta^q)_{k \in \mathbb{N}}$ is uniformly distributed modulo 1 if and only if either θ is not a Pisot number and $x \neq 0$, or θ is a Pisot number and $x \notin \mathbb{Q}(\theta)$.

In particular, if $x \neq 0$ and θ is not a Pisot number, then the set of finite sums $x \sum \theta^{n_j}$ is dense modulo 1. The same conclusion holds if θ is a Pisot number and $x \notin \mathbb{Q}(\theta)$.

Proof. We first recall that θ is a Pisot number if and only if there exists a nonzero real number x such that $\sum_q \|x \theta^q\|^2 < \infty$, see [15, p. 238]; furthermore any such x must necessarily belong to $\mathbb{Q}(\theta)$ see [16, Théorème 2, p. 153–154] or [17, Theorem A, p. 4]. Now, since the sequence $j \rightarrow u(j, \lambda)$ is uniformly distributed modulo 1 if and only if

$$\lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{0 \leq j < N} \exp(2i\pi \ell u(j, \lambda)) \right) = 0$$

for all integers $\ell \neq 0$, we see from Lemma 1 that it suffices to prove that conditions (i) and (ii) below are equivalent:

$$(i) \quad \forall \ell \neq 0, \quad \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{0 \leq j < N} \exp(2i\pi \ell u(j, \lambda)) \right) = 0$$

$$(ii) \quad \forall \ell \neq 0, \quad \prod_{q \geq 0} |\cos \pi \ell \lambda_q| = 0.$$

Now Identity 2 with $a := 1$ and $Z := 2i\pi \ell$ (where ℓ is an integer) reads

$$\prod_{0 \leq k < n} \left(\frac{1 + e^{2i\pi \ell \lambda_k}}{2} \right) = \frac{1}{2^n} \sum_{0 \leq j < 2^n} e^{2i\pi \ell u(j, \lambda)}.$$

Hence

$$\prod_{0 \leq k < n} |\cos \pi \ell \lambda_k| = \left| \frac{1}{2^n} \sum_{0 \leq j < 2^n} e^{2i\pi \ell u(j, \lambda)} \right|$$

showing that (i) implies (ii).

Conversely, suppose that Condition (ii) holds. Note that $u(2j, \lambda) = u(j, T\lambda)$ and that $u(2j+1, \lambda) = \lambda_0 + u(j, T\lambda)$, where $T\lambda$ is the shifted sequence $T\lambda := (\lambda_{q+1})_{q \geq 0}$. Hence, defining $\mathcal{M}(N, \lambda)$ by

$$\mathcal{M}(\lambda) := \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{0 \leq k < N} \exp(2i\pi \ell u(k, \lambda)) \right|$$

we have (see Lemma 3 of [11])

$$\mathcal{M}(\lambda) = |\cos(\pi \ell \lambda_0)| \mathcal{M}(T\lambda) = |\cos(\pi \ell \lambda_0)| |\cos(\pi \ell \lambda_1)| \mathcal{M}(T^2\lambda) = \dots$$

hence

$$\mathcal{M}(\lambda) \leq \prod_{q \geq 0} |\cos(\pi \ell \lambda_q)| = 0. \quad \square$$

6 Multigrade equalities and the Prouhet-Tarry-Escott problem

We alluded at the beginning of this paper to multigrade equalities. The pioneering work in this area is the very short Note aux Comptes-Rendus de l'Académie des Sciences of Prouhet in 1851: for a survey on the Prouhet problem see for example [3, 18]. In this section we sketch the construction of multigrade equalities à la Prouhet using our fundamental identity.

Replacing λ_k by $x\lambda_k$ in Identity 6 gives

$$\begin{aligned} \prod_{0 \leq k < 2^m} \sinh x\lambda_k &= 2^{-2^m} \sum_{0 \leq j < 2^{2^m}} (-1)^{s(j)} \cosh \left(x \sum_{0 \leq q < 2^m} \sigma_q(j) \lambda_q \right) \\ &= 2^{-2^m} \sum_{0 \leq j < 2^{2^m}} (-1)^{s(j)} \sum_{\ell \geq 0} \frac{1}{(2\ell)!} x^{2\ell} \left(\sum_{0 \leq q < 2^m} \sigma_q(j) \lambda_q \right)^{2\ell} \\ &= 2^{-2^m} \sum_{\ell \geq 0} \frac{1}{(2\ell)!} x^{2\ell} \sum_{0 \leq j < 2^{2^m}} (-1)^{s(j)} \left(\sum_{0 \leq q < 2^m} \sigma_q(j) \lambda_q \right)^{2\ell}. \end{aligned}$$

Since the left-hand term equals $\lambda_0 \lambda_1 \cdots \lambda_{2m-1} x^{2m} + \mathcal{O}(x^{2m+2})$ in the neighborhood of 0, we have for $\ell \in [0, m-1]$

$$\sum_{0 \leq j < 2^{2m}} (-1)^{s(j)} \left[\left(\sum_{0 \leq q < 2m} \sigma_q(j) \lambda_q \right)^2 \right]^\ell = 0.$$

Choosing the λ_j 's adequately gives a family of multigrade equalities.

