The Random Planar Graph

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Abstract

We construct a Markov chain whose stationary distribution is uniform over all planar subgraphs of a graph. In the case of the complete graph our experiments suggest that the random simple planar graph on n vertices is connected but not 2-connected and has approximately $2n$ edges. We present a first attack on the problem of describing what the random planar graph looks like.

1 Introduction

The basic questions which we will be considering are the following.

Problem 1. How does one generate a random simple planar graph uniformly at random from the set of simple planar graphs on n vertices?

Problem 2. What does this random planar graph look like?

First we clarify the issue. While there is a vast literature and long history of methods of generating random plane configurations such as Voronoi polygons, Delauney triangulations and the like, these are *not* random in the sense of being *uniformly* at random over the set of all planar graphs and are just ad hoc, fast, appealing methods of generating random plane configurations.

There is an intimate relationship between problems of counting and uniform generation and there is considerable literature on the problems of counting plane graphs and maps with a prescribed number of edges (see for example Tutte [13], Cori [2], Liskovets [7, 8], Cori and Vauquelin [3], Wormald [14, 15]). On the other hand, a few works are devoted to the random generation of certain types of planar maps and graphs (see [4, pp 74–83], [11]). However as far as we can see there is very little known about the two fundamental questions raised above.

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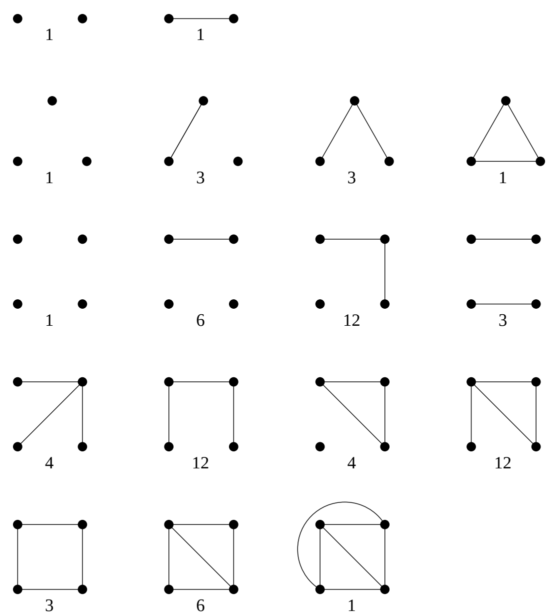

Figure 1: The (planar) graphs with 2, 3 and 4 vertices.

Consider first the collection of unlabelled simple planar graphs on n vertices. Call this set $\mathcal{U}(n)$ and denote its cardinality by $u(n)$. For example $u(2) = 2$, $u(3) = 4$ and $u(4) = 11$. Similarly let $\mathcal{L}(n)$ and $l(n)$ denote the set (respectively number) of simple planar *labelled* graphs on n vertices. Clearly $l(n) \geq u(n)$, for example the members of $\mathcal{U}(3)$ shown above give rise to respectively $1, 3, 3, 1$ distinct member of $\mathcal{L}(3)$ so that $l(3) = 8$. Figure 1 shows the set of unlabelled planar graphs with 2, 3 and 4 vertices. The number below each graph counts the associated labelled graphs.

In general, it is the labelled structures which are easier to deal with and it is these on which we shall be concentrating here.

In the recent book [10] the first terms of the sequence $(u(n))_{n\geq 1}$ are given: 1, 2, 4, 11, 33, 142, 822, 6910. It is easy to find the very first terms of $(l(n))_{n\geq 1}$: 1, 2, 8, 64, 1023.

The precise formulation of the question raised in problem 1 and to which we shall devote most of our attention is the following.

Problem 1(a). Does there exist an algorithm A which outputs a random planar subgraph of K_n and runs in time bounded by some polynomial function of n (written in unary)?

We are doubtful whether such an algorithm exists. First consider the exhaustive algorithm of listing all planar subgraphs of K_n and choosing one at random. There are $2^{\binom{n}{2}}$ subgraphs of K_n and $l(n)$, the number of these which are planar, is exponential (see section 6 below) so this approach cannot provide an answer. We next consider a randomised, slightly speeded up version of the above which can be applied to any input graph G and actually works well provided G is "close to planar".

(1) Generate a random subgraph of G by deleting each edge independently with probability $\frac{1}{2}$. Call the resulting graph R.

(2) If R is planar then $R_p = R$ else repeat.

(3) Output R_p.

When $G = K_n$ this randomised algorithm certainly gives a random planar graph. However it is extremely slow.

A more general version of problem 1 is the following:

Problem 1(b). Does there exist a polynomial time algorithm which for any input graph G will output a planar subgraph $R = R(G)$ chosen uniformly at random from all planar subgraphs of G?

