Università degli Studi di Roma "La Sapienza"

Dottorato di Ricerca in Matematica
XII ciclo

tesi

Cellular Automata and Finitely Generated Groups

di

Francesca Fiorenzi

Dipartimento di Matematica
Università di Roma "La Sapienza"
Piazzale Aldo Moro 2 - 00185 Roma
e–mail fiorenzi@mat.uniroma1.it

Contents

Introduction

The notion of a cellular automaton has been introduced by Ulam [U] and von Neumann [vN]. In this classical setting, the "universe" is the lattice of integers $\mathbf{Z}^n$ of Euclidean space $\mathbf{R}^n$. The *set of states* is a finite set A (also called the *alphabet*) and a *configuration* is a function $c : \mathbf{Z}^n \to A$. Time t goes on in discrete steps and represents a *transition function* $\tau : A^{\mathbf{Z}^n} \to A^{\mathbf{Z}^n}$ (if c is the configuration at time t, then $\tau(c)$ is the configuration at time $t+1$), which is deterministic and *local*. *Locality* means that the new state at a point $\gamma \in \mathbf{Z}^n$ at time $t+1$ only depends on the states of certain fixed points in the neighborhood of γ at time t. More precisely, if c is the configuration reached from the automaton at time t then $\tau(c)_{|\gamma} = \delta(c_{|N_\gamma})$, where $\delta : A^N \to A$ is a function defined on the configurations with support the finite set N (the neighborhood of the point $\mathbf{0} \in \mathbf{Z}^n$), and $N_\gamma = \gamma + N$ is the neighborhood of γ obtained from N by translation. For these structures, Moore [Moo] has given a sufficient condition for the existence of the so–called *Garden of Eden (GOE) patterns*, that is those configurations with finite support that cannot be reached at time t from another configuration starting at time $t-1$ and hence can only appear at time $t = 0$. Moore's condition (i.e. the existence of *mutually erasable patterns* – a sort of non–injectivity of the transition function on the "finite" configurations) was also proved to be necessary by Myhill [My]. This equivalence between "local injectivity" and "local surjectivity" of the transition function is the classical well–known *Garden of Eden theorem.*

The purpose of this work is to consider this kind of problems in the more general framework of symbolic dynamic theory, with particular regard to *surjunctivity* theorems (and, in this case, to the density of the *periodic configurations*) and to GOE–like theorems restricted to the *subshifts* of the space A^Γ (where Γ is a finitely generated group and A is a finite alphabet). Indeed for these sets is still possible to define a structure of a cellular automaton.

More precisely, given a finitely generated group Γ, one can consider as "universe" the *Cayley graph* of Γ. A *configuration* is an element of the space A^Γ (the so–called *full A–shift*), that is a function defined on Γ with values in a finite alphabet A. The space A^Γ is naturally endowed with a metric and hence with an induced topology, this latter being equivalent to the usual product topology, where the topology in A is the discrete one. A subset X of A^Γ which is Γ–invariant and topologically closed is called *subshift*, *shift space* or simply *shift*. In this setting a *cellular automaton* (CA) on a shift space X is given by specifying a *transition function* $\tau : X \to X$ which is *local* (that is the value of $\tau(c)$, where $c \in X$ is a configuration, at a point $\gamma \in \Gamma$ only depends on the values of c at the points of a fixed neighborhood of γ).

In Chapter 1 we formally define all these notions, also proving that many basic results for the subshifts of $A^{\mathbf{Z}}$ given in the book of Lind and Marcus [LinMar], can be generalized to the subshifts of A^Γ. For example, we prove that the topological definition of shift is equivalent to the combinatorial one of set of configurations avoiding some *forbidden blocks*.

Moreover we prove that a function between shift spaces is local if and only if it is continuous and commutes with the Γ–action. This fact, together with the compactness of the shift spaces (notice that in A^Γ closeness and compactness are equivalent), implies that the inverse of an invertible cellular automaton is also a cellular automaton and allows us to call *conjugacy* each invertible local function between two shifts. The *invariants* are those properties which are invariant under conjugacy.

A fundamental notion we give is that of *irreducibility* for a shift; in the one–dimensional case, this means that given any pair of words u, v in the *language* of the shift (i.e. the set of all finite words appearing in some bi–infinite configuration), there is a word w such that the concatenation uwv still belongs to the language. We generalize this definition to a generic shift, using the patterns and their supports rather than the words.

Then, in terms of forbidden blocks the notion of *shift of finite type* is given and we prove that, as in the one–dimensional case, such a shift has a useful "overlapping" property that will be necessary in later chapters. Moreover we recall from [LinMar] the notions of *edge shift* and of *sofic shift*, and restate many of the basic properties of these one–dimensional shifts strictly connected with the one–dimensional shifts of finite type.

As we have noticed, cellular automata have mainly been investigated in the Euclidean case and for the full shifts. The difference between the one–dimensional cellular automata and the higher dimensional ones, is very deep. For example, Amoroso and Patt have shown in [AP] that there are algorithms to decide surjectivity and injectivity of local maps for one–dimensional cellular automata. On the other hand Kari has shown in [K1] and [K2] that both the injectivity and the surjectivity problems are undecidable for Euclidean cellular automata of dimension at least two. Here we extend the Amoroso–Patt's results to the one–dimensional cellular automata over shifts of finite type.

Finally, the notion of *entropy* for a generic shift as defined by Gromov in [G] is also given; we see that this definition involves the existence of a suitable sequence of sets and we prove that, in the case of non–exponential growth of the group, these sets can be taken as balls centered at 1 and with increasing radius. Moreover, we see that the entropy is an invariant of the shifts. Then, following Lind and Marcus [LinMar], we recall its basic properties in the one–dimensional case, also stating the principal result of the Perron–Frobenius theory to compute it.

A selfmapping $\tau : X \to X$ on a set X is *surjunctive* if it is either non–injective or surjective. In other words a function is surjunctive if the implication injective $\Rightarrow$ surjective holds. Using the GOE theorem and the compactness of the space $A^{\mathbf{Z}^n}$, Richardson has proved in [R] that a transition function $\tau : A^{\mathbf{Z}^n} \to A^{\mathbf{Z}^n}$ is surjunctive. In Chapter 2, we consider the surjunctivity of the transition function in a general cellular automaton over a group Γ.

A configuration of a shift is *periodic* if its Γ–orbit is finite; after proving some generalities about periodic configurations, we recall the class of *residually finite* groups, proving that a group Γ is residually finite if and only if the periodic configurations are dense in A^Γ.

Hence if Γ is a residually finite group, a transition function τ of a cellular automaton on A^Γ is surjunctive. In fact, we prove more generally that if the periodic configurations of a subshift $X \subseteq A^\Gamma$ are dense, then a transition function on X is surjunctive. We also prove that the density of the periodic configurations is an invariant of the shifts, as is the number of the periodic configuration with a fixed period.

The remaining part of the chapter is devoted to establish for which class of shifts the periodic configurations are dense. We prove the density of the periodic configurations for an irreducible subshift of finite type of $A^{\mathbf{Z}}$ and hence, a sofic shift being the image under a local map of a shift of finite type, the density of the periodic configurations for an irreducible sofic subshift of $A^{\mathbf{Z}}$. We see that these results cannot be generalized to higher dimensions.

If the alphabet A is a finite group G and Γ is abelian, the periodic configurations of a subshift which is also a subgroup of G^Γ (namely a *group shift*), are dense (this result is a consequence of a more general theorem in [KitS2]). Moreover, this further hypothesis is not still included in the result about irreducible shifts of finite type; indeed a group shift is of finite type but there are examples of group shifts which are not irreducible. Finally, we list some other well–known decision problems for Euclidean shifts proving that in the special case of a one–dimensional shift they can be solved; more generally they can be solved for the class of group shifts using some results due to Wang [Wa] and Kitchens and Schmidt [KitS1].

In Chapter 3, we consider generalizations of the *Moore's property* and *Myhill's property* to a generic shift. In details, the GOE–theorem has been proved by Machì and Mignosi [MaMi] more generally for cellular automata in which the space of configurations is the whole A–shift A^Γ and the group Γ has *non–exponential growth*; more recently it has been proved by Ceccherini–Silberstein, Machì and Scarabotti [CeMaSca] for the wider class of the *amenable groups*.

Instead of the non–existence of mutually erasable patterns, we deal with the notion of *pre–injectivity* (a function $\tau : X \subseteq A^\Gamma \to A^\Gamma$ is *pre–injective* if whenever two configurations $c, \bar{c} \in X$ differ only on a finite non–empty subset of Γ, then $\tau(c) \neq \tau(\bar{c})$); this notion has been introduced by Gromov in [G]. In fact, we prove that these two properties are equivalent for local functions defined on the full shift, but in the case of proper subshifts the former may be meaningless. On the other hand, we give a proof of the fact that the non–existence of GOE patterns is equivalent to the non–existence of GOE configurations, that is to the surjectivity of the transition function. Hence, in this language, the GOE theorem states that τ is surjective if and only if it is pre–injective. We call *Moore's property* the implication surjective $\Rightarrow$ pre–injective and *Myhill's property* the inverse one.

Concerning one–dimensional shifts, from the works of Hedlund [H] and Coven and Paul [CovP] we prove that the Moore–Myhill property (MM–property) holds for irreducible shifts of finite type of $A^{\mathbf{Z}}$. Moreover, using this result we prove that Myhill's property holds for irreducible sofic shifts of $A^{\mathbf{Z}}$. On the other hand, we give a counterexample of an irreducible sofic shift $X \subseteq A^{\mathbf{Z}}$ but not of finite type for which Moore's property does not hold.

Concerning general cellular automata over amenable groups, from a result of Gromov [G] in a more general framework, it follows that the MM–property holds for shifts of *bounded propagation* contained in A^{Γ}. We generalize this result showing that the MM–property holds for *strongly irreducible* shifts of finite type of A^{Γ} (and we also show that strong irreducibility together with the finite type condition is strictly weaker that the bounded propagation property).

The main difference between irreducibility and strong irreducibility is easily seen in the one–dimensional case. Here the former property states that given any two words u, v in the language of a shift, there exists a third word w such that the concatenation uwv is still in the language. Strong irreducibility says that we can arbitrarily fix the length of this word (but it must be greater than a fixed constant only depending on the shift). The same properties for a generic shift refers to the way in which two different patterns in the language of the shift may appear simultaneously in a configuration. For irreducibility we have that two patterns always appear simultaneously in some configuration if we translate their supports. Strong irreducibility states that if the supports of the pattern are far enough, than it is not necessary to translate them in order to find a configuration in which both the patterns appear.

These two irreducibility conditions are not equivalent, not even in the one–dimensional case. Hence our general results about strongly irreducible shifts of finite type are strictly weaker than the one–dimensional ones. In the attempt of using weaker hypotheses in the latter case, we give a new notion of irreducibility, the *semi–strong irreducibility.* In the one–dimensional case, this property means that the above word w may "almost" be of the length we want (provided that it is long enough): we must allow it to be "a little" longer or "a little" shorter; the length of this difference is bounded and only depends on the shift. In general, semi–strong irreducibility states that if the supports of the patterns are far enough from each other, than translating them "a little" we find a configuration in which both the patterns appear. The reason for this choice lies in the fact that using the Pumping Lemma we can prove that a sofic subshift of $A^{\mathbf{Z}}$ is irreducible if and only if is semi–strongly irreducible. Moreover Myhill's property holds for semi–strongly irreducible shifts of finite type of A^{Γ} if Γ has non–exponential growth.

1. Cellular Automata

In this chapter we give the notion of a cellular automaton over a finitely generated group. This notion generalizes the definition given by Ulam [U] and von Neumann [vN]. In that classical setting, the "universe" is the lattice of integers $\mathbf{Z}^n$ of Euclidean space $\mathbf{R}^n$. The *set of states* is a finite set A (also called the *alphabet*) and a *configuration* is a function $c : \mathbf{Z}^n \to A$. Time t goes on in discrete steps and represents a *transition function* $\tau : A^{\mathbf{Z}^n} \to A^{\mathbf{Z}^n}$ (if c is the configuration at time t, then $\tau(c)$ is the configuration at time $t+1$), which is deterministic and *local*. *Locality* means that the new state at a point $\gamma \in \mathbf{Z}^n$ at time $t+1$ only depends on the states of certain fixed points in the neighborhood of γ at time t. More precisely, if c is the configuration reached from the automaton at time t then $\tau(c)_{|\gamma} = \delta(c_{|N_\gamma})$, where $\delta : A^N \to A$ is a function defined on the configurations with support the finite set N (the neighborhood of the point $\mathbf{0} \in \mathbf{Z}^n$), and $N_\gamma = \gamma + N$ is the neighborhood of γ obtained from N by translation.

Our purpose is, given a finitely generated group Γ, to consider as "universe" the *Cayley graph* of Γ. A *configuration* is an element of the space A^Γ, that is a function defined on Γ with values in a finite alphabet A. The space A^Γ is naturally endowed with a metric and hence with an induced topology, this latter being equivalent to the usual product topology, where the topology in A is the discrete one. A subset X of A^Γ which is Γ–invariant and topologically closed is called *subshift*, *shift space* or simply *shift*. In this setting a *cellular automaton* (*CA*) on a shift space X is given by specifying a *transition function* $\tau : X \to X$ which is *local* (that is the value of $\tau(c)$, where $c \in X$ is a configuration, at a point $\gamma \in \Gamma$ only depends on the values of c at the points of a fixed neighborhood of γ).

