Contributions to the Domino Problem:

Seeding, Recurrence and Satisfiability

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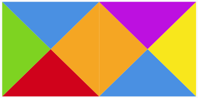
Wang Tiles

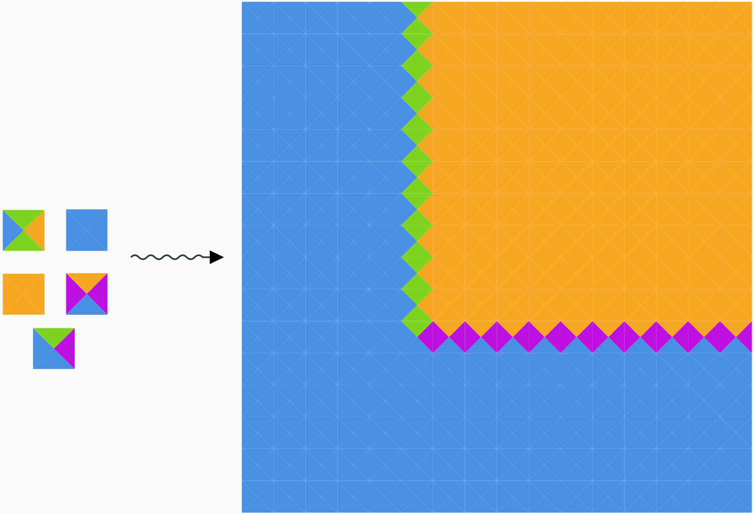
- Our story begins with Wang tiles:

  First introduced by Hao Wang in 1961.

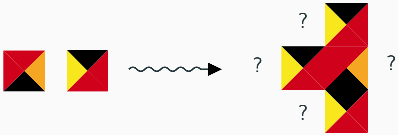
- They can be placed side by side if they share the same color along their common border.



Tilings



Tiling?



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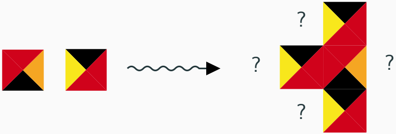
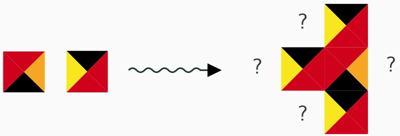


Domino Problem

Given a finite set τ of Wang tiles, does τ tile the plane?

Tiling?



Domino Problem

Given a finite set τ of Wang tiles, does τ tile the plane?

Theorem (Berger '64)

The Domino Problem is undecidable (Π^0_1-comp.).

Consequences

The Domino Problem has been used to prove the undecidability of

- Seeded Domino Problem,
- Recurrent Domino Problem,
- k-SAT on $\mathbb{Z}^2$,
- Injectivity and Surjectivity of 2D CA,
- Infinite Snake Problem,
- Translational Monotilings,
- Spectral gap of quantum many-body systems.

... and more!

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Planting the Seeds of Undecidability

Seeded Domino Problem

Given a Wang tileset τ and a tile t_0, does there exist a tiling $x : \mathbb{Z}^2 \to \tau$ such that $x(0,0) = t_0$?

Theorem (Kahr, Moore and Wang '62)

The Seeded Domino Problem is undecidable (Π_1^0-complete)

Reduction: Halting Problem from blank tape.

Déjà Vu

Recurrent Domino Problem

Given a Wang tileset τ and a tile t_0, does there exist a tiling $x : \mathbb{Z}^2 \to \tau$ such that t_0 appear infinitely often?

Theorem (Harel '83)

The Recurrent Domino Problem is undecidable (Σ_1^1-complete)

Reduction: State Recurrence problem for non-deterministic Turing Machines.

k-SAT

Freedman defined the following infinite generalization of SAT:

Take elements $\{v_{ij}\} \subseteq \mathbb{Z}^2$ and define the formula

$$\phi = \bigwedge_{i=1}^{m} ((v_{i1})' \vee ... \vee (v_{ik})'),$$

where v' represents v or the negation $\neg v$.