We conjecture not.

2 A Markov chain algorithm

Let $G = (V, E)$ be any simple graph. We define a Markov chain $M(G)$ with state space all planar subgraphs of G and with transitions defined as follows.

A *position* of G consists of an unordered pair of distinct vertices of G. If X_t denotes the state $M(G)$, at time t, then X_{t+1} is chosen as follows. A position f of G is chosen uniformly at random.

(a) If the position f contains an edge e of X_t then $X_{t+1} = X_t \backslash e$.

(b) If the position $f = (i, j)$ does not contain an edge in X_t then X_{t+1} is formed from X_t by adding an edge (i, j) provided this addition preserves planarity,

(c) otherwise $X_{t+1} = X_t$.

It is easy to verify that (X_t) is an irreducible aperiodic Markov chain whose transition matrix is symmetric. Thus, X_t has a limiting stationary distribution which is uniform over the set of planar subgraphs of G. In principle therefore it gives an easily implemented algorithm for generating a planar subgraph of G which will be approximately uniformly at random. The closeness of the approximation will be governed by the mixing rate of the chain, and this will depend on the graph G. In particular, when $G = K_n$ it gives what appears to be a fairly effective way of generating a random planar graph.

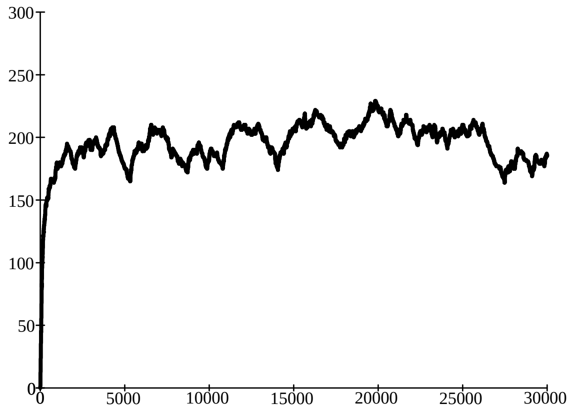

Figure 2: A typical simulation. Number of edges versus time-steps.

3 Experimental results

We present here the results of our experiments with this Markov Chain. The program was written in C++ using the LEDA library [9].

For practical reasons, we have usually chosen the empty graph as the initial state of the simulation. In each simulation, given n the number of vertices, we arbitrarily fix the number of time-steps to $3n^2$, which from our earlier pilot studies seems sufficiently large for the chain to settle down to what we believe is its equilibrium state.

Figure 2 shows the evolution of the number of edges during one execution of the program on a graph with 100 vertices. The curve increases rapidly then oscillates around a value near to 200. This can be seen more precisely in Figure 3 which presents average values of 50 simulations. The same experiment has been repeated on graphs with a number of vertices varying between 1 and 100. Figure 4 clearly suggests a linear relation between the number of vertices and the number of edges of a random planar graph. This result was obtained by computing the average number of edges of 10 graphs for each value of n varying from 1 to 100.

The next questions which we consider are the probabilities of a random planar graph being connected or biconnected. Figure 5 suggests that almost all planar graphs are connected: the probability of being connected seems to tend to 1 or to a value very near to 1 when n goes to infinity. On the contrary, the proportion of biconnected graphs decreases rapidly, as shown in Figure 6. These two experiments were done on 100 graphs for each value of n.

Finally we present in Figure 7 the distribution of the degrees of the vertices of 50 random planar graphs with 100 vertices. More precisely, in the random

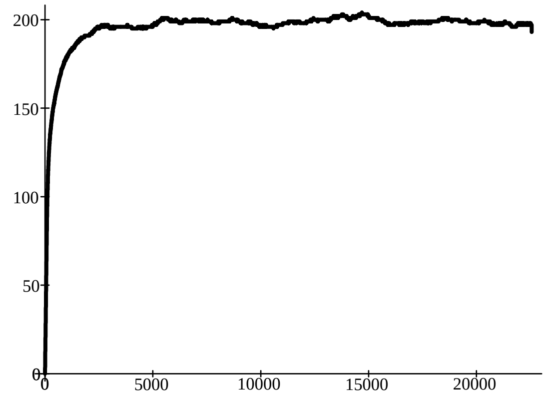

Figure 3: Number of edges versus time-steps: average values of 50 simulations.