In the same way, using the second part of Remark 3, one can obtain an identity for $\prod_{0 \leq k < 2m+1} \sinh x \lambda_k$, and hence another family of multigrade identities.

7 Back to iterating the duplication of sines

7.1 A natural occurrence of morphisms of the free monoid

We recall the notation: $P_n(x) := \sin x \sin 2x \sin 4x \cdots \sin 2^n x$. We will denote by $\text{sgn } y$ the sign of the real number y , i.e., $\text{sgn } y := y/|y|$ if $y \neq 0$. We will show that the signs of $P_n(x)$, for $x \in (0, \pi)$, form a word of length 2^n that is obtained by iterating the morphism φ of the free monoid generated by $\{0, 1\}$ defined by

$$\begin{aligned} \varphi(+) &= +- \\ \varphi(-) &= -+ \end{aligned}$$

(in particular when n goes to infinity, the sequence of signs of $P_n(x)$ converges to the Prouhet-Thue-Morse sequence).

Proposition 3 *The finite sequence of (consecutive distinct) signs of the function $P_{n+1}(x)$ for $x \in (0, \pi)$ can be obtained from the sequence of signs of $P_n(x)$ for $x \in (0, \pi)$ by applying the morphism of monoid $+ \rightarrow + -, - \rightarrow - +$. In particular the sequence of signs of $P_n(x)$ for $x \in (0, \pi)$ is the prefix of length 2^n of the Prouhet-Thue-Morse sequence.*

Proof. For $n \geq 0$ and $j \in [0, 2^n)$, let $p_{n,j}$ be the sign of $P_n(x)$ for $x \in (\frac{j\pi}{2^n}, \frac{(j+1)\pi}{2^n})$ (it is clear that $P_n(x)$ has constant sign on such an interval).

Let now j belong to $[0, 2^{n+1})$ and let us determine the sign of $P_{n+1}(x)$ for $x \in (\frac{j\pi}{2^{n+1}}, \frac{(j+1)\pi}{2^{n+1}})$.

- If j is even, say $j := 2u$, then the interval $(\frac{j\pi}{2^{n+1}}, \frac{(j+1)\pi}{2^{n+1}})$ is contained in the interval $(\frac{u\pi}{2^n}, \frac{(u+1)\pi}{2^n})$ and $2^{n+1}x$ belongs to the interval $(j\pi, (j+1)\pi) = (2u\pi, (2u+1)\pi)$. Hence $\sin 2^{n+1}x > 0$ and $p_{n+1,j} = p_{n+1,2u} = p_{n,u}$.
- If j is odd, say $j := 2u+1$, then the interval $(\frac{j\pi}{2^{n+1}}, \frac{(j+1)\pi}{2^{n+1}})$ is contained in the interval $(\frac{u\pi}{2^n}, \frac{(u+1)\pi}{2^n})$ and $2^{n+1}x$ belongs to the interval $(j\pi, (j+1)\pi) = ((2u+1)\pi, (2u+2)\pi)$. Hence $\sin 2^{n+1}x < 0$ and $p_{n+1,j} = p_{n+1,2u+1} = -p_{n,u}$.

The formulas giving $p_{n+1,j}$ in terms of $p_{n,j}$ exactly mean that each of the intervals $(\frac{j\pi}{2^n}, \frac{(j+1)\pi}{2^n})$ where $P_n(x)$ has constant sign splits into the two subintervals $(\frac{2j\pi}{2^{n+1}}, \frac{(2j+1)\pi}{2^{n+1}})$ and $(\frac{(2j+1)\pi}{2^{n+1}}, \frac{(2j+2)\pi}{2^{n+1}})$ where $P_{n+1}(x)$ has constant sign, this sign being the same as the sign of $P_n(x)$ on the first interval and the opposite on the second one. This can be described by

$$\begin{array}{ccc} + & \rightarrow & + - \\ - & \rightarrow & - + \end{array}$$

at each splitting. \square

Remark 4 What seems interesting in this (easy) result is the natural occurrence of a morphism of monoid in a simple question that is not explicitly related to combinatorics on words. For more material on morphisms of (free) monoids, the reader might want to read, for example, the book [2].