Definition

The k-SAT problem for $\mathbb{Z}^2$ asks, given a formula with k literals ϕ and $(p, q) \in \mathbb{N}^2$, if there is an assignation of truth values $\alpha : \mathbb{Z}^2 \to \{0, 1\}$ such that

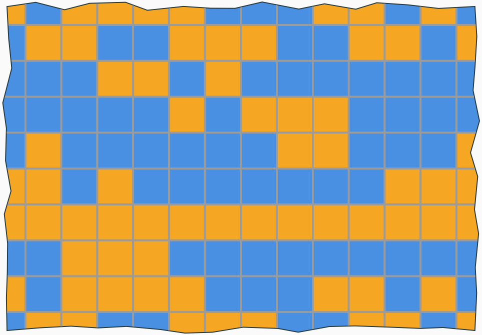
$$\bigwedge_{u \in H} \bigwedge_{i=1}^{m} (\alpha(u + v_{i1})' \vee ... \vee \alpha(u + v_{ik})') = 1,$$

where $H = p\mathbb{Z} \times q\mathbb{Z}$.

Take $v_1 = (0,0)$, $v_2 = (1,0)$ and $v_3 = (0,1)$.

$$\phi = (\neg v_1 \vee v_2 \vee v_3)$$

If we take $H = (2\mathbb{Z})^2$:



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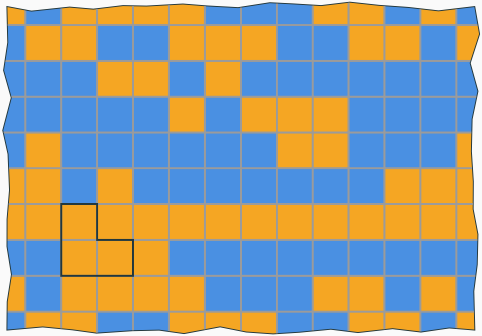

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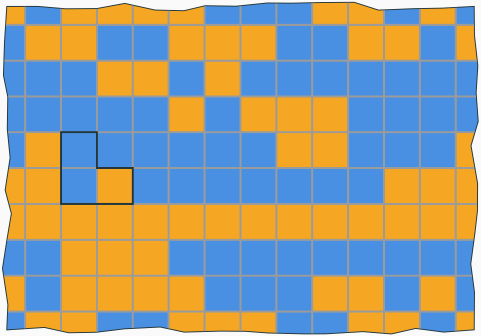

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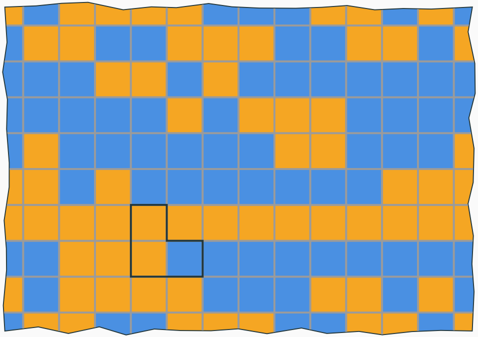

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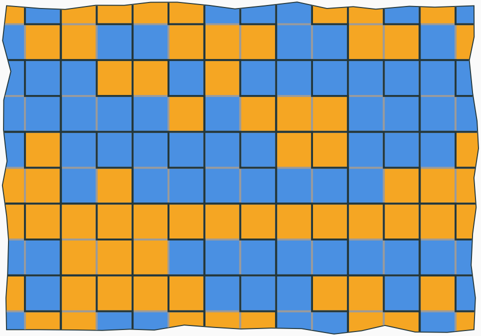

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Take $v_1 = (0, 0)$, $v_2 = (1, 0)$ and $v_3 = (0, 1)$.

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If we take $H = (2\mathbb{Z})^2$:

Theorem (Freedman '99)

The 3-SAT problem on $\mathbb{Z}^2$ is undecidable.

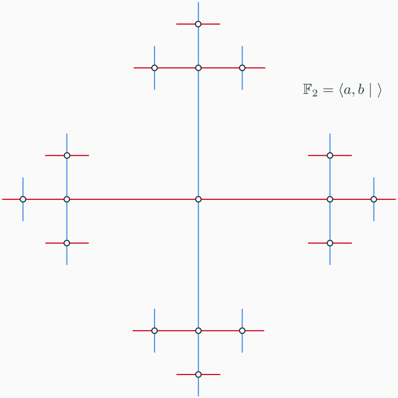
Reduction from the Domino Problem.

Where is the Undecidability?