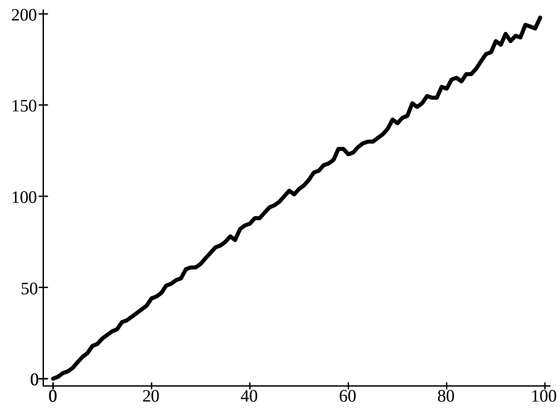

Figure 4: Average number of edges versus number of vertices.

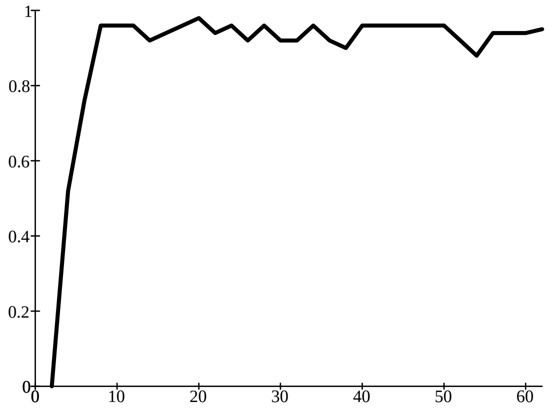

Figure 5: Experimental probability of being connected versus number of vertices.

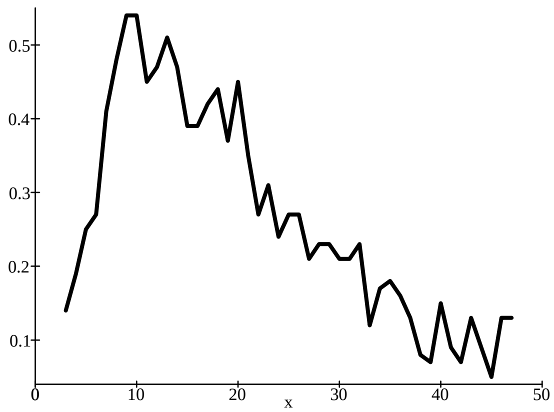

Figure 6: Experimental probability of being biconnected versus number of vertices.

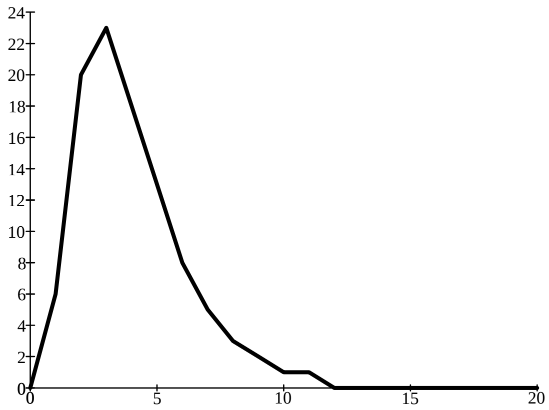

Figure 7: Distribution of the degrees of vertices.

planar graph on 100 vertices we would expect the values below:

degree	0	1	2	3	4	5	6	7	8	9	10	11	> 11
#vertices	0	6	20	23	18	13	8	5	3	2	1	1	0

In order to verify that our results do not depend on the initial state, other experiments were done with maximal planar graphs as initial states. Results seem to be equivalent. For instance, Figure 8 presents such a simulation for a graph with 100 vertices.

4 Properties of the random planar graph

We will denote by $R(G)$ the random planar subgraph of G, and when $G = K_n$, will abbreviate $R(K_n)$ to R_n.

If $e(R_n)$ denotes the expected number of edges in R_n, then our experimental evidence suggests that

$$\lim_{n\to\infty} n^{-1}e(R_n) = C$$

exists and that C is a constant fairly close to 2.

There is also a heuristic but wrong argument in support of the constant C being exactly 2. It runs as follows: the expected number of vertices of a planar map (as defined in [13]) with k edges is $\frac{k}{2}+1$ (by duality and Euler's formula). Also, the expected number of vertices of a 3-connected unlabelled planar graph with k edges is $\frac{k}{2}+1$ (because 3-connected planar graphs are in 1-to-1 correspondence with 3-connected planar maps, and the set of 3-connected planar maps is closed under duality).

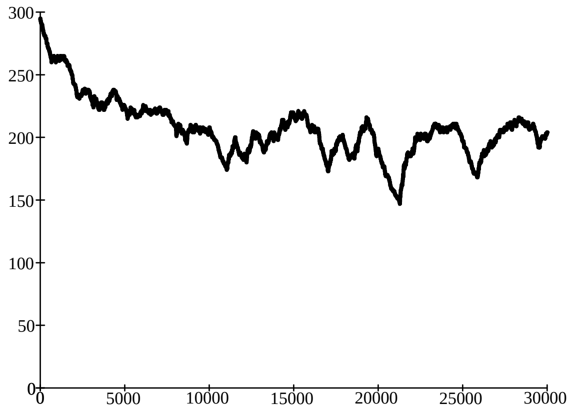

Figure 8: A typical simulation with a maximal planar graph as initial state.