Sections 1.1 and 1.2 of this chapter are dedicated to give a formal definition of all these notions, also proving that many basic results for the subshifts of $A^{\mathbf{Z}}$ given in the book of Lind and Marcus [LinMar], can be generalized to the subshifts of A^Γ. For example, we prove that the topological definition of shift is equivalent to the combinatorial one of set of configurations avoiding some *forbidden blocks*. Moreover we prove that a function between shift spaces is local if and only if it is continuous and commutes with the Γ–action. This fact, together with the compactness of the shift spaces (notice that in A^Γ closeness and compactness are equivalent), implies that the inverse of an invertible cellular automaton is also a cellular automaton and allows us to call *conjugacy* each invertible local function between two shifts. The *invariants* are those properties which are invariant under conjugacy. Finally we extend to a general shift the notion of *irreducibility* and of *being of finite type*.

In Sections 1.4 and 1.5, we recall from [LinMar] the notions of *edge shift* and of *sofic shift*, and restate many of the basic properties of these one–dimensional shifts strictly connected with the one–dimensional shifts of finite type.

In Section 1.6, we generalize the Amoroso–Patt's results about the decision problem of the surjectivity and the injectivity of local maps for one–dimensional cellular automata on the full shift (see [AP] and [Mu]). Our results concern one–dimensional cellular automata over shifts of finite type.

Finally, in Section 1.7 the notion of *entropy* for a generic shift as defined by Gromov in [G] is given. We see that this definition involves the existence of a suitable sequence of sets and we prove that, in the case of non–exponential growth of the group, these sets can be taken as balls centered at 1 and with increasing radius. Moreover, we see that the entropy is an invariant of the shifts.

Then, following Lind and Marcus [LinMar, Chapter 4], we recall basic properties of the entropy in the one–dimensional case.

1.1 Cayley Graphs of Finitely Generated Groups

In this section, we give the basic notion of *Cayley graph* of a finitely generated group; as we have said, this graphs will constitute the "universe" of our class of cellular automata. Moreover, we recall the definition of *growth* of a finitely generated group Γ.

Let Γ be a finitely generated group and $\mathcal{X}$ a fixed finite set of generators for Γ; then each $\gamma \in \Gamma$ can be written as

$$\gamma = x_{i_1}^{\delta_1} x_{i_2}^{\delta_2} \dots x_{i_n}^{\delta_n} \tag{1.1}$$

where the x_{i_j}'s are generators and $\delta_j \in \mathbf{Z}$.

We define the *length of* γ *(with respect to* $\mathcal{X}$*)* as the natural number

$$\|\gamma\|_{\mathcal{X}} := \min\{|\delta_1| + |\delta_2| + \cdots + |\delta_n| \mid \gamma \text{ is written as in (1.1)}\}$$

so that Γ is naturally endowed with a metric space structure, with the distance given by

$$\mathrm{dist}_{\mathcal{X}}(\alpha, \beta) := \|\alpha^{-1}\beta\|_{\mathcal{X}} \tag{1.2}$$

and

$$D_n^{\mathcal{X}} := \{\gamma \in \Gamma \mid \|\gamma\|_{\mathcal{X}} \le n\}$$

is the disk of radius n centered at 1. Notice that $D_1^{\mathcal{X}}$ is the set $\mathcal{X} \cup \mathcal{X}^{-1} \cup \{1\}$. The asintotic properties of the group being independent on the choice of the set of generators $\mathcal{X}$, from now on we fix a set $\mathcal{X}$ which is also symmetric (i.e. $\mathcal{X}^{-1} = \mathcal{X}$) and we omit the index $\mathcal{X}$ in all the above definitions.

For each $\gamma \in \Gamma$, the set D_n provides, by left translation, a *neighborhood of* γ, that is the set $\gamma D_n = D(\gamma, n)$. Indeed, if $\alpha \in \gamma D_n$ then $\alpha = \gamma\beta$ with $\|\beta\| \leq n$. Hence $\mathrm{dist}(\alpha, \gamma) = \|\alpha^{-1}\gamma\| = \|\beta^{-1}\| \leq n$. Conversely, if $\alpha \in D(\gamma, n)$ then $\|\gamma^{-1}\alpha\| \leq n$ (that is $\gamma^{-1}\alpha \in D_n$), and $\alpha = \gamma\, \gamma^{-1}\alpha$.

Now we define, for each $E \subseteq \Gamma$ and for each $n \in \mathbf{N}$, the following subsets of Γ: the *n–closure of E*

$$E^{+n} := \bigcup_{\alpha \in E} D(\alpha, n);$$

the *n–interior of E*

$$E^{-n} := \{\alpha \in E \mid D(\alpha, n) \subseteq E\};$$

the *n–boundary of E*

$$\partial_n E := E^{+n} \backslash E^{-n};$$

the *n–external boundary of E*

$$\partial_n^+ E := E^{+n} \backslash E$$

and, finally, the *n–internal boundary of E*

$$\partial_n^- E := E \backslash E^{-n}.$$

For all these sets, we will omit the index n if $n = 1$.

The *Cayley graph* of Γ, is the graph in which Γ is the set of vertices and there is an edge from γ to $\bar{\gamma}$ if there exists a generator $x \in \mathcal{X}$ such that $\bar{\gamma} = \gamma x$. Obviously this graph depends on the presentation of Γ. For example, we may look at the classical cellular decomposition of Euclidean space $\mathbf{R}^n$ as the Cayley graph of the group $\mathbf{Z}^n$ with the presentation $\langle a_1, \ldots, a_n \mid a_i a_j = a_j a_i \rangle$.

If $\mathbf{G} = (\mathcal{V}, \mathcal{E})$ is a graph with set of vertices $\mathcal{V}$ and set of edges $\mathcal{E}$, the *graph distance* (or *geodetic distance*) between two vertices $v_1, v_2 \in \mathcal{V}$ is the minimal length of a path from v_1 to v_2. Hence the distance defined in (1.2) coincides with the graph distance on the Cayley graph of Γ.

We recall that the function $g : \mathbf{N} \to \mathbf{N}$ defined by

$$g(n) := |D_n|$$

which counts the elements of the disk D_n, is called *growth function* of Γ (with respect to $\mathcal{X}$). One can prove that the limit

$$\lambda := \lim_{n \to \infty} g(n)^{\frac{1}{n}}$$

always exists; if $\lambda > 1$ then, for all sufficiently large n,

$$g(n) \geq \lambda^n,$$

and the group Γ has *exponential growth*. If $\lambda = 1$, we distinguish two cases. Either there exists a polynomial $p(n)$ such that for all sufficiently large n

$$g(n) \leq p(n),$$

in which case Γ has *polynomial growth*, or Γ has *intermediate growth* (i.e. $g(n)$ grows faster than any polynomial in n and slower then any exponential function x^n with $x > 1$). Moreover, it is possible to prove that the type of growth is a property of the group Γ (i.e. it does not depend on the choice of a set of generators); for this reason we deal with the *growth of a group*. This notion has been indipendently introduced by Milnor [Mil], Efremovič [E] and Švarc [Š]; it is very useful in the theory of cellular automata.

1.2 Shift Spaces and Cellular Automata

Let A be a finite *alphabet*; in the classical theory of cellular automata, the "universe" is the Cayley graph of the free abelian group $\mathbf{Z}^n$ and a *configuration* is an element of $A^{\mathbf{Z}^n}$, that is a function $c : \mathbf{Z}^n \to A$ assigning to each point of the graph a letter of A. We generalize this notion taking as universe a Cayley graph of a generic finitely generated group Γ and taking suitable subsets of configurations in A^Γ.

Let A be a finite set (with at least two elements) called *alphabet*; on the set A^Γ (i.e. the set of all functions $c : \Gamma \to A$), we have a natural metric and hence a topology. This latter is equivalent to the usual product topology, where the topology in A is the discrete one. By Tychonoff's theorem, A^Γ is also compact. An element of A^Γ is called a *configuration*.

If $c_1, c_2 \in A^\Gamma$ are two configurations, we define the distance

$$\mathrm{dist}(c_1, c_2) := \frac{1}{n+1}$$

where n is the least natural number such that $c_1 \neq c_2$ in D_n. If such an n does not exist, that is if $c_1 = c_2$, we set their distance equal to zero.

Observe that the group Γ acts on A^Γ on the right as follows:

$$(c^\gamma)_{|\alpha} := c_{|\gamma\alpha}$$

for each $c \in A^\Gamma$ and each $\gamma, \alpha \in \Gamma$ (where $c_{|\alpha}$ is the value of c at α).

Now we give a topological definition of a *shift space* (briefly *shift*); we will see in the sequel that this definition is equivalent (in the Euclidean case) to the classical combinatorial one.

Definition 1.2.1 A subset X of A^Γ is called a *shift* if it is topologically closed and Γ–invariant (i.e. $X^\Gamma = X$).

For every $X \subseteq A^\Gamma$ and $E \subseteq \Gamma$, we set

$$X_E := \{c_{|E} \mid c \in X\};$$

a *pattern of* X is an element of X_E where E is a non–empty finite subset of Γ. The set E is called the *support of* the pattern; a *block of* X is a pattern of X with support a disk. The *language of* X is the set $L(X)$ of all the blocks of X. If X is a subshift of $A^{\mathbf{Z}}$, a configuration is a bi–infinite word and a block of X is a finite word appearing in some configuration of X.

Hence a pattern with support E is a function $p : E \to A$. If $\gamma \in \Gamma$, we have that the function $\bar{p} : \gamma E \to A$ defined as $\bar{p}_{|\gamma\alpha} = p_{|\alpha}$ (for each $\alpha \in E$), is the pattern obtained copying p on the translated support γE. Moreover, if X is a shift, we have that $\bar{p} \in X_{\gamma E}$ if and only if $p \in X_E$. For this reason, in the sequel we do not make distinction between p and $\bar{p}$ (when the context makes it possible). For example, a word $a_1 \dots a_n$ is simply a finite sequence of symbols for which we do not specify (if it is not necessary), if the support is the interval $[1, n]$ or the interval $[-n, -1]$.

Definition 1.2.2 Let X be a subshift of A^Γ; a function $\tau : X \to A^\Gamma$ is *M–local* if there exists $\delta : X_{D_M} \to A$ such that for every $c \in X$ and $\gamma \in \Gamma$

$$(\tau(c))_{|\gamma} = \delta((c^\gamma)_{|D_M}) = \delta(c_{|\gamma\alpha_1}, c_{|\gamma\alpha_2}, \dots, c_{|\gamma\alpha_m}),$$

where $D_M = \{\alpha_1, \dots, \alpha_m\}$.

In this definition, we have assumed that the alphabet of the shift X is the same as the alphabet of its image $\tau(X)$. In this assumption there is no loss of generality because if $\tau : X \subseteq A^\Gamma \to B^\Gamma$, one can always consider X as a shift over the alphabet $A \cup B$.

Definition 1.2.3 Let Γ be a finitely generated group with a fixed symmetric set of generators $\mathcal{X}$, let A be a finite alphabet with at least two element and let D_M the disk in Γ centered at 1 and with radius M. A *cellular automaton* is a triple (X, D_M, τ) where X is a subshift of the compact space A^Γ, D_M is the neighborhood of 1 and $\tau : X \to X$ is an M–local function.

Let $\tau : X \to A^\Gamma$ be a M–local function; if c is a configuration of X and E is a subset of Γ, $\tau(c)_{|E}$ only depends on $c_{|E^{+M}}$. Thus we have a family of functions $(\tau_{E^{+M}} : X_{E^{+M}} \to \tau(X)_E)_{E \subseteq \Gamma}$.

The following characterization of local functions is, in the one–dimensional case, known as the Curtis–Lyndon–Hedlund theorem. Here we generalize it to a general local function.

Proposition 1.2.4 *A function $\tau : X \to A^\Gamma$ is local if and only if it is continuous and commutes with the Γ–action (i.e. for each $c \in X$ and each $\gamma \in \Gamma$, one has $\tau(c^\gamma) = \tau(c)^\gamma$).*

PROOF Suppose that τ is M–local and that $D_M = \{\alpha_1, \ldots, \alpha_m\}$; then for $\gamma \in \Gamma$ and $c \in X$,

$$(\tau(c^\gamma))_{|\alpha} = \delta((c^\gamma)_{|\alpha\alpha_1}, (c^\gamma)_{|\alpha\alpha_2}, \ldots, (c^\gamma)_{|\alpha\alpha_m}) = \delta(c_{|\gamma\alpha\alpha_1}, \ldots, c_{|\gamma\alpha\alpha_m}).$$

On the other hand

$$((\tau(c))^\gamma)_{|\alpha} = (\tau(c))_{|\gamma\alpha} = \delta(c_{|\gamma\alpha\alpha_1}, \ldots, c_{|\gamma\alpha\alpha_m}).$$

Hence τ commutes with the Γ–action. We prove that τ is continuous. A generic element of a sub–basis of A^Γ is

$$\xi := \{c \in A^\Gamma \mid c_{|\alpha} = a\}$$

with $\alpha \in \Gamma$ and $a \in A$. It suffices to prove that the set

$$\bar{\xi} := \tau^{-1}(\xi) = \{c \in X \ \mid \tau(c)_{|\alpha} = a\}$$

is open in X. Actually, if $c \in \bar{\xi}$ and

$$r := \max(\|\alpha\alpha_1\|, \ldots, \|\alpha\alpha_m\|), \tag{1.3}$$

we claim that the ball $B_X(c, \frac{1}{r+1})$ is contained in $\bar{\xi}$. Indeed, if $\bar{c} \in B_X(c, \frac{1}{r+1})$ then

$$\mathrm{dist}(c, \bar{c}) < \frac{1}{r+1}$$

and on D_r we have $c_{|D_r} = \bar{c}_{|D_r}$; since, from (1.3), $\alpha\alpha_i \in D_r$, we have $\tau(\bar{c})_{|\alpha} = \delta(\bar{c}_{|\alpha\alpha_1}, \ldots, \bar{c}_{|\alpha\alpha_m}) = \delta(c_{|\alpha\alpha_1}, \ldots, c_{|\alpha\alpha_m}) = \tau(c)_{|\alpha} = a$.