7.2 Asymptotics of the duplicating sine product

In this section we will study the asymptotic behavior of the product

$$P_n(x) := \sin x \sin 2x \sin 4x \cdots \sin 2^n x,$$

in particular the asymptotic behavior of $\|P_n\|_\infty := \sup_{x \in \mathbb{R}} |P_n(x)|$ and $\|P_n\|_1 := \int_0^\pi |P_n(x)| dx$.

Theorem 2 (Gelfond [8], Newman-Slater [14]) *The following inequality holds*

$$\|P_n\|_\infty \leq \left(\frac{\sqrt{3}}{2} \right)^n.$$

Remark 5

– Note that this inequality is nearly optimal since $|P_n(\pi/3)| = (\sqrt{3}/2)^{n+1}$.

– This result seems to have been proved for the first time by Gelfond who used a simple and ingenious lemma [8, p. 262]. Gelfond applied this bound to obtain an asymptotic estimation of the number of integers $\leq x$ belonging to a given class of congruence and whose sum of digits in some integral base belongs to a given class of congruence. Another proof was given by Newman and Slater in [14, p. 73]: they used this result to establish the preponderance of integers having an even number of 1's in their binary expansion over integers having an odd number of 1's in natural sequences of integers (for example the sequence $(M \cdot n)_{n \geq 0}$, or the sequence of squarefree integers). Among other occurrences of this result in the literature we cite the paper of Fouvry and Mauduit [7, p. 583] where they were interested *inter alia* in almost-primes in sequences of integers with automatic characteristic function; their proof is the same as the proof of Newman and Slater in [14].

Note that $\log((\sqrt{3}/2)^n) = n(\log 3)/(2 \log 2)$ and that the exponent $\frac{\log 3}{2 \log 2}$ as well as the summatory function of the sequence $((-1)^{s(3n)})_{n \geq 0}$ also occur in [13, 4].

For the sake of completeness we give below the proof of Newman and Slater slightly generalized to the case where duplication is replaced by multiplication by any integer. (For recovering Theorem 2 above take $r = 2$ in Proposition 4 below.)

Proposition 4 *Let r be a positive integer. Define $P_{r,n}(x)$ by*

$$P_{r,n}(x) := |\sin x \sin rx \cdots \sin r^n x|.$$

If r is odd, then $\|P_{r,n}\|_\infty = 1$,

If r is even, then $\left(\cos\left(\frac{\pi}{2r+2}\right)\right)^{n+1} \leq \|P_{r,n}\|_\infty \leq \left(\cos\left(\frac{\pi}{2r+2}\right)\right)^n$.

Proof. To avoid both absolute values and taking real powers of negative numbers, we will study the quantity $P_{r,n}^2(x)$. The following trick is adapted from [14]:

$$P_{r,n}^2(x) = (\sin^2 x)^{1/r+1} \left(\prod_{0 \leq j \leq n-1} (\sin^2 r^j x)^{r/r+1} (\sin^2 r^{j+1} x)^{1/r+1} \right) (\sin^2 r^n x)^{r/r+1}.$$

Hence, defining the function φ by

$$\varphi(x) := (\sin^2 x)^{r/r+1} (\sin^2 rx)^{1/r+1},$$

we have

$$P_{r,n}^2(x) \leq \prod_{0 \leq j \leq n-1} \varphi(r^j x).$$

Now the derivative of the function φ is given by:

$$\varphi'(x) = \frac{2r}{r+1} (\sin^2 x)^{-1/r+1} (\sin^2 rx)^{-r/r+1} \sin x \sin rx \sin((r+1)x).$$

In particular the maximum of φ must occur for x such that $\sin((r+1)x) = 0$ i.e., $x \in \{k\pi/(r+1), k \geq 0\}$. It is thus immediate that the maximum of φ occurs at $x = \frac{\lfloor \frac{r+1}{2} \rfloor}{r+1} \pi$ and that this maximum is equal to $\sin^2 \left(\frac{\lfloor \frac{r+1}{2} \rfloor}{r+1} \pi \right)$.

Hence

$$P_{r,n}^2(x) \leq \left(\sin^2 \left(\frac{\lfloor \frac{r+1}{2} \rfloor}{r+1} \pi \right) \right)^n$$

which yields an upper bound for $\|P_{r,n}\|_\infty$.