It is obvious that $e(R_n) \leq 3n - 6$ but better upper bounds seem difficult to find. What we can prove is:

Theorem 1 *The expected number of edges in R_n is at least $(3n-6)/2$.*

The proof is an almost immediate consequence of a more general result.

Theorem 2 *Let E be a finite set, and $\mathcal{D}$ a family of subsets of E such that*

1. $X \in \mathcal{D}\ , Y \subset X \ \Rightarrow\ Y \in \mathcal{D}$,

2. *all maximal members of $\mathcal{D}$ have same cardinality m.*

Then the expected number of elements of a random member of $\mathcal{D}$ is at least $m/2$.

Proof of Theorem 2. Let $\mathcal{A} = (A_1, ..., A_k)$ be the collection of maximal members of $\mathcal{D}$. We will prove the theorem by induction on k. Clearly, since each A_i has cardinality m, the theorem is true (with equality) when $k = 1$. Now assume it is true for families with k or fewer maximal members and consider the family $\mathcal{D}$ having $A_1, ..., A_{k+1}$ as maximal members (all of cardinality m).

Let $\mathcal{D}'$ be the family defined by

$$X \in \mathcal{D}' \Leftrightarrow \{X \subset A_i : 1 \leq i \leq k\}.$$

Then if $R(\mathcal{D})$ denotes a random member of $\mathcal{D}$, the expected size of R, written $\langle R(\mathcal{D})\rangle$, is given by

$$\begin{aligned} \langle R(\mathcal{D})\rangle &= \sum_{X \in \mathcal{D}} |X|/|\mathcal{D}| \\ &= \frac{\sum_{X\in\mathcal{D}'} |X| + \sum_{X\in\mathcal{D}\backslash\mathcal{D}'} |X|}{|\mathcal{D}'| + |\mathcal{D}\backslash\mathcal{D}'|}. \end{aligned}$$

We say that a collection of subsets $\mathcal{U}$ is *closed above* or *monotone increasing* if $X \in \mathcal{U},\ Y \supseteq X \ \Rightarrow\ Y \in \mathcal{U}$. We now use the following easy application of the FKG inequality [6].

Lemma 3 *Let E be any finite set, $\mathcal{U}$ any collection of subsets of E closed above, then*

$$\left(\sum_{X \in \mathcal{U}} |X| \right) / |\mathcal{U}| \geq \frac{1}{2}|E|.$$

Proof. Define f, g on 2^E by

$$\begin{aligned} f(Y) &= |Y| \quad Y \subseteq E, \\ g(Y) &= \begin{cases} 1 & Y \in \mathcal{U} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then the FKG inequality gives for any positively correlated measure $\mu : 2^E \to \mathbf{R}^+$

$$\Sigma\mu(Y)\Sigma f(Y)g(Y)\mu(Y) \geq \Sigma f(Y)\mu(Y)\Sigma g(Y)\mu(Y)$$

where in all cases the sum is over all subsets of E. Taking $\mu(Y) = 1$ forall Y gives

$$2^{|E|} \sum_{Y \in \mathcal{U}} |Y| \geq \left(\sum_{Y \subset E} |Y| \right) |\mathcal{U}|$$

as required. $\square$

Applying the lemma with $E = A_{k+1}$ gives

$$\frac{\sum_{X \in \mathcal{D} \setminus \mathcal{D}'} |X|}{|\mathcal{D} \setminus \mathcal{D}'|} \geq \frac{|A_{k+1}|}{2} = \frac{m}{2}.$$

Now, by the induction hypothesis,

$$\sum_{X \in \mathcal{D}'} \frac{|X|}{|\mathcal{D}'|} = \frac{c}{d} \geq \frac{m}{2}.$$

But if $c/d \geq m/2$ and $u/v \geq m/2$ then

$$\frac{c+u}{d+v} \geq \frac{m}{2}$$

which completes the proof of Theorem 2. $\square$

Proof of Theorem 1 Take E to be the edge set of K_n in Theorem 2 and let a subset $X \in \mathcal{U}$ iff X is the edge set of a planar subgraph of K_n. $\square$

We now turn to the relationship between R_n, the random planar graph, and the well understood random graph $G(n,p)$, (see Bollobás [1]).