Conversely, suppose that τ is continuous and commutes with the action of Γ. Since X is compact, τ is uniformly continuous; fix M in $\mathbf{N}$ such that for every $c, \bar{c} \in X$,

$$\mathrm{dist}(c, \bar{c}) < \frac{1}{M+1} \Rightarrow \ \mathrm{dist}(\tau(c), \tau(\bar{c})) < 1.$$

Hence, if c and $\bar{c}$ agree on D_M, then $\tau(c)$ and $\tau(\bar{c})$ agree at 1, that is, for every $c \in X$,

$$(\tau(c))_{|1} = \delta(c_{|\alpha_1}, \ldots, c_{|\alpha_m})$$

where $D_M = \{\alpha_1, \ldots, \alpha_m\}$.

In general, since τ commutes with the action of Γ, we have

$$(\tau(c))_{|\alpha} = (\tau(c))_{|\alpha 1} = ((\tau(c))^\alpha)_{|1} = (\tau(c^\alpha))_{|1} = \delta(c_{|\alpha\alpha_1}, \ldots, c_{|\alpha\alpha_m})$$

showing that τ is M–local. □

From the previous theorem, it is clear that *the composition of two local functions is still local*. In any case, this can be easily seen by a direct proof that follows Definition 1.2.2.

Now, fix $\gamma \in \Gamma$ and consider the function $X \to A^\Gamma$ that associates with each $c \in X$ its translated configuration c^γ. In general, this function does not commute with the Γ–action (and therefore it is not local). Indeed, if Γ is not abelian and $\gamma\alpha \neq \alpha\gamma$, then $(c^\gamma)^\alpha \neq (c^\alpha)^\gamma$. However, this function is continuous. In order to see this, if $n \geq 0$, fix a number $m \geq 0$ such that $\gamma D_n \subseteq D_m$; if $\mathrm{dist}(c, \bar{c}) \leq \frac{1}{m+1}$, then c and $\bar{c}$ agree on D_m and hence on γD_n. If $\alpha \in D_n$, we have $c_{|\gamma\alpha} = \bar{c}_{|\gamma\alpha}$ and then ${c^\gamma}_{|\alpha} = {\bar{c}^\gamma}_{|\alpha}$ that is c^γ and $\bar{c}^\gamma$ agree on D_n so that $\mathrm{dist}(c^\gamma, \bar{c}^\gamma) < \frac{1}{n+1}$.

This result will be necessary in the proof of Theorem 1.2.6, below.

Observe that if X is a subshift of A^Γ and $\tau : X \to A^\Gamma$ is a local function, then, by Proposition 1.2.4, the image $Y := \tau(X)$ is still a subshift of A^Γ. Indeed Y is closed (or, equivalently, compact) and Γ–invariant:

$$Y^\Gamma = (\tau(X))^\Gamma = \tau(X^\Gamma) = \tau(X) = Y.$$

Moreover, if τ is injective then $\tau : X \to Y$ is a homeomorphism; if $c \in Y$ then $c = \tau(\bar{c})$ for a unique $\bar{c} \in X$ and we have

$$\tau^{-1}(c^\gamma) = \tau^{-1}(\tau(\bar{c})^\gamma) = \tau^{-1}(\tau(\bar{c}^\gamma)) = \bar{c}^\gamma = (\tau^{-1}(c))^\gamma$$

that is, τ^{-1} commutes with the Γ–action. By Proposition 1.2.4, τ^{-1} is local. Hence the well–known theorem (see [R]), stating that the inverse of an invertible Euclidean cellular automaton is a cellular automaton, holds also in this more general setting. In the one–dimensional case, Lind and Marcus [LinMar, Theorem 1.5.14] give a direct proof of this fact.

This result leads us to give the following definition.

Definition 1.2.5 Two subshifts $X, Y \subseteq A^\Gamma$ are *conjugate* if there exists a local bijective function between them (namely a *conjugacy*). The *invariants* are the properties of a shift invariant under conjugacy.

Now we prove that the topological definition of a shift space is equivalent to a combinatorial one involving the avoidance of certain *forbidden blocks*: this fact is well–known in the Euclidean case.

Theorem 1.2.6 *A subset $X \subseteq A^\Gamma$ is a shift if and only if there exists a subset $\mathcal{F} \subseteq \bigcup_{n\in\mathbf{N}} A^{D_n}$ such that $X = X_\mathcal{F}$, where*

$$X_\mathcal{F} := \{c \in A^\Gamma \mid {c^\alpha}_{|D_n} \notin \mathcal{F} \text{ for every } \alpha \in \Gamma, n \in \mathbf{N}\}.$$

In this case, $\mathcal{F}$ is a set of forbidden blocks *of X.*

PROOF Suppose that X is a shift. X being closed, for each $c \notin X$ there exists an integer $n_c > 0$ such that the ball

$$B_{A^\Gamma}(c, \frac{1}{n_c}) \subseteq \complement X.$$

Let $\mathcal{F}$ be the set

$$\mathcal{F} := \{c_{|D_{n_c}} \mid c \notin X\};$$

we prove that $X = X_{\mathcal{F}}$. If $c \notin X_{\mathcal{F}}$, there exists $\bar{c} \notin X$ such that ${c^{\alpha}}_{|D_{n_{\bar{c}}}} = \bar{c}_{|D_{n_{\bar{c}}}}$ for some $\alpha \in \Gamma$. Then $\mathrm{dist}(c^{\alpha}, \bar{c}) < \frac{1}{n_{\bar{c}}}$ and hence $c^{\alpha} \in B_{A^{\Gamma}}(\bar{c}, \frac{1}{n_{\bar{c}}}) \subseteq \complement X$ which implies $c \notin X$, by the Γ–invariance of X. This proves that $X \subseteq X_{\mathcal{F}}$. For the other inclusion, if $c \notin X$ then, by definition of $\mathcal{F}$, $c_{|D_{n_c}} \in \mathcal{F}$ and hence $c \notin X_{\mathcal{F}}$.

Now, for the converse, we have to prove that a set of type $X_{\mathcal{F}}$ is a shift. Observe that

$$X_{\mathcal{F}} = \bigcap_{p \in \mathcal{F}} X_{\{p\}}$$

and, if $\mathrm{supp}(p) = D_n = \{\alpha_1, \ldots, \alpha_N\}$,

$$X_{\{p\}} = \bigcap_{\alpha \in \Gamma} \{c \in A^{\Gamma} \mid {c^{\alpha}}_{|D_n} \neq p\} = \bigcap_{\alpha \in \Gamma} \Big(\bigcup_{\alpha_i \in D_n} \{c \in A^{\Gamma} \mid {c^{\alpha}}_{|\alpha_i} \neq p_{|\alpha_i}\} \Big).$$

Thus, in order to prove that $X_{\mathcal{F}}$ is closed, it suffices to prove that for any $i = 1, \ldots, N$ the set

$$\{c \in A^{\Gamma} \mid {c^{\alpha}}_{|\alpha_i} \neq p_{|\alpha_i}\} \tag{1.4}$$

is closed. We have

$$(\{c \in A^{\Gamma} \mid {c^{\alpha}}_{|\alpha_i} \neq p_{|\alpha_i}\})^{\alpha} = \{c \in A^{\Gamma} \mid c_{|\alpha_i} \neq p_{|\alpha_i}\}$$

and then the set in (1.4) is closed being the pre–image of a closed set under a continuous function. Finally we have to prove that $X_{\mathcal{F}}$ is Γ–invariant. If $\gamma \in \Gamma$ and $c \in X_{\mathcal{F}}$, for every $\alpha \in \Gamma$ and every $n \in \mathbf{N}$ we have ${c^{\gamma\alpha}}_{|D_n} \notin \mathcal{F}$ that is ${(c^{\gamma})^{\alpha}}_{|D_n} \notin \mathcal{F}$ and hence $c^{\gamma} \in X_{\mathcal{F}}$ □

Now we give the first, fundamental notion of irreducibility for a one–dimensional shift and we see how to generalize this notion to a generic shift.

Definition 1.2.7 A shift $X \subseteq A^{\mathbf{Z}}$ is *irreducible* if for each pair of words $u, v \in L(X)$, there exists a word $w \in L(X)$ such that the concatenated word $uwv \in L(X)$.

The natural generalization of this property to any group Γ is the following.

Definition 1.2.8 A shift $X \subseteq A^{\Gamma}$ is *irreducible* if for each pair of patterns $p_1 \in X_E$ and $p_2 \in X_F$, there exists an element $\gamma \in \Gamma$ such that $E \cap \gamma F = \emptyset$ and a configuration $c \in X$ such that $c_{|E} = p_1$ and $c_{|\gamma F} = p_2$.

In other words, a shift is irreducible if whenever we have $p_1, p_2 \in L(X)$, there exists a configuration $c \in X$ in which these two blocks appear simultaneously on disjoint supports. This definition could seem weaker than Definition 1.2.7, in fact in this latter we establish that each word $u \in L(X)$ must always appear in a configuration on the left of each other word of the language. In order to prove that the two definitions agree, suppose that $X \subseteq A^{\mathbf{Z}}$ is an irreducible shift according with Definition 1.2.8. If u, v are words in $L(X)$, there exists a configuration $c \in X$ such that $c_{|E} = u$ and $c_{|F} = v$ where E and F are finite and disjoint intervals. If $\max E < \min F$ then there exists a word w such that $uwv \in L(X)$ (where $w = c_{|I}$ and I is the interval $[\max E+1, \min F-1]$). If, otherwise, $\max F < \min E$ there exists a word w such that $vwu \in L(X)$; consider the word vwu two times, there exists another word x such that $vwu\ x\ vwu \in L(X)$ and hence $uxv \in L(X)$.

1.3 Shifts of Finite Type

Now we give the fundamental notion of shift of finite type. The basic definition is in terms of forbidden words; in a sense we may say that a shift is of finite type if we can decide whether or not a configuration belongs to the shift only checking its words of a fixed (and only depending on the shift) length. This fact implies an useful characterization of the one–dimensional shifts of finite type, a sort of "overlapping" property for the words of the language. We prove that this overlapping property still holds for a generic shift of finite type.

Definition 1.3.1 A shift is *of finite type* if it admits a finite set of forbidden blocks.

If X is a shift of finite type, since a finite set $\mathcal{F}$ of forbidden blocks of X has a maximal support, we can always suppose that each block of $\mathcal{F}$ has the disk D_M as support (indeed each block that contains a forbidden block is forbidden). In this case the shift X is called *M–step* and the number M is called the *memory of X*. If X is a subshift of $A^{\mathbf{Z}}$, we define the memory of X as the number M, where $M+1$ is the maximal length of a forbidden word.

For the shifts of finite type in $A^{\mathbf{Z}}$, we have the following useful property.

Proposition 1.3.2 [LinMar, Theorem 2.1.8] *A shift $X \subseteq A^{\mathbf{Z}}$ is an M–step shift of finite type if and only if whenever $uv, vw \in L(X)$ and $|v| \geq M$, then $uvw \in L(X)$.*

Now we prove that this "overlapping" property holds more generally for subshifts of finite type of A^{Γ}.

Proposition 1.3.3 *Let X be an M–step shift of finite type and let E be a subset of Γ. If $c_1, c_2 \in X$ are two configurations that agree on $\partial^+_{2M} E$, then the configuration $c \in A^{\Gamma}$ that agrees with c_1 on E and with c_2 on $\complement E$ is still in X.*

Proof Let c_1, c_2 and c be three configurations as in the statement; if $\mathcal{F}$ is a finite set of forbidden blocks for X, each having $D_M = \{\alpha_1, \ldots, \alpha_m\}$ as support, we have to prove that for each $\gamma \in \Gamma$, $c^{\gamma}{}_{|D_M} \notin \mathcal{F}$. Observe that either

$$\gamma D_M \subseteq E^{+2M} \quad \text{or} \quad \gamma D_M \subseteq \complement E;$$

indeed, if $\gamma \in E^{+M}$ then by definition $\gamma D_M \subseteq (E^{+M})^{+M} = E^{+2M}$. If $\gamma \notin E^{+M}$, suppose that there exists $\bar{\gamma} \in \gamma D_M \cap E$. Then, for some i, $\bar{\gamma} = \gamma\alpha_i$ and hence $\gamma = \bar{\gamma}\alpha_i^{-1} \in \bar{\gamma} D_M \subseteq E^{+M}$ which is excluded; then $\gamma D_M \subseteq \complement E$.

Now, if γ is such that $\gamma D_M \subseteq E^{+2M}$, we have $c^{\gamma}{}_{|D_M} = (c_1{}^{\gamma})_{|D_M} \notin \mathcal{F}$. If γ is such that $\gamma D_M \subseteq \complement E$, we have $c^{\gamma}{}_{|D_M} = (c_2{}^{\gamma})_{|D_M} \notin \mathcal{F}$. □

Corollary 1.3.4 *Let X be an M–step shift of finite type and let E be a finite subset of Γ; if $p_1, p_2 \in X_{E^{+2M}}$ are two patterns that agree on $\partial^+_{2M} E$, than there exist two extensions $c_1, c_2 \in X$ of p_1 and p_2, respectively, that agree on $\complement E$.*

Proof   Indeed, if $\bar{c}_1$ and $\bar{c}_2$ are two configurations extending p_1 and p_2 respectively, then they agree on $\partial^+_{2M} E$. The configuration c of Proposition 1.3.3 and the configuration $\bar{c}_2$, coincide on $\complement E$ and they are extensions of p_1 and p_2, respectively. □

1.4 Edge Shifts

A relevant class of one–dimensional subshifts of finite type, is that of *edge shifts*. This class is strictly tied up the one of finite graphs. This relation allows us to study the properties of an edge shift (possible quite complex) studying the properties of its graph. On the other hand, each one–dimensional shift of finite type is conjugate to an edge shift and hence they have the same invariants; thus also in this case the properties of the shift depend on the structure of the accepting graph.