All the previous problems become decidable on $\mathbb{Z}$! Why?

Where is the Undecidability?

As was done for the Domino Problem, we study our problems on finitely generated groups.

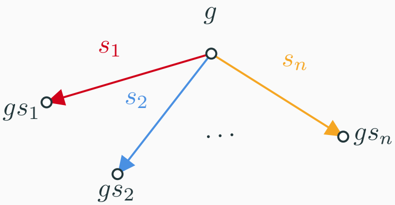


We can understand these groups by their Cayley graphs. These graphs are infinite, locally finite, regular, transitive, edge labelled.

Groups as Graphs

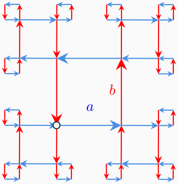
A Cayley graph is defined from a group G along with a finite generating set S:

- Vertices are elements of G,
- There is an edge from g to h if $h = gs^{\pm 1}$.

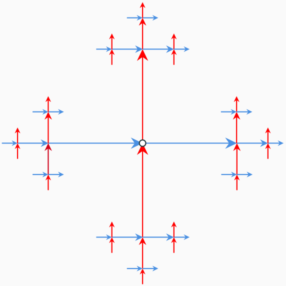


Examples

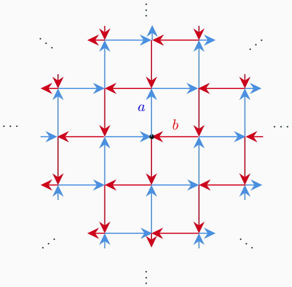
$\langle a, b \mid (ab)^2 \rangle$



$\mathbb{F}_2 = \langle a, b \mid \rangle$



$\langle a, b \mid a^2 b^2 \rangle$

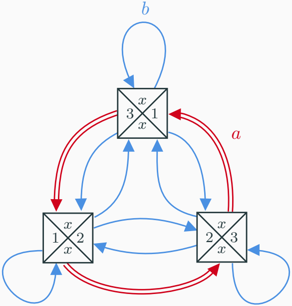


A generalization of Wang Tiles

We generalize Wang tiles through the concept of tileset graphs.

Definition

A tileset graph is a finite graph $\Gamma = (V_\Gamma, E_\Gamma)$ where $E_\Gamma \subseteq V_\Gamma \times S \times V_\Gamma$. We say a tiling $x : G \to V_\Gamma$ respects Γ if for every $g \in G$, $(x(g), x(gs), s) \in E_\Gamma$.



$\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$

Our Favorite Problems

Seeded Domino Problem

Given a tileset graph Γ and a tile t_0, does there exist a tiling $x : G \to V_\Gamma$ that respects Γ such that $x(1_G) = t_0$?

Recurrent Domino Problem

Given a tileset graph Γ and a tile t_0, does there exist a tiling $x : G \to V_\Gamma$ that respects Γ such that t_0 appear infinitely often?

We denote these decision problems by $\mathrm{SDP}(G, S)$ and $\mathrm{RDP}(G, S)$ respectively

Results

For the Seeded and Recurrent Domino Problem:

- Their decidability does not depend on the generating set,
- For a subgroup $H \leq G$, $\mathrm{SDP}(H) \leq_p \mathrm{SDP}(G)$,
- For a finite index subgroup, $\mathrm{SDP}(H) \equiv_p \mathrm{SDP}(G)$,
- The Domino Problem reduces to $\mathrm{SDP}(G)$.

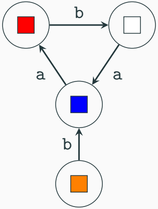
Theorem

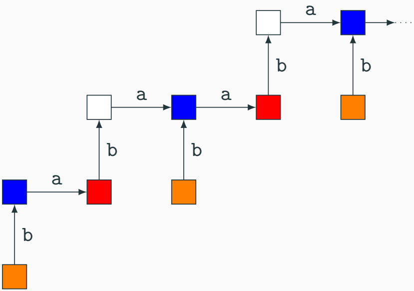
$\mathrm{RDP}(\mathbb{F}_n)$ is decidable.

Balloons

Theorem

$\mathrm{RDP}(\mathbb{F}_n)$ is decidable.





The algorithm consists on finding "balloon" like stuctures on the tileset graph.