First an elementary result which may be intuitively obvious but which we feel is worth spelling out. If π is any property of graphs, then we write $G \in \pi$ to signify that G has π. In other words we are identifying a property π with a class of graphs closed under isomorphism.

Lemma 4 *For any graph property π,*

$$Pr\{R_n \in \pi\} = Pr\{G(n, \frac{1}{2}) \in \pi \mid G(n, \frac{1}{2}) \text{ is planar}\}$$

Proof. Let us call $\pi(n)$ the set of graphs with n vertices having property π. Then

$$\begin{aligned}
&Pr\{G(n, \frac{1}{2}) \in \pi \mid G(n, \frac{1}{2}) \ \text{ is planar}\} \\
&\quad = \frac{Pr\{G(n, \frac{1}{2}) \in \pi \ \text{ and } \ G(n, \frac{1}{2}) \ \text{ is planar}\}}{Pr\{G(n, \frac{1}{2}) \ \text{ is planar}\}} \\
&\quad = \frac{|\pi(n) \cap \mathcal{L}(n)|}{2^{\binom{n}{2}}} . \frac{2^{\binom{n}{2}}}{l(n)} \\
&\quad = Pr\{R_n \in \pi\}
\end{aligned}$$

□

An immediate consequence of this is the following.

We say that a property π is *monotone increasing* (respectively *decreasing*) if for any graph $G \in \pi$ any supergraph (respectively subgraph) of G having the same set of vertices also has π. Then using the FKG inequality we get:

Proposition 5 *Let π be any monotone property of graphs then*

(a) $Pr\{R_n \in \pi\} \geq Pr\{G(n, \frac{1}{2}) \in \pi\}$ *if π is decreasing*

(b) $Pr\{R_n \in \pi\} \leq Pr\{G(n, \frac{1}{2}) \in \pi\}$ *if π is increasing.*

For example, taking π to be the property of being connected, all it tells us is the intuitively obvious result that

$$Pr\{R_n \text{ is connected}\} \leq Pr\{G(n, \frac{1}{2}) \text{ is connected}\}$$

and it is well known that the right hand side tends to 1 as $n \to \infty$.

A more interesting comparison is between the behaviour of R_n, which we believe typically has about $2n$ edges, and the random graph $G(n, p(n))$, where $p(n) \sim 4/n$ is chosen so that the number of edges agree.

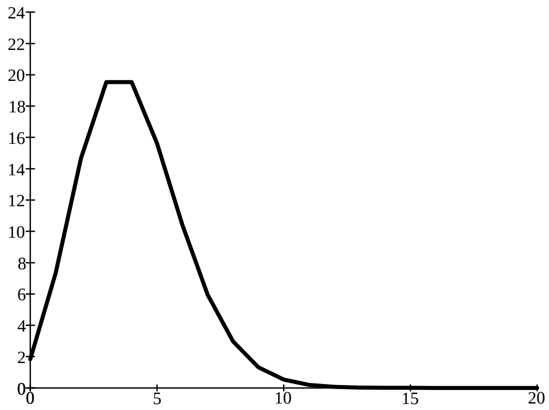

Figure 9: Distribution of the degrees of vertices in $G(n, 4/n)$.

Elementary results from random graph theory, see Bollobas [1, p.57], show that the number of vertices of degree k in $G(n,p)$ is asymptotically Poisson with parameter λ_k, and that

$$\lambda_k = n \binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

Thus the expected number of vertices of degree k in $G(n, 4/n)$ is

$$D_k(n) \sim n \frac{4^k}{k!} e^{-4} \text{ as } n \to \infty.$$

It is interesting to compare these, as shown in Figure 9, with our experimental results on degrees of R_n (Figure 7).

A more striking difference between R_n and $G(n, \frac{4}{n})$ is that almost certainly $G(n, \frac{4}{n})$ is disconnected for large n, whereas our simulations suggest R_n is connected. Intuitively this can be explained as follows, with about $2n$ edges to distribute, they have to be far more "spread out" in R_n than in $G(n, \frac{4}{n})$. This helps connectivity.

5 Connectivity properties

First we consider the probability $i(n)$ that a specific vertex of R_n, say vertex 1, is isolated. This is given by

$$i(n) = \frac{l(n-1)}{l(n)}.$$

We believe that $i(n)$ is monotone decreasing but note that showing this is equivalent to showing that

$$l(n)^2 \leq l(n+1)l(n-1),$$

in other words that the sequence $l(n)$ is log concave. Proving such inequalities tends to be extremely difficult.