Definition 1.4.1 Let $\mathbf{G}$ be a finite directed graph with edge set $\mathcal{E}$. The *edge shift* $X_{\mathbf{G}}$ is the subshift of $\mathcal{E}^{\mathbf{Z}}$ defined by

$$X_{\mathbf{G}} := \{(e_z)_{z\in\mathbf{Z}} \mid \mathbf{t}(e_z) = \mathbf{i}(e_{z+1}) \text{ for all } z \in \mathbf{Z}\}$$

where, for $e \in \mathcal{E}$, $\mathbf{i}(e)$ denotes the initial vertex of e and $\mathbf{t}(e)$ the terminal one.

The structure of a finite graph (and hence of an edge shift) can be easily described by a matrix, the so–called *adjacency matrix of* $\mathbf{G}$.

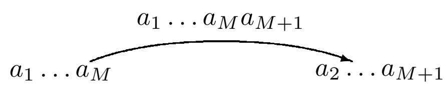
More precisely, let $\mathbf{G}$ be a graph with vertex set $\mathcal{V} = \{1, \ldots, n\}$; the *adjacency matrix of* $\mathbf{G}$ is the matrix $\mathbf{A}$ such that $\mathbf{A}_{ij}$ is the number of edges in $\mathbf{G}$ with initial state i and terminal state j.

It is easily seen that the (i, j)–entry of the matrix $\mathbf{A}^m$ (the product of $\mathbf{A}$ with itself m times), is the number of paths in $\mathbf{G}$ with length m from i to j.

The edge shift $X_{\mathbf{G}}$ is sometimes denoted as $X_{\mathbf{A}}$, where $\mathbf{A}$ is the adjacency matrix of $\mathbf{G}$. The fundamental role of the adjacency matrix will be clarified in the sequel.

Concerning the edge shifts, it is easy to see that every edge shift is a 1–step shift of finite type with set of forbidden blocks $\{ef \mid \mathbf{t}(e) \neq \mathbf{i}(f)\}$. On the other hand, each shift of finite type is conjugate to an edge shift accepted by a suitable graph $\mathbf{G}$. Now we give an effective procedure to construct it.

Let X be a shift of finite type X with memory M, the vertices of $\mathbf{G}$ are the words of $L(X)$ of length M and the edges are the words of $L(X)$ of length $M+1$. There is an edge named $a_1 \dots a_M a_{M+1} \in L(X)$ from $a_1 \dots a_M$ to $a_2 \dots a_{M+1}$.



The edge shift accepted by this graph is the $(M+1)$th *higher block shift* of X and is denoted by $X^{[M+1]}$. Consider the function $\tau : X^{[M+1]} \to X$ defined setting, for each $c \in X^{[M+1]}$ and each $z \in \mathbf{Z}$, $\tau(c)_{|z}$ equal to the first letter of the word $c_{|z}$; τ is a bijective local function and hence a conjugacy.

$\dots$	$a_{-1}a_0 \dots a_M$	$a_0 a_1 \dots a_{M+1}$	$a_1 a_2 \dots a_{M+2}$	$\dots$
$\dots$	a_{-1}	a_0	a_1	$\dots$

The above table points out the behavior of the function τ.

1.5 Sofic Shifts

The class of *sofic shifts* has been introduced by Weiss in [Wei1] as the smallest class of shifts containing the shifts of finite type and closed under *factorization* (i.e. the image under a local map of a sofic shift is sofic). Equivalently, one can see that a shift is sofic if it is the set of all *labels* of the bi–infinite paths in a finite *labeled graph* (or *finite–state automaton*). In automata theory, this corresponds to the notion of *regular language*. Indeed a *language* (i.e. a set of finite words over a finite alphabet) is *regular* if it is the set of all labels of finite paths in a labeled graph.

Definition 1.5.1 Let A be a finite alphabet; a *labeled graph* $\mathcal{G}$ is a pair $(\mathbf{G}, \mathcal{L})$, where $\mathbf{G}$ is a finite graph with edge set $\mathcal{E}$, and the *labeling* $\mathcal{L} : \mathcal{E} \to A$ assigns to each edge e of $\mathbf{G}$ a label $\mathcal{L}(e)$ from A.

We recall that a *finite–state automaton* is a triple $(A, \mathcal{Q}, \Delta)$ where A is a finite alphabet, $\mathcal{Q}$ is a finite set whose elements are called *states* and $\Delta : \mathcal{Q} \times A \to \mathcal{P}(\mathcal{Q})$ is a function defined on the Cartesian product $\mathcal{Q} \times A$ with values in the family of all subsets of $\mathcal{Q}$. Suppose that $\bar{Q} \in \Delta(Q, a)$; this means that when the automaton “is” in the state Q and “reads” the letter a, then “it can pass” to the state $\bar{Q}$. The automaton is *deterministic* when $\Delta : \mathcal{Q} \times A \to \mathcal{Q} \cup \{\emptyset\}$, that is the state $\bar{Q}$ above (if it exists) is determined from Q and a; the automaton is *complete* when $\Delta : \mathcal{Q} \times A \to \mathcal{P}(\mathcal{Q})\backslash\{\emptyset\}$, that is for each state Q and each letter a there is at least a state $\bar{Q}$ such that $\bar{Q} \in \Delta(Q, a)$.

Clearly each general finite–state automaton determines a labeled graph setting $\mathcal{V} := \mathcal{Q}$ and drawing an edge labeled a from Q to $\bar{Q}$ if and only if $\bar{Q} \in \Delta(Q, a)$. Also the converse holds: given a labeled graph $\mathcal{G}$ with set of vertices $\mathcal{V}$, we can set $\mathcal{Q} := \mathcal{V}$ and $\Delta(i, a)$ is the set of all vertices j of the graph for which there is an edge e such that $\mathbf{i}(e) = i$, $\mathbf{t}(e) = j$ and $\mathcal{L}(e) = a$.

In the sequel we will not distinguish between a labeled graph and the related finite–state automaton.

If $\xi = \dots e_{-1}e_0e_1 \dots$ is a configuration of the edge shift $X_{\mathbf{G}}$, define the *label of the path* ξ to be

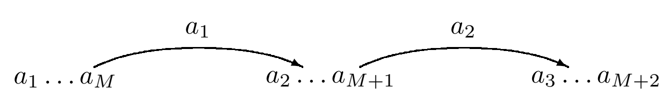
$$\mathcal{L}(\xi) = \dots \mathcal{L}(e_{-1})\mathcal{L}(e_0)\mathcal{L}(e_1) \dots \in A^{\mathbf{Z}}.$$

The set of labels of all configurations in $X_{\mathbf{G}}$ is denoted by

$$X_{\mathcal{G}} := \{c \in A^{\mathbf{Z}} \mid c = \mathcal{L}(\xi) \text{ for some } \xi \in X_{\mathbf{G}}\} = \mathcal{L}(X_{\mathbf{G}}).$$

Definition 1.5.2 A shift $X \subseteq A^{\mathbf{Z}}$ is *sofic* if $X = X_{\mathcal{G}}$ for some labeled graph $\mathcal{G}$. A *presentation* of a sofic shift X is a labeled graph $\mathcal{G}$ for which $X_{\mathcal{G}} = X$.

Obviously each edge shift is sofic. Indeed if $\mathbf{G}$ is a graph with edge set $\mathcal{E}$, we can consider the identity function on $\mathcal{E}$ as a labeling $\mathcal{L} : \mathcal{E} \to \mathcal{E}$ so that $X_{\mathbf{G}} = X_{(\mathbf{G}, \mathrm{id}_{\mathcal{E}})}$. Moreover, each shift of finite type is sofic. Consider the graph $\mathbf{G}$ introduced in the previous section that accepts the edge shift $X^{[M+1]}$; labeling the edges of $\mathbf{G}$ setting $\mathcal{L}(a_1 \dots a_M a_{M+1}) = a_1$ (that is $\mathcal{L}$ is the previous function $\tau : X^{[M+1]} \to X$), it can be easily seen that the labeled graph $\mathcal{G}$ so obtained is such that $X = X_{\mathcal{G}}$.



Notice that in a graph $\mathbf{G}$ (or in the labeled graph $\mathcal{G}$ having $\mathbf{G}$ as support), there can be a vertex from which no edges start or at which no edges end. Such a vertex is called *stranded*. Clearly no bi–infinite paths in $X_{\mathbf{G}}$ (or in $X_{\mathcal{G}}$), involve a stranded vertex, hence the stranded vertices and the edges starting or ending at them are inessential for the edge shift $X_{\mathbf{G}}$ or for the sofic shift $X_{\mathcal{G}}$.

Following Lind and Marcus [LinMar, Definition 2.2.9], a graph is *essential* if no vertex is stranded. Removing step by step the stranded vertices of $\mathbf{G}$, we get an essential graph $\bar{\mathbf{G}}$ that gives rise to the same edge shift. This procedure is effective, because $\mathbf{G}$ has a finite number of vertices. Moreover, this "essential form" of $\mathbf{G}$ is unique. This property of the edge shift $X_{\mathbf{G}}$ holds true for the sofic shift $X_{\mathcal{G}}$: the labeled graph $\bar{\mathcal{G}} = (\bar{\mathbf{G}}, \mathcal{L}_{|\mathcal{E}(\bar{\mathbf{G}})})$ is such that $X_{\mathcal{G}} = X_{\bar{\mathcal{G}}}$ (indeed $X_{\mathcal{G}} = \mathcal{L}(X_{\mathbf{G}}) = \mathcal{L}(X_{\bar{\mathbf{G}}}) = X_{\bar{\mathcal{G}}}$).

If we deal only with essential graphs, it is easy to see that *the language of a sofic shift is regular*.

Among the subshifts of $A^{\mathbf{Z}}$, *the sofic shifts are the images under a local function* (briefly *factors*) *of a shift of finite type*. Indeed each labeling is a local function defined on an edge shift and, on the other hand, the factor of a shift of finite type is also the image under a local function of a suitable edge shift (each shift of finite type being conjugate to an edge shift). This characterization allows us to call *sofic* a shift $X \subseteq A^{\Gamma}$ which is factor of a shift of finite type.

1.5.1 Minimal Deterministic Presentations of a Sofic Shift

In automata theory, a finite–state automaton is *deterministic* if, given a state Q and a letter a, there is at most one successive state $\bar{Q}$ (determined by Q and a). This corresponds, in the finite graph representing the automata, to the fact that from a vertex i (the state) there is at most one edge carrying the label a. Although this restrictive condition, one can prove that for each regular language there is a deterministic automaton accepting it. This property holds true for sofic shifts: each sofic shift admits a deterministic presentation. A *minimal deterministic presentation* of a sofic shift is a deterministic presentation with the least possible number of vertices. We will see that the irreducibility of the sofic shift implies the existence of only one (up to labeled graphs isomorphism) minimal deterministic presentation of it.

Definition 1.5.3 A labeled graph $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ is *deterministic* if, for each vertex i of $\mathbf{G}$, the edges starting at i carry different labels.

Proposition 1.5.4 [LinMar, Theorem 3.3.2] *Every sofic shift has a deterministic presentation.*

Definition 1.5.5 A *minimal deterministic presentation* of a sofic shift X is a deterministic presentation of X having the least number of vertices among all deterministic presentations of X.

One can prove that any two minimal deterministic presentations of an irreducible sofic shift are isomorphic as labeled graphs (see [LinMar, Theorem 3.3.18]), so that one can speak of *the* minimal deterministic presentation of an irreducible sofic shift.

In the following propositions, we clarify the relation between the irreducibility of a sofic (or edge) shift and the strong connectedness of its presentations.

On the other hand, it can be easily seen that the strong connectedness of $\mathbf{G}$ is equivalent to the following property of the $n \times n$ adjacency matrix $\mathbf{A}$ of $\mathbf{G}$: for each $i, j \in \{1, \ldots, n\}$, there exists an $m \in \mathbf{N}$ such that the (i, j)–entry of $\mathbf{A}^m$ is not zero.

Proposition 1.5.6 [LinMar, Lemma 3.3.10] *Let X be an irreducible sofic shift and $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ the minimal deterministic presentation of X. Then $\mathbf{G}$ is a strongly connected graph.*

Proposition 1.5.7 [LinMar, Proposition 2.2.14] *If $\mathbf{G}$ is a strongly connected graph, then the edge shift $X_{\mathbf{G}}$ is irreducible.*

As a consequence of this two facts, we have the following corollary that will be useful in the sequel.

Corollary 1.5.8 *Let X be an irreducible sofic shift and $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ the minimal deterministic presentation of X. Then the edge shift $X_{\mathbf{G}}$ is irreducible.*

As we have seen, a shift is sofic if and only if it is the image under a local function of a shift of finite type. From the previous result it follows a stronger property.

Corollary 1.5.9 *A shift is an irreducible sofic shift if and only if it is the image under a local function of an irreducible shift of finite type.*

Notice that, in general, the image under a local function of an irreducible shift is also irreducible.

1.6 Decision Problems

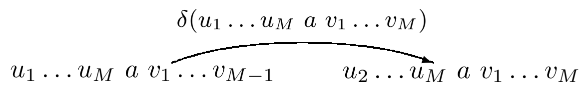
Two natural decision problems arising in the theory of cellular automata concern the existence of effective procedures to establish the surjectivity and the injectivity of the transition function. For Euclidean cellular automata over full shifts, Amoroso and Patt have shown in [AP] that there are algorithms to decide surjectivity and injectivity of local maps for one–dimensional cellular automata (see also [Mu]). On the other hand Kari has shown in [K1] and [K2] that both the injectivity and the surjectivity problems are undecidable for Euclidean cellular automata of dimension at least two.