Expanding the Conjecture

Joining these facts with

The Domino Conjecture

$\mathrm{DP}(G)$ is decidable iff G is virtually free

we obtain

Corollary

If the Domino Conjecture is true then, TFAE:

- G is virtually free,
- $\mathrm{DP}(G)$ is decidable,
- $\mathrm{SDP}(G)$ is decidable,
- $\mathrm{RDP}(G)$ is decidable.

Our Other Favorite Problem

Take words $\{w_{ij}\} \subseteq (S \cup S^{-1})^*$ and define the formula

$$\phi = \bigwedge_{i=1}^{m} ((w_{i1})' \vee ... \vee (w_{ik})'),$$

where w' represents v or the negation $\neg w$.

Definition

The k-SAT problem for G asks, given a formula with k literals ϕ and $\{u_i\}_{i=1}^n \in (S \cup S^{-1})^*$, if there is an assignation of truth values $\alpha : G \to \{0, 1\}$ such that

$$\bigwedge_{h \in H} \bigwedge_{i=1}^{m} (\alpha(hg_{i1})' \vee ... \vee \alpha(hg_{ik})') = 1,$$

where $H = \langle u_1, ..., u_n \rangle$, and $g_{ij} = \overline{w}_{ij}$.

Results

Lemma

The subgroup membership problem (SMP) reduces to 2-SAT(G).

Lemma

If G has decidable SMP, then 2-SAT(G) reduces to $\mathrm{DP}(G)$. In particular, 2-SAT(G) is decidable for virtually free groups.

Results

Theorem

Suppose G admits a strict finite index subgroup $H \lneq G$ such that $H \simeq G$. Then, $\mathrm{DP}(G)$ reduces to 3-SAT(G).

Corollary

3-SAT(G) is undecidable for

- $\mathbb{Z}^d$, $d \geq 2$,
- Solvable Baumslag-Solitar groups,
- the Lamplighter group,
- the Heisenberg group,
- $\mathbb{Z}^d \rtimes \mathsf{GL}(d, \mathbb{Z})$.

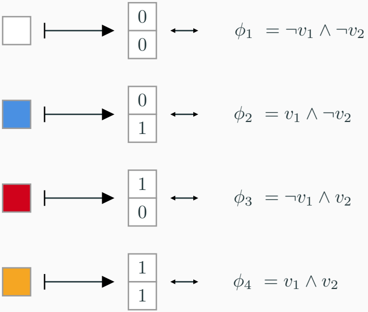
Coding Dominos

Theorem

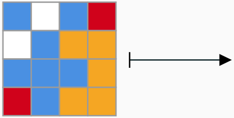
Suppose G admits a strict finite index subgroup $H \lneq G$ such that $H \simeq G$. Then, $\mathrm{DP}(G)$ reduces to 3-SAT(G).

For a tileset of size n, take a subgroup of index $\geq \lceil \log_2(n) \rceil$ and code each tile.

If $G = \mathbb{Z}^2$ and $n = 4$, take $H = \mathbb{Z} \times 2\mathbb{Z}$ and $v_0 = (0,0)$, $v_1 = (0,1)$:



Coding Dominos

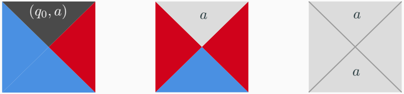


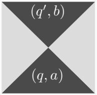
0	0	0	1
1	0	1	0
0	0	1	1
0	1	1	1
0	0	0	1
1	1	1	1
1	0	1	1
0	1	1	1

$$\varphi = \left(\bigvee_{v \in V_\Gamma} \phi_v(1_G) \right) \wedge \left(\bigwedge_{(a,b,s) \notin E_\Gamma} \neg\phi_a(1_G) \vee \neg\phi_b(f_m(s)) \right)$$

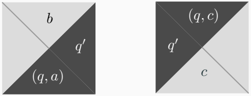
Thank you for listening!

Embedding computation

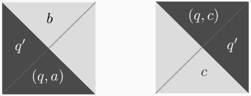




$$\delta(q,a) \;=\; (q', b, 0)$$



$$\delta(q,a) \;=\; (q', b, 1)$$



$$\delta(q,a) \;=\; (q', b, -1)$$

Embedding computation