Elementary computations for small n indicate also that

$$p_I(n) = Pr\{R_n \text{ has an isolated vertex}\}$$

decreases fairly rapidly. For example we have

n	1	2	3	4	5
$p_I(n)$	1	1/2	1/2	23/64	256/1023

We believe that $lim_{n\to\infty} p_I(n) = 0$ but have only been able to show:

Theorem 6 *The probability that R_n has an isolated vertex is $\Omega(n^{-10})$.*

Proof Suppose that (X_t) the Markov chain on n-vertex planar graphs is in equilibrium. Let (Z_t) be the Markov process defined by

$$Z_t = \begin{cases} 1 & \text{if } X_t \text{ has an isolated vertex} \\ 0 & \text{otherwise.} \end{cases}$$

There must be at least one vertex of degree ≤ 5 in X_t. Let it be v and let $i_1, ..., i_k$ be the neighbours of v. Then $Z_{t+5} = 1$ if the random mechanism governing X_t chooses, in some order, the positions $(v, i_1)...(v, i_k)$ in its next k transitions and avoids them in the remaining $5-k$ transitions. The probability of this is at least Cn^{-10}. $\square$

We now consider the probability that R_n, the random planar graph, is connected. If we denote this probability by $p_c(n)$ then clearly

$$p_c(n) = l_c(n)/l(n),$$

where $l_c(n)$ denotes the number of connected members of $\mathcal{L}(n)$, and for small n we get

n	2	3	4	5
$p_c(n)$	1/2	1/2	19/32	727/1023

From Theorem 6 $p_c(n) \leq 1 - Cn^{-10}$ as $n \to \infty$ but we believe that, as with the general random graph,

$$\lim_{n\to\infty} p_c(n) = 1.$$

We are unable to prove this but the following result indicates a certain drift towards there being only one connected component in R_n:

Proposition 7 *Let (Z_t) be the Markov process which counts the number of connected components in the graph X_t (having n vertices). Let us denote, for any $t \geq 0$ and for any $1 \leq i, j \leq n-1$, $P_t\{i \to j\} = Pr\{Z_{t+1} = j | Z_t = i\}$. Then, for $t \geq 0$ and $1 \leq k \leq n-1$,*

$$P_t\{k \to k+1\} \leq \frac{n-k}{\binom{n}{2}},$$

$$P_t\{k+1 \to k\} \geq \frac{k(n-k) + \frac{k(k-1)}{2}}{\binom{n}{2}}$$

and, for $1 \leq i, j \leq n$ and $|i-j| > 1$,

$$P_t\{i \to j\} = 0.$$

Proof. Suppose that X_t has k connected components and let i be the number of isthmi in X_t. Then obviously $P_t\{k \to k+1\} = i/\binom{n}{2}$. Now the number of isthmi in a graph with n vertices and k connected components is at most $n-k$ (this value can be reached only if the graph is a forest of trees). This gives the first inequality of the proposition.

On the other hand, if X_t has $k+1$ connected components with respective cardinalities $c_1, c_2, \ldots, c_{k+1}$, then the number of ways to add an edge in order that X_{t+1} has k connected components is

$$\sum_{\substack{i=1\ldots k+1 \\ j<i}} c_i c_j$$

The minimum of this expression is reached when $c_1 = c_2 = \ldots = c_k = 1$ and $c_{k+1} = n-k$ subject to the obvious constraints that $1 \leq c_i$ and $\Sigma c_i = n$ or some permutation of these values. To see this write

$$2\sum_{i<j} c_i c_j = (\Sigma c_i)^2 - \Sigma c_1^2 = n^2 - \Sigma c_i^2.$$

Hence the problem reduces to maximising Σc_i^2 subject to the same constraints. The result follows by standard dynamic programming arguments. Then

$$\sum_{\substack{i=1\ldots k+1 \\ j<i}} c_i c_j \geq k(n-k) + \frac{k(k-1)}{2}$$

and this gives the second inequality. □

We deduce immediately the corollary:

Corollary 8

$$\frac{P_t\{k \to k+1\}}{P_{t'}\{k+1 \to k\}} \le \frac{1}{k} \qquad \forall 1 \le k \le n-1, \quad t, t' \ge 0.$$

Proposition 9 *Let (Y_t) be an ergodic Markov chain having state space $\{1, \ldots, n\}$ and transition probabilities as in Proposition 7. Then, at equilibrium,*

$$Pr\{Y_t = 1\} \ge \frac{1}{e}.$$

Proof. Let $\pi = (\pi_1, \pi_2, \ldots \pi_n)$ denote the stationary distribution of (Y_t), and $M = (a_{ij})$ its transition matrix. Then π satisfies the following system of equations:

$$\begin{cases} \pi_1 &= a_{11}\pi_1 + a_{21}\pi_2 \\ \pi_2 &= a_{12}\pi_1 + a_{22}\pi_2 + a_{32}\pi_3 \\ \ldots & \\ \pi_k &= a_{k-1,k}\pi_{k-1} + a_{k,k}\pi_k + a_{k+1,k}\pi_{k+1} \quad (2 \le k \le n-1) \\ \ldots & \\ \pi_n &= a_{n-1,n}\pi_{n-1} + a_{n,n}\pi_n \end{cases}$$