In this section we extend the problem to cellular automata over subshifts of finite type of $A^{\mathbf{Z}}$, giving in both cases a positive answer to the existence of decision procedures.

1.6.1 A Decision Procedure for Surjectivity

If X is a shift of finite type, the problem of deciding whether or not a function $\tau : X \to X$ given in terms of local map is surjective, is decidable. In fact we can decide if a local function $\tau : X \to A^{\mathbf{Z}}$ is such that $\tau(X) \subseteq X$.

Consider a transition function $\tau : X \to A^{\Gamma}$ defined by a local rule δ; suppose (in this assumption there is no loose of generality) that X has memory $2M$ and that τ is M–local. The function τ can be represented in this way. Consider the presentation $\mathbf{G}$ of the edge shift $X^{[2M+1]}$ constructed in Section 1.4, the label of the edge $u\ a\ v \in L(X)$ (where u, v are two words in the language of X of length M), is the letter $\delta(u\ a\ v)$, that is the letter to write under a in the output tape in correspondence with $u\ a\ v$:



In this way we get a labeled graph $\mathcal{G}$ which is the presentation of the (sofic) shift $\tau(X)$. On this presentation we can check if any forbidden block of X appears in $\tau(X)$; if none of them appears, we have that $\tau : X \to X$ and hence we deal with a genuine cellular automaton.

To see whether or not the function τ is surjective, first suppose that the shift X is irreducible. Hence it can be easily seen that also $\tau(X)$ is irreducible and hence we can construct the minimal deterministic presentation of X and $\tau(X)$, respectively. These two presentations are isomorphic (as labeled graphs) if and only if $X = \tau(X)$.

In the general case, Lind and Marcus give in [LinMar, Theorem 3.4.13] an effective procedure to decide whether two labeled graph generate the same shifts.

1.6.2 A Decision Procedure for Injectivity

If X is a shift of finite type, the problem of deciding whether or not a function $\tau : X \to X$ given in terms of local map is injective, is decidable.

As we have seen in the previous subsection, we can construct a labeled graph $\mathcal{G}$ which is the presentation of the shift $\tau(X)$. From $\mathcal{G}$, we construct another graph $\mathcal{G}*\mathcal{G}$. The vertices of it are the couples (i, j) where $i, j \in \mathcal{V}(\mathcal{G})$ are vertices of $\mathcal{G}$. There is an edge $(i, j) \xrightarrow{a} (h, k)$ labeled a, if in $\mathcal{G}$ there are two edges in $\mathcal{E}(\mathcal{G})$ labeled a of kind:

$$i \xrightarrow{a} h \quad \text{and} \quad j \xrightarrow{a} k.$$

Notice that, in general, $X_{\mathcal{G}} = X_{\mathcal{G}*\mathcal{G}}$ and hence, in our case, $\mathcal{G}*\mathcal{G}$ is a presentation of $\tau(X)$.

A vertex (i, j) of $\mathcal{G} * \mathcal{G}$ is *diagonal* if $i = j$. Now, notice that the function τ is non–injective if and only if on the graph $\mathcal{G}$ there are two different bi–infinite paths with the same label. This fact is equivalent to the existence of a bi–infinite path on $\mathcal{G} * \mathcal{G}$ that involves a non–diagonal vertex. Hence, starting from the graph $\mathcal{G} * \mathcal{G}$ we construct an essential graph that accepts the same bi–infinite paths . Now it suffices to check, on this latter graph, if some non–diagonal vertex is involved.

1.7 Entropy

The *entropy* of a shift is the first invariant we deal with in the present work. It is a concept introduced by Shannon [Sha] in information theory that involves probabilistic concepts. Later Adler, Konheim and McAndrew [AdlKoM] introduced the *topological entropy* for *dynamical systems*. The entropy we deal with is a special case of topological entropy and is independent on probabilities.

In this section we give the general definition of entropy for a generic shift. We will see that this definition involves the existence of a suitable sequence of sets that, in the case of non–exponential growth of the group can be taken as balls centered at 1 and with increasing radius.

Then, following Lind and Marcus [LinMar, Chapter 4], we recall its basic properties in the one–dimensional case, also stating the principal result of the Perron–Frobenius theory to compute it.

Let $(E_n)_{n\geq 1}$ a sequence of subsets of Γ such that $\bigcup_{n\in\mathbf{N}} E_n = \Gamma$ and

$$\lim_{n\to\infty} \frac{|\partial_M E_n|}{|E_n|} = 0; \tag{1.5}$$

if $X \subseteq A^\Gamma$ is a shift, the *entropy of X respect to* $(E_n)_n$ is defined as

$$\mathrm{ent}(X) := \limsup_{n\to\infty} \frac{\log |X_{E_n}|}{|E_n|}.$$

Condition (1.5) will be necessary in the proof of Theorem 1.7.1 and other aspects of its importance will be clarified in Chapter 3. Notice that (1.5) implies $\lim_{n\to\infty} \frac{|\partial_M^+ E_n|}{|E_n|} = 0$ and $\lim_{n\to\infty} \frac{|\partial_M^- E_n|}{|E_n|} = 0$; moreover it is equivalent to the condition $\lim_{n\to\infty} \frac{|\partial E_n|}{|E_n|} = 0$.

If X is a subshift of $A^{\mathbf{Z}}$, we choose as E_n the interval $[1, n]$ (or equivalently, in order to have $\bigcup_{n\in\mathbf{N}} E_n = \Gamma$, the interval $[-n, n]$), so that X_{E_n} is the set of the words of X of length n. One can prove that in this one–dimensional case, the above maximum limit is a limit. Indeed $|X_{E_{n+m}}| \leq |X_{E_n}||X_{E_m}|$ and hence setting $a_n := \log |X_{E_n}|$, we have that the sequence $(a_n)_{n\in\mathbf{N}}$ is sub–additive, namely $a_{n+m} \leq a_n + a_m$. We want to prove that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}.$$

Fix $m \geq 1$; then there exist $q, r \in \mathbf{N}$ such that $n = mq + r$ and we have $a_n \leq qa_m + a_r$ so that $\frac{a_n}{n} \leq \frac{qa_m}{n} + \frac{a_r}{n}$. Now

$$\lim_{n\to\infty} \left(\frac{qa_m}{n} + \frac{a_r}{n}\right) = \lim_{n\to\infty} \left(\frac{q}{n} a_m + \frac{a_r}{n}\right) = \frac{a_m}{m}$$

and then $\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m}$. Hence

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \inf_{m\geq 1} \frac{a_m}{m} \leq \liminf_{n\to\infty} \frac{a_n}{n}.$$

In general, if Γ is a group of non–exponential growth, we choose as E_n a suitable disk centered at $1 \in \Gamma$; indeed, setting $a_h = |D_h|$, we have that $\lim_{h\to\infty} \sqrt[h]{a_h} = 1$ hence $\liminf_{h\to\infty} \frac{a_{h+1}}{a_h} = 1$. From this fact it follows that for a suitable sequence $(a_{h_k})_k$ we have that $\lim_{k\to\infty} \frac{a_{h_k+1}}{a_{h_k}} = 1$. Hence

$$\liminf_{k\to\infty} \frac{a_{h_k+1}}{a_{h_k-1}} = \lim_{k\to\infty} \frac{a_{h_k+1}}{a_{h_k}} \liminf_{k\to\infty} \frac{a_{h_k}}{a_{h_k-1}} = 1.$$

Then, for a suitable sequence $(a_{h_{k_n}})_n$ we have $\lim_{n\to\infty} \frac{a_{h_{k_n}+1}}{a_{h_{k_n}-1}} = 1$, that is we find a sequence of disks $E_n := D_{h_{k_n}}$ such that

$$\lim_{n\to\infty} \frac{|E_n^+|}{|E_n^-|} = 1.$$

Hence $\frac{|\partial E_n|}{|E_n|} = \frac{|E_n^+ \setminus E_n^-|}{|E_n|} \leq \frac{|E_n^+ \setminus E_n^-|}{|E_n^-|} = \frac{|E_n^+|}{|E_n^-|} - 1 \longrightarrow_{n\to\infty} 0$.

In the following theorem we prove that the entropy is an invariant of the shift.

Theorem 1.7.1 *Let X be a shift and $\tau : X \to A^\Gamma$ a local function. Then* $\mathrm{ent}(\tau(X)) \leq \mathrm{ent}(X)$ *(that is, the entropy is invariant under conjugacy).*

Proof Let τ be M–local and let $Y := \tau(X)$; we have that the function $\tau_{E_n^{+M}} : X_{E_n^{+M}} \to Y_{E_n}$ is surjective and hence

$$|Y_{E_n}| \leq |X_{E_n^{+M}}| \leq |X_{E_n}||X_{\partial_M^+ E_n}| \leq |X_{E_n}||A^{\partial_M^+ E_n}|.$$

From the previous inequalities we have

$$\frac{\log |Y_{E_n}|}{|E_n|} \leq \frac{\log |X_{E_n}|}{|E_n|} + \frac{|\partial_M^+ E_n| \log |A|}{|E_n|}$$

and hence, taking the maximum limit, $\mathrm{ent}(Y) \leq \mathrm{ent}(X)$. □

Now we see how to compute the entropy for an irreducible sofic subshift of $A^{\mathbf{Z}}$.

Let X be a sofic shift; given a presentation $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ of X, there is an effective procedure to construct, from $\mathcal{G}$ a deterministic presentation $\bar{\mathcal{G}}$ of X. This procedure is called *subset construction.* The vertices of $\bar{\mathcal{G}}$ are the non–empty subsets of the vertices of $\mathcal{G}$ and there is an edge in $\bar{\mathcal{G}}$ labeled a from I to J if J is the set of terminal vertices of edges in $\mathcal{G}$ starting at some vertex in I and labeled a.

Using combinatorial methods, it is easy to see that the entropy of a sofic shift X coincides with the entropy of the edge shift $X_{\mathbf{G}}$ of a deterministic presentation of X, as stated in the following theorem.

Proposition 1.7.2 [LinMar, Proposition 4.1.13] *Let X be a sofic shift and let $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ be a deterministic presentation of X. Then* $\operatorname{ent}(X) = \operatorname{ent}(X_{\mathbf{G}})$*.*

Given a finite graph $\mathbf{G}$, it is obvious that the adjacency matrix $\mathbf{A}$ of $\mathbf{G}$ is non–negative and, by the Perron–Frobenius Theorem, has a maximum eigenvalue $\lambda_{\mathbf{A}} > 0$, the so–called *Perron eigenvalue of* $\mathbf{A}$. This eigenvalue gives us a way to compute the entropy of an irreducible sofic shift, as stated in the following theorem.

Proposition 1.7.3 [LinMar, Theorem 4.3.3] *Let X be an irreducible sofic shift and let $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ be a strongly connected deterministic presentation of X. Then* $\operatorname{ent}(X) = \log(\lambda_{\mathbf{A}})$*.*

If X and $\mathcal{G}$ are as in the above theorem, it is clear that the graph $\mathbf{G}$ is a strongly connected deterministic presentation of $X_{\mathbf{G}}$. Hence we have another proof of Proposition 1.7.2 (as far as irreducible sofic shifts are concerned).

In fact, in [LinMar, Section 4.4] the above results are used to give a general method of computing the entropy of a generic sofic shift.

A fundamental result given in that section whose proof is based on the proposition above, is the following theorem.

Theorem 1.7.4 [LinMar, Corollary 4.4.9] *If X is an irreducible sofic shift and Y is a proper subshift of X, then* $\operatorname{ent}(Y) < \operatorname{ent}(X)$*.*

In Chapter 3, we will prove a theorem of this kind in a much more general setting.

2. Surjunctivity and Density of Periodic Configurations

A selfmapping $\tau : X \to X$ on a set X is *surjunctive* if it is either non–injective or surjective. In other words a function is surjunctive if the implication injective $\Rightarrow$ surjective holds. In this chapter, we consider the surjunctivity of the transition function in a general cellular automaton over a group Γ.

A configuration of a shift is *periodic* if its Γ–orbit is finite; after proving in Section 2.2 some generalities about periodic configurations, we recall in Section 2.3 the class of *residually finite* groups, proving that a group Γ is residually finite if and only if the periodic configurations are dense in A^Γ.

Hence if Γ is a residually finite group, a transition function τ of a cellular automaton on A^Γ is surjunctive. In fact, in Section 2.4 we prove that if the periodic configurations of a subshift $X \subseteq A^\Gamma$ are dense, then a transition function on X is surjunctive. We also prove that the density of the periodic configurations is an invariant of the shifts, as is the number of the periodic configuration with a fixed period.

The remaining part of the chapter is devoted to establish for which class of shifts the periodic configurations are dense. We prove in Section 2.5 the density of the periodic configurations for an irreducible subshift of finite type of $A^{\mathbf{Z}}$ and hence, a sofic shift being the image under a local map of a shift of finite type, the density of the periodic configurations for an irreducible sofic subshift of $A^{\mathbf{Z}}$. We see that these results cannot be generalized to higher dimensions.

In Section 2.6 we introduce the notion of a *group shift* and (as a consequence of a more general theorem in [KitS2]), we prove that for this class of shifts the periodic configurations are dense. Finally, we list some well–known decision problems for Euclidean shifts proving that in the special case of a one–dimensional shift they can be solved; more generally they can be solved for the class of group shifts using some results due to Wang [Wa] and Kitchens and Schmidt [KitS1].