Taking $x_0 := (\pi \lfloor \frac{r+1}{2} \rfloor)/(r+1)$, we see that, for any integer $k \geq 0$,

$$r^{k+1}x_0 + r^kx_0 = r^k\pi \left\lfloor \frac{r+1}{2} \right\rfloor \in \pi\mathbb{Z}^+$$

hence

$$|\sin r^{k+1}x_0| = |\sin r^kx_0| = \dots = |\sin x_0|.$$

This implies that $P_{r,n}(x_0) = |\sin x_0|^{n+1}$, hence

$$\left(\sin \left(\frac{\lfloor \frac{r+1}{2} \rfloor}{r+1} \pi \right) \right)^{n+1} \leq \|P_{r,n}\|_\infty \leq \left(\sin \left(\frac{\lfloor \frac{r+1}{2} \rfloor}{r+1} \pi \right) \right)^n$$

which gives the desired result. \square

Remark 6 Proposition 4 shows in particular that $\lim_{r \rightarrow \infty} \|P_{r,n}\|_\infty = 1$, for all n .

The asymptotic behavior of the L_1 -norm of $P_n(x) := P_{2,n}(x)$ also occurs in the literature. We have the following result.

Theorem 3 (Fouvry-Mauduit [7]) *The following equality holds*

$$\|P_n\|_1 = \int_0^\pi |P_n(t)| dt = C\rho^n(1 + o(1))$$

where C is a positive constant and ρ satisfies $0.654336 < \rho < 0.663198$.

Remark 7 Theorem 3 above is proved in [7] where the authors are interested in almost-primes (i.e., integers having at most a given number of primes) in sequences of integers whose characteristic functions are automatic; their proof uses transfer operators. Note that Èminyan gave earlier in [5] a bound for $\|P_{2n-1}\|_1$ of the form $C'\mu^{2n-1}$ where $\mu = (2 + \sqrt{2})^{1/4}/2$. This result is used to compute the average value of the number of divisors of n for n having an even binary sum of digits. This result is also used in [6] to compute the average value of the number of representations of n as a sum of two squares for n having an even binary sum of digits.

A rough upper bound for ρ in Theorem 3 can easily be obtained from Identity 11 and Parseval's equality (orthogonality of sines or cosines). We state this result in a more general framework.

Proposition 5 *Let $\lambda = (\lambda_0, \dots, \lambda_N)$ be an $(N+1)$ -tuple of integers such that, for all choices of $\varepsilon_q = \pm 1$, and $\varepsilon'_q = \pm 1$, we have*

$$\sum_{0 \leq q \leq N} \varepsilon_q \lambda_q = \sum_{0 \leq q \leq N} \varepsilon'_q \lambda_q \quad \Rightarrow \quad \varepsilon_q = \varepsilon'_q \text{ for all } q.$$

Define

$$P_N^\lambda(x) := \left| \prod_{0 \leq j \leq N} \sin(\lambda_j x) \right|.$$

Then

$$\int_0^\pi |P_N^\lambda(x)| dx \leq \frac{\pi}{2^{1+N/2}}.$$

Remark 8 If $\lambda_q := 2^q$ for all q , the condition on ε_q and ε'_q holds. Proposition 5 yields for ρ in Theorem 3 the bound $\rho \leq \frac{\sqrt{2}}{2} = 0.707106 \dots$. This bound is of course less precise than the result of [7], but more precise than the bound obtained from the inequality $\|P_n\|_1 \leq C\|P_n\|_\infty$ which gives $\rho \leq \frac{\sqrt{3}}{2} = 0.866025 \dots$.

Proof. The Fourier expansion of P_N^λ depends on the parity of N : we have from Identity 7 and the two parts of Remark 3, where λ_k is replaced by $\lambda_k x$:

$$P_N^\lambda(x) = \frac{1}{2^{N+1}} \left| \sum_{\varepsilon \in \{-1, +1\}^{N+1}} \pm F \left(x \sum_{0 \leq q \leq N} \varepsilon_q \lambda_q \right) \right|$$

where the \pm symbol equals $\varepsilon_0 \varepsilon_1 \cdots \varepsilon_N$, and where $F(\xi) = \cos \xi$ if N is odd and $F(\xi) = \sin \xi$ if N is even. In either case, Parseval's equality reads

$$\frac{1}{\pi} \int_0^\pi |P_N^\lambda(x)|^2 dx = \frac{1}{2} \sum_{0 \leq k \leq 2^{N+1}-1} \frac{1}{2^{2(N+1)}} = \frac{1}{2^{N+2}}.$$

Applying the inequality of Cauchy-Schwarz yields

$$\frac{1}{\pi} \int_0^\pi |P_N^\lambda(x)| dx \leq \left(\frac{1}{\pi} \int_0^\pi |P_N^\lambda(x)|^2 dx \right)^{1/2} = \frac{1}{2^{1+N/2}}. \quad \square$$

A last comment. The sup norm of the product $|\prod \cos(\lambda_j x)|$ is obviously 1. Its L^2 -norm can be computed as above: the graph of the product of sines or cosines L^2 -collapses onto the horizontal axis as N increases.

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