Moreover, we know that M is stochastic, that is $a_{i,i-1} + a_{i,i} + a_{i,i+1} = 1 \ \forall i$. Then by induction we can prove that

$$\pi_{k+1} = \frac{a_{k,k+1}}{a_{k+1,k}}\pi_k \quad 1 \le k \le n-1,$$

and we deduce from Corollary 8 that

$$\pi_{k+1} \le \frac{\pi_k}{k} \qquad 1 \le k \le n-1.$$

Since $\sum_{i=1}^n \pi_i = 1$, we get, as required,

$$\begin{aligned} 1 &\le \pi_1 \sum_{i=1}^{n} \frac{1}{(i-1)!} \\ &\le \pi_1 e. \end{aligned}$$

$\square$

6 Associated counting problems

In order to be more precise in our estimates above we need to understand better the behaviour of quantities such as $l_c(n)$ and $l(n)$. Crude counting arguments show that $\log l(n) = \theta(n \log n)$ as $n \to \infty$ and similarly for $l_c(n)$. Greater precision seems difficult. What we can prove is:

Lemma 10 $l_c(n) \geq (6n-16)l_c(n-1)$.

Proof. Let G be a graph of $\mathcal{L}_c(n-1)$. First, suppose that G is maximal planar. Let us count the number of ways to create a graph of $\mathcal{L}_c(n)$ by adding the vertex n. We can

- attach n to one vertex of G : there are $n-1$ possibilities;
- or attach n to the two extremities of one edge of G : there are $3n-9$ possibilities;
- or attach n to the three vertices of one face (in the unique planar representation of G): there are $2n-6$ possibilities if $n > 4$.

There are no more possibilities. This gives $6n-16$ ways of constructing a graph of $\mathcal{L}_c(n)$ from a maximal member of $\mathcal{L}_c(n-1)$.

If G is not maximal, then add some "virtual" edges to obtain a maximal graph G' which contains G, and then apply the previous constructions.

So the formula is true for $n > 4$. It is true too for $n \leq 4$ since $l_c(1) = 1$, $l_c(2) = 1$, $l_c(3) = 4$ and $l_c(4) = 38$. □

Corollary 11 $l(n) \geq (6n-15)l(n-1)$.

Proof. It suffices to add the case where n is connected to none of the other vertices. □

More generally, we can prove

Lemma 12 *If n is large enough that s exists satisfying the equation*

$$\left(\frac{n-1}{s-1}\right)^{\left(\frac{n-1}{s-1}-2\right)} \geq 6s-16.$$

then $l_c(n) \geq (6s-16)l(n-1)$.

For example this gives $l_c(n) \geq 20l(n-1)$ provided $n \geq 26$.

Proof. Let $G \in \mathcal{L}(n-1)$. First, fix $s > 1$ and suppose that there exists in G a connected component with at least s vertices. We can construct a graph of $\mathcal{L}_c(n)$ by attaching the vertex n to each connected component of G, using the method given in the proof of lemma 10. This gives at least $6s-16$ ways to construct the new graph.

Now suppose that all connected components of G have less than s vertices. We construct a graph of $\mathcal{L}_c(n)$ as follows: attach n to one vertex of each component of G, then attach together the neighbours of n in a way such that the subgraph containing only the neighbours of n is a tree. Let k be the number of connected components of G: $k \geq \frac{n-1}{s-1}$. The number of labelled trees (Cayley trees) with k nodes is equal to k^{k-2}.

So, we have $l_c(n) \geq (6s-16)l(n-1)$, provided that $(\frac{n-1}{s-1})^{(\frac{n-1}{s-1}-2)} \geq 6s-16$. $\square$

We believe that

$$\lim_{n\to\infty} \frac{l_c(n)}{l(n)} = 1$$

but cannot prove it. However, it would not be surprising if, for large n, the random planar graph had very few automorphisms, see for example the remark [13, page 138]. If this is the case it would be useful to have better understanding of the unlabelled counting problem. For this we obtain the following results:

Theorem 13 *There exists $\theta > 0$ such that*

$$\lim_{n\to\infty} (u_c(n))^{\frac{1}{n}} = \theta$$

and

$$\frac{256}{27} \leq \theta \leq 8\frac{256}{27}.$$

Proof. Let $\mathcal{M}$ be the set of *maximal* planar (unlabelled) graphs. Given an integer n, any graph of $\mathcal{U}(n)$ can be constructed from some graph of $\mathcal{M}(n)$ by deleting some edges. Thus, since there are $3n-6$ edges in a maximal planar graph, $u(n) \leq 2^{3n-6}m(n)$.