2.1 Surjunctivity

In this section we investigate under which hypotheses an injective selfmapping of a set is also surjective, that is we study the class of selfmappings that are either surjective or non–injective. This property is the so–called *surjunctivity* and is due to Gottschalk (see [Gott]). Here we prove a sufficient condition for which a selfmapping of a topological space is surjunctive.

Definition 2.1.1 Let X be a set and $\tau : X \to X$ a function; τ is *surjunctive* if it is either surjective or non–injective.

The simplest example is that of a *finite* set X and a selfmapping $\tau : X \to X$; clearly every function of this kind is surjunctive; in other words, for a selfmapping of X we have injectivity $\Rightarrow$ surjectivity. Another example of surjunctive function is given by an endomorphism of a finite–dimensional vector space and by a regular selfmapping of a complex algebraic variety (see [Ax]); many others examples are given in [G]. Moreover, Richardson proves in [R] that each transition function of an Euclidean cellular automaton on the full shift is surjunctive.

Lemma 2.1.2 *Let X be a topological space, let $\tau : X \to X$ be a closed function and let $(X_i)_{i\in I}$ be a family of subsets of X such that*

- $X = \overline{\bigcup_{i\in I} X_i}$
- $\tau(X_i) \subseteq X_i$
- $\tau_{|X_i} : X_i \to X_i$ *is surjunctive*

then τ is surjunctive.

Proof If τ is injective then, for every $i \in I$, the restriction $\tau_{|X_i}$ is also injective; by the hypotheses we have $\tau(X_i) = X_i$ and hence $\bigcup_{i\in I} X_i = \bigcup_{i\in I} \tau(X_i) = \tau(\bigcup_{i\in I} X_i) \subseteq \tau(\overline{\bigcup_{i\in I} X_i}) = \tau(X)$. Then $X = \overline{\bigcup_{i\in I} X_i} \subseteq \overline{\tau(X)}$, and τ being closed we have $X \subseteq \tau(X)$. □

2.2 Periodic Configurations of a Shift

In this section we point out the fundamental subset of the *periodic* configurations of A^Γ. In the Euclidean case, we imagine that a periodic configuration is obtained "repeating" in each direction the same finite block. Hence translating such a configuration, we get only a finite number of new configurations; this property leads us to define periodic a configuration whose Γ–orbit is finite.

Definition 2.2.1 A configuration $c \in A^\Gamma$ is *n–periodic* if its orbit $c^\Gamma = \{c^\gamma \mid \gamma \in \Gamma\}$ consists of n elements; in this case n is the *period of c*. A configuration is *periodic* if it is n–periodic for some $n \in \mathbf{N}$.

From now on, P_n denotes the set of the periodic configurations whose period divides n and C_p is the set $\bigcup_{n\geq 1} P_n$ of all the periodic configurations.

In general, a configuration $c \in A^\Gamma$ is constant on the right cosets of its own stabilizer H_c (i.e. the subgroup of all $h \in \Gamma$ such that $c^h = c$). Indeed, if $h \in H_c$, we have

$$c_{|h\gamma} = (c^h)_{|\gamma} = c_{|\gamma}.$$

Hence, if c is periodic, c is constant on the right cosets of a subgroup of finite index. Now we prove that this property characterizes periodic configurations.

Proposition 2.2.2 *A configuration $c \in A^\Gamma$ belongs to P_n if and only if there exists a subgroup H of Γ with finite index dividing n, such that c is constant on the right cosets of H.*

PROOF Let c be m–periodic with $m|n$; by definition, the stabilizer H_c has finite index m and c is constant on the right cosets of H_c. Conversely, if H has finite index dividing n and c is constant on the right cosets of H, we prove that $H \subseteq H_c$. Indeed, if $h \in H$ and $\gamma \in \Gamma$ we have

$$(c^h)_{|\gamma} = c_{|h\gamma} = c_{|\gamma}$$

and hence $c^h = c$ so that $h \in H_c$. H being of finite index, H_c also has finite index and the index of H_c divides that of H so that it divides n. Hence $c \in P_n$. $\square$

The following result is well known (see, for example, [Rot]).

Lemma 2.2.3 *Let Γ be a finitely generated group; for every $n \in \mathbf{N}$ there are finitely many subgroups of Γ of finite index n.*

PROOF If $H \subseteq \Gamma$ has finite index n, fix a set $\{\gamma_1, \ldots, \gamma_n\}$ of right coset representatives of H and consider the function $\Phi : \Gamma \to S_n$ from Γ to the symmetric group on n elements defined by:

$$\Phi_\gamma(\gamma_i) = \gamma_j$$

where $H\gamma_i\gamma = H\gamma_j$, that is

$$H\gamma_i\gamma = H\Phi_\gamma(\gamma_i).$$

This function is a group homomorphism, indeed from the above equality

$$H\gamma_i\gamma\bar{\gamma} = H\Phi_\gamma(\gamma_i)\bar{\gamma} = H\Phi_{\bar{\gamma}}(\Phi_\gamma(\gamma_i));$$

on the other hand we have, by definition,

$$H\gamma_i\gamma\bar{\gamma} = H\Phi_{\gamma\bar{\gamma}}(\gamma_i)$$

and then $\Phi_{\bar{\gamma}} \circ \Phi_\gamma = \Phi_{\gamma\bar{\gamma}}$.

Notice that $\ker(\Phi) \subseteq H$. Indeed, if $\Phi_\gamma = \mathrm{id}_{S_n}$, then $\Phi_\gamma(1) = 1$ that is $H\gamma = H$.

Now, the subgroups of Γ containing $\ker(\Phi)$ are as many as the subgroups of $\Gamma/\ker(\Phi)$ which is a finite group.

On the other hand, there are finitely many homomorphisms from Γ to S_n because such an homomorphism is completely determined by its value on the (finitely many) generators of Γ. $\square$

Corollary 2.2.4 *The set P_n is finite.*

Proof By Proposition 2.2.2, a configuration $c \in X$ belongs to $P_n(X)$ if and only if it is constant on the right cosets of a subgroup H with finite index dividing n. By Lemma 2.2.3, these subgroups H are in finite number. For a fixed H, there are finitely many functions from the right cosets of H to A, that is $|A|^{[\Gamma:H]}$. $\square$

2.3 Residually Finite Groups

In this section we deal with the basic properties of *residually finite* groups. We will see that the (finitely generated) residually finite groups are precisely those groups such that for each finite set A, the set C_p of periodic configurations is dense in A^Γ.

Definition 2.3.1 A group Γ is *residually finite* if for every $\gamma \in \Gamma$ with $\gamma \neq 1$, there exists $H \leq \Gamma$ of finite index such that $\gamma \notin H$.

In other words, a group if residually finite if

$$\bigcap_{\substack{H \leq \Gamma \\ [\Gamma:H]<\infty}} H = \{1\}.$$

Examples of residually finite groups are the groups $\mathbf{Z}^n$ $(n = 1, 2, \dots)$ and, in general, all finitely generated abelian groups. The free group $\mathbf{F}_n$ of rank n is an example of residually finite, non–abelian group. The additive group of rational numbers $\mathbf{Q}$ is an example of abelian, non–finitely generated and non–residually finite group.

The proofs of the following Lemma 2.3.2 and Theorem 2.3.3 are due to T. Ceccherini–Silberstein and A. Machì.

Lemma 2.3.2 *If Γ is a residually finite group and $F = \{\gamma_1, \dots, \gamma_n\}$ is a finite subset of Γ with $1 \notin F$, then there exists a subgroup $H_F \leq \Gamma$ of finite index such that $\gamma_i \notin H_F$ and $H_F\gamma_i \neq H_F\gamma_j$ (if $i \neq j$).*

PROOF For every $i = 1, \ldots, n$ let H_i be a subgroup of finite index such that $\gamma_i \notin H_i$ and let H_{ij} be a subgroup of finite index such that $\gamma_i \gamma_j^{-1} \notin H_{ij}$ (where $i \neq j$). The intersection H_F of all these subgroups also has finite index; moreover $\gamma_i \notin H_F$ (for every i) and $\gamma_i \gamma_j^{-1} \notin H_F$ $(i \neq j)$. □

In particular, the set F of this Lemma can be extended to a set of right coset representatives of the subgroup H_F.

Theorem 2.3.3 *Let Γ be a finitely generated group and A a finite alphabet. If Γ is residually finite, then the set C_p of periodic configurations is dense in A^Γ.*

PROOF Suppose that Γ is residually finite; we have to prove that

$$A^\Gamma = \overline{C_p}.$$

Fix $c \in A^\Gamma$ and let H_{D_n} be the subgroup of finite index whose existence is guaranteed by Lemma 2.3.2 with $F := D_n \backslash \{1\}$, and let D be a set of right coset representatives of H_{D_n} containing D_n. If $\gamma \in \Gamma$ and $\gamma = hd$ with $h \in H_{D_n}$ and $d \in D$, define a configuration c_n such that $(c_n)_{|\gamma} = c_{|d}$. This configuration being constant on the right cosets of H_{D_n}, is periodic. Moreover c and c_n agree on D_n and hence $\text{dist}(c, c_n) < \frac{1}{n+1}$. Then the sequence of periodic configurations $(c_n)_n$ converges to c. □

The same result is also given by Yukita [Y]. The converse of this theorem also holds.

Theorem 2.3.4 *Let Γ be a finitely generated group and A a finite alphabet. Then Γ is residually finite if and only if the set C_p of periodic configurations is dense in A^Γ.*

PROOF If Γ is not residually finite, let $1 \neq \gamma \in \Gamma$ be an element belonging to all the subgroups of Γ of finite index; in particular $\gamma \in \bigcap_{c \in C_p} H_c$ so that, for $c \in C_p$, $c^\gamma = c$ and hence $c_{|\gamma} = c_{|1}$. Let $\bar{c} \in A^\Gamma$ such that $\bar{c}_{|\gamma} \neq \bar{c}_{|1}$, then for each n such that $\gamma \in D_n$ and each $c \in C_p$ we have $\bar{c}_{|D_n} \neq c_{|D_n}$. Hence $\text{dist}(\bar{c}, c) \geq \frac{1}{n+1}$ and $\bar{c} \notin \overline{C}_p$. □

2.4 Density of Periodic Configurations

In this section we prove that the density of the periodic configuration is a sufficient (but not necessary) condition for the surjunctivity of the transition function in a cellular automaton. By Theorem 2.3.4, we have that from the residual finiteness of Γ it follows that a transition function $\tau : A^\Gamma \to A^\Gamma$ on a full shift surjunctive. The groups $\mathbf{Z}^n$ being residually finite, we have that this result generalizes Richardson's theorem.

Theorem 2.4.1 *Let $X \subseteq A^\Gamma$ be a shift whose set $C_p(X) := C_p \cap X$ of periodic configurations of X is dense in X. Then every transition function $\tau : X \to X$ is surjunctive.*

Proof By Corollary 2.2.4, we have that the set $P_n(X) := P_n \cap X$ is finite; now we prove that if τ is local then $\tau(P_n(X)) \subseteq P_n(X)$. Indeed if $c \in P_n(X)$ then $H_c \subseteq H_{\tau(c)}$ because if $h \in H_c$

$$(\tau(c))^h = \tau(c^h) = \tau(c).$$

Hence, the index of H_c being a divisor of n, the index of $H_{\tau(c)}$ also divides n and $\tau(c) \in P_n(X)$.

By Lemma 2.1.2, τ is surjunctive. □

Corollary 2.4.2 *If Γ is a residually finite group and $\tau : A^\Gamma \to A^\Gamma$ is a transition function, then τ is surjunctive.*

In general, the implication injective $\Rightarrow$ surjective of Theorem 2.4.1 is not invertible; the standard example is the following. Let $A = \{0, 1\}$ and $\Gamma = \mathbf{Z}$; let τ be the local function given by the local rule $\delta : A^3 \to A$ such that

$$\delta(a_1, a_2, a_3) = a_1 + a_3 \ (\mathrm{mod} 2).$$

Then τ is surjective and not injective. Indeed if $(a_z)_{z\in\mathbf{Z}}$ is a configuration in $A^{\mathbf{Z}}$, define a pre–image of it in this way:

$$\begin{cases} b_0 = 0 \\ b_1 = 0 \\ b_{n+1} = a_n - b_{n-2} \ (\mathrm{mod}2) \text{ if } n \geq 2 \\ b_{-n} = a_{-n+1} - b_{-n+2} \ (\mathrm{mod}2) \text{ if } n \leq 0 \end{cases}$$

that is

$\dots$	$a_{-2} - a_0$	a_{-1}	a_0	0	0	a_1	a_2	$a_3 - a_1$	$a_4 - a_2$	$\dots$
$\dots$	a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3	a_4	a_5	$\dots$

.

With $b_0 = 1 = b_1$ we can construct a pre–image in an analogous way.

Proposition 2.4.3 *If $X \subseteq A^\Gamma$ is a shift such that $C_p(X)$ is dense in X and $\tau : X \to A^\Gamma$ is a local function, then $C_p(\tau(X))$ is dense in $\tau(X)$.*

Proof Set $Y := \tau(X)$; we first prove that $\tau(C_p(X)) \subseteq C_p(Y)$. Indeed if $c \in X$ the stabilizer H_c is contained in $H_{\tau(c)}$ and if H_c has finite index, then also $H_{\tau(c)}$ has finite index. Then $\overline{C_p(Y)} \supseteq \overline{\tau(C_p(X))} \supseteq \tau(\overline{C_p(X)}) = \tau(X) = Y$. □

Corollary 2.4.4 *The density of the periodic configurations is an invariant of the shift.*

In general, given a shift X, denote with $Q_n(X)$ the set of the periodic configurations of X with period n and let $q_n(X)$ be the cardinality of $Q_n(X)$. We have that *$q_n(X)$ is an invariant of X*. Indeed, suppose that $\tau : X \to Y$ is a conjugacy and let $c \in Q_n(X)$. We prove that $H_c = H_{\tau(c)}$; as we have already seen, $H_c \subseteq H_{\tau(c)}$. If $h \in H_{\tau(c)}$, we have $\tau(c^h) = \tau(c)^h = \tau(c)$; τ being injective, we have $c^h = c$.