Tutte [12] has given the number of planar triangulations with n vertices. Let us consider the triangulations whose external face has degree 3. Their number is (using Tutte's notation)

$$\psi_{n-3,0} = \frac{2(4n-11)!}{(n-2)!(3n-7)!} \sim \frac{729\sqrt{2}\sqrt{3}}{2097152\sqrt{\pi}n^{5/2}}\left(\frac{256}{27}\right)^n.$$

Such a triangulation can be considered as a maximal planar graph in which one face[1] is distinguished (the external one), and the three vertices of this face are labelled a, b and c (see [12]). Then

$$u_c(n) \leq u(n) \leq 2^{3n-6}m(n) \leq 2^{3n-6}\psi_{n-3,0}$$

and finally we get

$$u_c(n) = O\left(n^{-\frac{5}{2}}\left(8\frac{256}{27}\right)^n\right).$$

Now let $\mathcal{B}$ Be the set of connected birooted graphs constructed as follows: take a graph which is not maximal planar in $\mathcal{U}_c$, choose two non adjacent vertices

[1]The faces are well defined here because there exists only one planar representation on the sphere of any maximal planar graph.

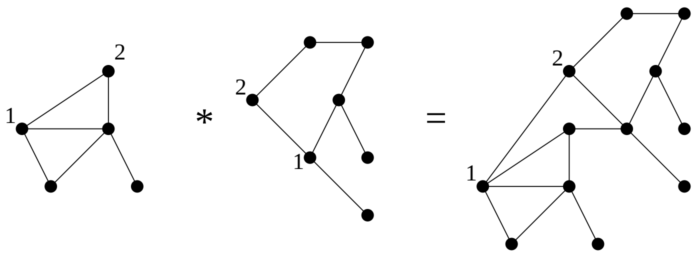

Figure 10: The operation $*$.

which lie in the same face in a planar representation of the graph and distinguish them as the first root r_1 and the second root r_2. Then create an edge between r_1 and r_2. It is easy to see that

$$b(n) = O\left(\left(8\frac{256}{27}\right)^n\right). \tag{1}$$

Now we define a binary operation in $\mathcal{B}$, as illustrated in Figure 10. Let G_1 and G_2 be in $\mathcal{B}$. The graph $G = G_1 * G_2$ is defined as follows: create an edge between the first root of G_1 and the second root of G_2 and an edge between the first root of G_2 and the second root of G_1. The first root of G_1 becomes the first root of G while the second root of G_2 becomes the second root of G.

We easily see that G belongs to $\mathcal{B}$: indeed, if we remove the edge between the two roots and forget the rooting, the resulting graph belongs to $\mathcal{U}_c$. On the other hand, $G_1 * G_2 = G'_1 * G'_2 \Rightarrow G_1 = G'_1$ and $G_2 = G'_2$. To see this, observe that, given $G = G_1 * G_2$, we find G_1 and G_2 by deleting the edge between the two roots of G and then, in the resulting graph, deleting the unique isthmus crossed by any path between the two roots of G. The second root of G_1 and the first root of G_2 are the extremities of this isthmus. Now notice that if G_1 and G_2 belong respectively to $\mathcal{B}(n_1)$ and $\mathcal{B}(n_2)$, then $G_1 * G_2$ belongs to $\mathcal{B}(n_1 + n_2)$. Thus

$$b(n_1 + n_2) \geq b(n_1)b(n_2) \quad \forall n_1, n_2. \tag{2}$$

From expressions (1) and (2) and the fundamental theorem of supermultiplicative functions we deduce that there exists θ such that $\lim_{n\to\infty} (b(n))^{1/n} = \theta$. Since the number of maximal planar graphs is such that $\lim_{n\to\infty} (m(n))^{1/n} = \frac{256}{27}$ we get

$$\lim_{n\to\infty} (u_c(n))^{1/n} = \theta.$$

□

We deduce the following corollary by standard asymptotic considerations (see for example [5]):

Corollary 14

$$\lim_{n\to\infty} u(n)^{1/n} = \theta.$$

This lends greater credence to our belief that as $n \to \infty$ the probability that R_n is connected tends to 1.

7 Conclusion

Of the many problems left open in the above the most pressing is deciding whether or not the Markov chain we propose is indeed rapidly mixing. Settling this would be greatly helped by a better knowledge of the random planar graph. However this seems a difficult combinatorial problem, and even a good upper bound on its number of edges is elusive.

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