From this fact it is clear that also the number $p_n(X)$ (that is, the cardinality of $P_n(X)$), *is an invariant of* X.

2.5 Periodic Configurations of Euclidean Shifts

In this section we concentrate our attention on the density of the periodic configurations for a Euclidean shift. In the one–dimensional case we prove this density for the irreducible sofic shifts. The situation in the two–dimensional case is deeply different: there are counterexamples of irreducible shifts of finite type for which the set C_p is not dense. At the end of the section also the notion of *mixing* shift is given and this property is strictly stronger than irreducibility. Nevertheless, Weiss have proved in [Wei2]) the existence of mixing two–dimensional shifts of finite type X and of local functions $\tau : X \to X$ which are injective and not surjective.

Proposition 2.5.1 *If* $X \subseteq A^{\mathbf{Z}}$ *is an irreducible shift of finite type, then the set* $C_p(X)$ *of periodic configurations of* X *is dense in* X.

Proof Suppose that X has memory M; let $c \in X$ and let $u_n := c_{|[-n,n]}$. Fix $a \in L(X)$ with $|a| = M$; X being irreducible, there exist two words $v_n, w_n \in L(X)$ such that

$$a\ v_n u_n w_n\ a \in L(X).$$

Let c_n be the periodic configuration

$$\ldots a\ v_n u_n w_n\ a\ v_n u_n w_n\ a \cdots = \overline{a\ v_n u_n w_n};$$

by Proposition 1.3.2, $c_n \in X$. Moreover $c_{n|[-n,n]} = c_{|[-n,n]}$ and hence $\lim_{n\to\infty} c_n = c$. □

Corollary 2.5.2 *If* $X \subseteq A^{\mathbf{Z}}$ *is an irreducible sofic shift, then the set* $C_p(X)$ *of periodic configurations of* X *is dense in* X.

Proof By Corollary 1.5.9, we have that every irreducible sofic shift is the image under a local map of an irreducible shift of finite type. Hence Propositions 2.5.1 and 2.4.3 apply. □

Counterexample 2.5.3 We now define a reducible shift $X \subseteq A^{\mathbf{Z}}$ which is of finite type but whose set $C_p(X)$ is not dense.

Let $A = \{0, 1\}$ and let X be the shift of finite type with set of forbidden blocks $\{01\}$. Then the elements of X are the configurations $\bar{0}, \bar{1}$ constant in 0 and 1 and the configurations of the type $\ldots 1111110000000 \ldots$; clearly X is not irreducible because there are no words $u \in L(X)$ such that $0u1 \in L(X)$. In this shift we have $C_p(X) = \{\bar{0}, \bar{1}\}$ which is closed (and so not dense) in X. Notice that, for this shift, a transition function is injective if and only if it is surjective and hence surjunctivity holds even if the set of periodic configurations is not dense. □

Observe that, if X is a subshift of $A^{\mathbf{Z}}$, it is always possible to define an irreducible subshift X^2 of $A^{\mathbf{Z}^2}$ consisting of copies of X; more precisely, a configuration c belongs to X^2 if and only if each horizontal line of c (i.e. the bi–infinite word $(c_{|(z,t)})_{z\in\mathbf{Z}}$, for each fixed $t \in \mathbf{Z}$), belongs to X. The irreducibility of X^2 can be easily seen: given two blocks of the language, it suffices to translate one of them in the vertical direction in such a way that the supports are far enough. Moreover, it is obvious that the shift X^2 is of finite type if X is.

Counterexample 2.5.4 We show, using the above example, that Proposition 2.5.1 no longer holds for the irreducible shifts of finite type of $A^{\mathbf{Z}^2}$.

Let X^2 be the shift over the alphabet $A = \{0, 1\}$ generated by the shift X of the previous counterexample; then X^2 is irreducible and of finite type. The set $C_p(X^2)$ is in this case contained in the set of all those configurations assuming constant value at each horizontal line. It is then clear that a configuration assuming the value 1 at $(0, 0)$ and 0 at $(1, 0)$, cannot be approximated with any sequence of periodic configurations. □

Definition 2.5.5 A shift $X \subseteq A^\Gamma$ is *mixing* if for each pair of blocks $p_1 \in X_E$ and $p_2 \in X_F$, there exists an $M > 0$ such that for each $\gamma \notin D_M$ there is a configuration $c \in X$ such that $c_{|E} = p_1$ and $c_{|\gamma F} = p_2$ (notice that if M is big enough, then $E \cap \gamma F = \emptyset$).

In other words, a shift X is mixing if and only if for each pair of open sets $U, V \subseteq X$ there is an $M > 0$ such that $U \cap V^\gamma \neq \emptyset$ for all $\gamma \notin D_M$. Indeed, given a pattern p with support E, consider the set $U := \{c \in X \mid c_{|E} = p\}$; U is open because if $E = \{\gamma_1, \dots, \gamma_n\}$ then $U = \bigcap_{i=1}^n \{c \in X \mid c_{|\gamma_i} = p_{|\gamma_i}\}$ which is a finite intersection of open sets.

Even if we strengthen the irreducibility hypothesis by assuming that the shift is mixing, there are examples of a mixing subshifts of finite type X and of local functions $\tau : X \to X$ which are injective and not surjective (see [Wei2, Section 4]).

2.6 Group Shifts

If the alphabet is a finite group G, the full shift G^Γ is also a group with product defined as in the direct product of infinitely many copies of G. Endowed with this operation the space G^Γ is a compact metric topological group.

Definition 2.6.1 If G is a finite group, a subset $X \subseteq G^\Gamma$ is a *group shift* if is a subshift and a subgroup of G^Γ.

Clearly a group shift is also a compact (metric) group. Hence it can be seen as an example of *dynamical system* (X, Γ), where X is a compact group and Γ is a subgroup of the group $\mathrm{Aut}(X)$ of the automorphisms of X which are also continuous. Indeed the action of Γ defines a subgroup of $\mathrm{Aut}(X)$: for a fixed $\gamma \in \Gamma$, the bijective function $c \mapsto c^\gamma$ from X to X is, as we have seen, continuous and it is also a group homomorphism because

$$(c_1c_2)^\gamma{}_{|\alpha} = (c_1c_2)_{|\gamma\alpha} = c_{1|\gamma\alpha}c_{2|\gamma\alpha} = c_1{}^\gamma{}_{|\alpha}c_2{}^\gamma{}_{|\alpha} = (c_1{}^\gamma c_2{}^\gamma)_{|\alpha}.$$

If (X, Γ) is a dynamical system, the group Γ *acts expansively* on X if there exists a neighborhood U of the identity in X such that $\bigcap_{\gamma\in\Gamma} \gamma(U) = \{1_X\}$; the set of Γ*–periodic points* is the set of points $x \in X$ such that $\{\gamma(x) \mid \gamma \in \Gamma\}$ is finite. Clearly it coincides with the set $C_p(X)$ if X is a group shift.

In the hypotheses that X is metrizable and Γ is an infinite and finitely generated abelian group, Kitchens and Schmidt prove in [KitS2, Theorem 3.2] that if Γ acts expansively on X then (X, Γ) satisfies the descending chain condition (i.e. each nested decreasing sequence of closed Γ–invariant subgroups is finite), if and only if (X, Γ) is conjugate to a dynamical system (Y, Γ), where Y is a group subshift of G^Γ and G is a compact Lie group. Notice that, in this context, a conjugation is a continuous groups isomorphism that commutes with the Γ–action.

A consequence of this fact is the following theorem.

Theorem 2.6.2 [KitS2, Corollary 7.4] *Let X be a compact group and $\Gamma \leq \mathrm{Aut}(X)$ a finitely generated, abelian group; if Γ acts expansively on X then the set of Γ–periodic points is dense in X.*

The following result gives an answer to the problems arising from Counterexample 2.5.4.

Corollary 2.6.3 *Let G be a finite group and let Γ be an abelian group; if $X \leq G^\Gamma$ is a group shift, then the set $C_p(X)$ of periodic configurations of X is dense in X.*

Proof We have to prove that the group Γ acts expansively on G^Γ; indeed the identity in X is the configuration c_1 assuming the constant value 1_G, where 1_G is the identity of G. Consider the neighborhood X_1 of c_1 consisting of all those configurations of X assuming the value 1_G at 1_Γ. Obviously $\bigcap_{\gamma\in\Gamma}\{c^\gamma \mid c \in X_1\} = \bigcap_{\gamma\in\Gamma}\{c \in X \mid c_{|\gamma} = 1_G\} = \{c_1\}$. □

In [KitS2] is also proved that if X is a group shift, then X is of finite type. Indeed the following theorem is proved.

Theorem 2.6.4 [KitS2, Corollary 3.9] *Let G be a compact Lie group. If $X \leq G^\Gamma$ is a closed Γ–invariant subgroup there exists a finite set $D \subseteq \Gamma$ such that*

$$X = \{c \in G^\Gamma \mid c^\gamma{}_{|D} \in H \ for\ every\ \gamma \in \Gamma\},$$

where H is a closed subgroup of G^D.

Hence if G is finite and X is a group shift, the set $G^D \setminus H$ is finite and is a set of forbidden blocks for X. Although this fact, X is not necessarily irreducible. For example, consider in $(\mathbf{Z}/2\mathbf{Z})^{\mathbf{Z}}$ the group shift $\{\bar{0}, \bar{1}, \overline{01}, \overline{10}\}$.

Notice that an abelian, finitely generated group Γ is also residually finite; we have another proof of this fact fixing a finite group G and applying Corollary 2.6.3 to the group shift G^Γ. By Theorem 2.3.4 we have that Γ is residually finite.

2.6.1 Decision Problems for Group Shifts

Now we list some other decision problems arising in the case of Euclidean subshifts of finite type.

- The *tiling problem*: given a finite list $\mathcal{F}$ of forbidden blocks is $X_{\mathcal{F}}$ empty or non–empty? In fact the tiling problem is an equivalent formulation of the *domino problem*, proposed by Wang [Wa].
- A problem strictly related with this latter is the following: given a finite list $\mathcal{F}$ of forbidden blocks, is there a periodic configuration in $X_{\mathcal{F}}$?
- Given a finite list $\mathcal{F}$ of forbidden blocks, are the periodic configurations dense in $X_{\mathcal{F}}$?
- The *extension problem*: given a finite list $\mathcal{F}$ of forbidden blocks and given an *allowable* block (that is a block in which does not appear any forbidden block), is there a configuration in $X_{\mathcal{F}}$ in which it appears? Clearly a positive answer to the extension problem would imply a positive answer to the tiling problem.

Now we prove that the answers for subshifts of finite type of $A^{\mathbf{Z}}$ are all positive: there are algorithms to decide, the tiling and the extension problems and there is an algorithm to decide whether or not the periodic configurations are dense in X. In order to see the first two algorithms, consider, more generally, a sofic shift. If $\mathcal{G}$ is a labeled graph $\mathcal{G}$ accepting X (and we may assume that $\mathcal{G}$ is essential), it can be easily seen that X is non–empty if and only if it exists a cycle on the graph. Hence the shift is non–empty if and only if it contains a periodic configuration. On the other hand, the language of X is the language accepted by $\mathcal{G}$ ($\mathcal{G}$ being essential); hence an allowable word is a word of the language if and only if it is accepted by $\mathcal{G}$.

To establish the density of the periodic configurations, suppose that X is of finite type with memory M; one has that $C_p(X)$ is dense in X if and only if $C_p(X^{[M+1]})$ is dense in $X^{[M+1]}$. The shift $X^{[M+1]}$ an edge shift accepted by the graph $\mathbf{G}$ constructed in Section 1.5 and hence the set $C_p(X^{[M+1]})$ is dense in $X^{[M]}$ if and only if each edge of $\mathbf{G}$ is contained in a strictly connected component of $\mathbf{G}$, that is if the graph $\mathbf{G}$ has no edges connecting two different connected components.

For the subshifts of finite type of $A^{\mathbf{Z}^2}$ the answers are quite different; in this setting Berger proved in [B] the existence of a non–empty shift of finite type containing no periodic configurations and the undecidability of the tiling problem. Sufficient conditions to the decidability of tiling and extension problems are the following.

Theorem 2.6.5 (Wang [Wa]) *If every non–empty subshift of finite type of $A^{\mathbf{Z}^2}$ contains a periodic configuration then there is an algorithm to decide the tiling problem.*

Theorem 2.6.6 (Kitchens and Schmidt [KitS1]) *If every subshift of finite type of $A^{\mathbf{Z}^2}$ has dense periodic configurations then there is an algorithm to decide the extension problem.*

As a consequence of these facts, we have from Corollary 2.6.3 that *if $X \leq G^{\mathbf{Z}^2}$ is a group shift, then the tiling and extension problems are decidable for X.*

3. The Moore–Myhill Property

For Euclidean cellular automata, Moore [Moo] has given a sufficient condition for the existence of the so–called *Garden of Eden (GOE) patterns*, that is those patterns that cannot be reached at time t from another configuration starting at time $t-1$ and hence can only appear at time $t=0$. Moore's condition (i.e. the existence of *mutually erasable patterns* – a sort of non–injectivity of the transition function on the "finite" configurations) was also proved to be necessary by Myhill [My]. This equivalence between "local injectivity" and "local surjectivity" of the transition function is the classical well–known *Garden of Eden theorem*.

In this chapter, we consider generalizations of the *Moore's property* and *Myhill's property* to a generic shift. In details, the GOE–theorem has been proved by Machì and Mignosi [MaMi] more generally for cellular automata in which the space of configurations is the whole A–shift A^Γ and the group Γ has non–exponential growth; more recently it has been proved by Ceccherini–Silberstein, Machì and Scarabotti [CeMaSca] for the wider class of the *amenable groups*.

Instead of the non–existence of mutually erasable patterns, we deal with the notion of *pre–injectivity* (a function $\tau : X \subseteq A^\Gamma \to A^\Gamma$ is *pre–injective* if whenever two configurations $c, \bar{c} \in X$ differ only on a finite non–empty subset of Γ, then $\tau(c) \neq \tau(\bar{c})$); this notion has been introduced by Gromov in [G]. In fact, we prove in Section 3.1 that these two properties are equivalent for local functions defined on the full shift, but in the case of proper subshifts the former may be meaningless. On the other hand, we give a proof of the fact that the non–existence of GOE patterns is equivalent to the non–existence of GOE configurations, that is to the surjectivity of the transition function. Hence, in this language, the GOE theorem states that τ is surjective if and only if it is pre–injective. We call *Moore's property* the implication surjective $\Rightarrow$ pre–injective and *Myhill's property* the inverse one.

In Section 3.2, we recall the notion of an *amenable group*. We give the more useful characterization of amenability in terms of *Følner's sequences*. It will follow from Section 1.7 that, if the group Γ has non–exponential growth, the Følner's sequence can be taken as a suitable sequence of disks centered at $1 \in \Gamma$.

Concerning one–dimensional shifts, in Section 3.3 we see that from the works of Hedlund [H] and Coven and Paul [CovP] it follows that the Moore–Myhill (MM) property holds for irreducible shifts of finite type of $A^{\mathbf{Z}}$. Moreover, using this result we prove that Myhill's property holds for irreducible sofic shifts of $A^{\mathbf{Z}}$. On the other hand, we give a counterexample of an irreducible sofic shift $X \subseteq A^{\mathbf{Z}}$ but not of finite type for which Moore's property does not hold.

Concerning general cellular automata over amenable groups, we see in Section 3.4 that from a result of Gromov [G] in a more general framework, it follows that the MM–property holds for shifts of *bounded propagation* contained in A^{Γ}.

In Section 3.5, we generalize this result showing that the MM–property holds for *strongly irreducible* shifts of finite type of A^{Γ} (and we also show that strong irreducibility together with the finite type condition is strictly weaker that the bounded propagation property).

The main difference between irreducibility and strong irreducibility is easily seen in the one–dimensional case. Here the former property states that given any two words u, v in the language of a shift, there exists a third word w such that the concatenation uwv is still in the language. Strong irreducibility says that we can arbitrarily fix the length of this word (but it must be greater than a fixed constant only depending on the shift). The same properties for a generic shift refers to the way in which two different patterns in the language of the shift may appear simultaneously in a configuration. For irreducibility we have that two patterns always appear simultaneously in some configuration if we translate their supports. Strong irreducibility states that if the supports of the pattern are far enough, than it is not necessary to translate them in order to find a configuration in which both the patterns appear.

These two irreducibility conditions are not equivalent, not even in the one–dimensional case. Hence our general results about strongly irreducible shifts of finite type are strictly weaker than the one–dimensional ones. In the attempt of using weaker hypotheses in the latter case, we give in Section 3.6 a new notion of irreducibility, the *semi–strong irreducibility*. In the one–dimensional case, this property means that the above word w may "almost" be of the length we want (provided that it is long enough): we must allow it to be "a little" longer or "a little" shorter; the length of this difference is bounded and only depends on the shift. In general, semi–strong irreducibility states that if the supports of the patterns are far enough from each other, than translating them "a little" we find a configuration in which both the patterns appear. The reason for this choice lies in the fact that using the Pumping Lemma we can prove that a sofic subshift of $A^{\mathbf{Z}}$ is irreducible if and only if is semi–strongly irreducible. Moreover Myhill's property holds for semi–strongly irreducible shifts of finite type of A^{Γ}if Γ has non–exponential growth.

3.1 The Garden of Eden Theorem and the Moore–Myhill Property

In this section we give the definition of a *pre–injective* function, proving that this property is equivalent to the notion of non–existence of *mutually erasable patterns* used in the original works of Moore [Moo] and Myhill [My]. Indeed they prove that a transition function of a Euclidean cellular automaton on a full shift admits two mutually erasable patterns if and only if it admits a *Garden of Eden pattern*, that is a pattern without pre–image. A result of this kind, concerning a local function τ between two shifts, is what we call a *Garden of Eden theorem*; we deal with the *Moore–Myhill property* when we have a shift such that for each transition function of a cellular automaton over it pre–injectivity is equivalent to surjectivity.

Definition 3.1.1 Let $\tau : A^\Gamma \to A^\Gamma$ be a transition function; two patterns p_1 and p_2 with the same support F are called *τ–mutually erasable* if $p_1 \neq p_2$ and if for every $c_1, c_2 \in A^\Gamma$ such that $c_{1|F} = p_1$, $c_{2|F} = p_2$ and $c_{1|\complement F} = c_{2|\complement F}$, one has $\tau(c_1) = \tau(c_2)$.

Definition 3.1.2 Let $X, Y \subseteq A^\Gamma$ be two shift and $\tau : X \to Y$ be a function; a pattern $p \in Y_E$ is *Garden of Eden* (briefly *GOE*) *with respect to* τ, if for every $c \in X$ one has $\tau(c)_{|E} \neq p$.

The *Garden of Eden (GOE) theorem* is the union of the following two theorems.

Theorem 3.1.3 (E. F. Moore - 1962) *If $\tau : A^{\mathbf{Z}^2} \to A^{\mathbf{Z}^2}$ is a transition function and there exist two τ–mutually erasable patterns, then there exists a GOE pattern.*

Theorem 3.1.4 (J. Myhill - 1963) *If $\tau : A^{\mathbf{Z}^2} \to A^{\mathbf{Z}^2}$ is a transition function and there exists a GOE pattern, then there exist two τ–mutually erasable patterns.*

In order to consider GOE–like theorems not in the whole of A^Γ but in a subshift $X \subseteq A^\Gamma$, notice first that two patterns of X are not necessarily extendible by the same configuration of X. Therefore it could happen that two patterns with support F for which there does not exist a common extension $c_{|\complement F}$, are τ–mutually erasable although the function τ is bijective. The notion that seems to be a good generalization of the non–existence of mutually erasable patterns, is that of *pre–injectivity*; we will see that if $X = A^\Gamma$ then the non–existence of τ–mutually erasable patterns is equivalent to the pre–injectivity of τ.

Definition 3.1.5 A function $\tau : X \subseteq A^\Gamma \to A^\Gamma$ is called *pre–injective* if whenever $c_1, c_2 \in X$ and $c_1 \neq c_2$ only on a finite non–empty subset of Γ, then $\tau(c_1) \neq \tau(c_2)$.

Proposition 3.1.6 *Let $\tau : A^\Gamma \to A^\Gamma$ be a transition function; then τ is pre–injective if and only if there are no τ–mutually erasable patterns.*

Proof Let p_1 and p_2 be two τ–mutually erasable patterns with support F. Fix $a \in A$ and define $c_1, c_2 \in A^\Gamma$ that coincide, respectively, with p_1 and p_2 on F and such that

$$(c_1)_{|\gamma} = (c_2)_{|\gamma} = a$$

if $\gamma \notin F$. Then c_1 and c_2 differ only on a non–empty finite set (since this set is contained in F), and $\tau(c_1) = \tau(c_2)$, so that τ is not pre–injective.

Suppose, conversely, that τ is not pre–injective; there exist two configurations c_1 and c_2 such that, for some non empty finite set F we have:

- $c_{1|F} \neq c_{2|F}$
- $c_{1|\complement F} = c_{2|\complement F}$
- $\tau(c_1) = \tau(c_2)$.

Set $p_1 := c_{1|F^{+2M}}$ and $p_2 := c_{2|F^{+2M}}$, where M is such that τ is M–local; then we prove that p_1 and p_2 are τ–mutually erasable. First $p_1 \neq p_2$ and if $\bar{c}_1$, $\bar{c}_2$ are two configurations such that

- $\bar{c}_{1|F^{+2M}} = p_1$
- $\bar{c}_{2|F^{+2M}} = p_2$
- $\bar{c}_1 = \bar{c}_2$ out of F^{+2M}

we have that $\tau(\bar{c}_1) = \tau(\bar{c}_2)$.

Indeed, set $D_M := \{\alpha_1, \ldots, \alpha_m\}$ and $\gamma_i := \gamma\alpha_i$; if $\gamma \in \Gamma$ and $\gamma D_M \subseteq F^{+2M}$ we have $(\tau(\bar{c}_1))_{|\gamma} = \delta(\bar{c}_{1|\gamma_1}, \ldots, \ ) = \delta(p_{1|\gamma_1}, \ldots, \ ) = \delta(c_{1|\gamma_1}, \ldots, \ ) = (\tau(c_1))_{|\gamma} = (\tau(c_2))_{|\gamma} = \delta(c_{2|\gamma_1}, \ldots, \ ) = \delta(p_{2|\gamma_1}, \ldots, \ ) = \delta(\bar{c}_{2|\gamma_1}, \ldots, \ ) = (\tau(\bar{c}_2))_{|\gamma}$.

If, otherwise, $\gamma D_M \subseteq \complement F$ and if we suppose, for example, that $\gamma_1, \ldots, \gamma_i \in F^{+2M}$ and $\gamma_{i+1}, \ldots, \gamma_n \in \complement(F^{+2M})$, then $(\tau(\bar{c}_1))_{|\gamma} = \delta(\bar{c}_{1|\gamma_1}, \ldots, \bar{c}_{1|\gamma_n}) = \delta(c_{1|\gamma_1}, \ldots, \bar{c}_{2|\gamma_n}) = \delta(c_{2|\gamma_1}, \ldots, \bar{c}_{2|\gamma_n}) = \delta(\bar{c}_{2|\gamma_1}, \ldots, \bar{c}_{2|\gamma_n}) = (\tau(\bar{c}_2))_{|\gamma}$. $\square$

One can prove (see [MaMi, Theorem 5]) that a transition function on A^Γ is surjective if and only if there are no GOE patterns. It is easy to prove that this property holds also for the local functions between shifts, as proved in the following Proposition.

Proposition 3.1.7 *Let $\tau : X \to Y$ a local function; then τ is surjective if and only if there are no GOE patterns.*

Proof It is clear that the surjectivity of τ implies the non–existence of GOE–patterns.

For the converse, suppose that for each finite set $E \subseteq \Gamma$ and each $p \in Y_E$ there is a configuration $c \in X$ such that $\tau(c)_{|E} = p$; we prove that τ is surjective.

If $\bar{c} \in Y$, let $c_n \in X$ be such that $\tau(c_n)_{|D_n} = \bar{c}_{|D_n}$; hence $\lim_{n\to\infty} \tau(c_n) = \bar{c}$. X being compact, there is a subsequence $(c_{n_k})_k$ that converges to a configuration $c \in X$. τ being continuous, we have that $\bar{c} = \lim_{k\to\infty} \tau(c_{n_k}) = \tau(c)$. $\square$

Hence we can restate the Garden of Eden Theorem [Moo] and [My] as follows.

Theorem 3.1.8 *If $\tau : A^{\mathbf{Z}^2} \to A^{\mathbf{Z}^2}$ is a transition function, then τ is pre–injective if and only if it is surjective.*

Definition 3.1.9 A shift $X \subseteq A^\Gamma$ has the *Moore–Myhill property* (briefly *MM–property*), if for every cellular automaton (X, D_M, τ) the transition function τ is pre–injective if and only if it is surjective. The *Moore–property* is surjective $\Rightarrow$ pre–injective and the *Myhill–property* is pre–injective $\Rightarrow$ surjective.

In the sequel we will distinguish between these properties and the GOE–theorems for a local function. Indeed the former are properties of a single shift but, on the other hand, we will speak of GOE–theorem whenever we have a GOE–like theorem for a local function between two possibly different shifts.

Notice that if a shift X has the Myhill–property, than each transition function $\tau : X \to X$ is surjunctive (because, obviously, injectivity $\Rightarrow$ pre–injectivity).

Proposition 3.1.10 *The MM–property is invariant under conjugacy.*

PROOF A conjugacy being both surjective and pre–injective, it suffices to prove that the composition of two local pre–injective function is still a (local) pre–injective function. Hence suppose that $\tau : X \to Y$ and $\phi : Y \to Z$ are pre–injective functions; if $c_1, c_2 \in X$ with $c_1 \neq c_2$ and there exists a finite subset $F \subseteq \Gamma$ such that $c_{1|\complement F} = c_{2|\complement F}$, we prove that if τ is M–local then $\tau(c_1)_{|\complement(F^{+M})} = \tau(c_2)_{|\complement(F^{+M})}$. Actually, if $\gamma \notin F^{+M}$ then $\gamma D_M \subseteq \complement F$ and hence $\tau(c_1)_{|\gamma} = \tau(c_2)_{|\gamma}$. The set F^{+M} being finite, we have $\phi(\tau(c_1)) \neq \phi(\tau(c_2))$ so that $\phi \circ \tau$ is pre–injective. $\square$

3.2 Amenable Groups

In this section we give the definition of *amenability* for a group Γ; using a characterization of it due to Følner (see [F], [Gr] and [N]), Ceccherini–Silberstein, Machì and Scarabotti have proved that the MM–property holds for the full shift A^Γ (see [CeMaSca]).

Definition 3.2.1 A group Γ is called *amenable* if it admits a Γ–invariant probability measure, that is a function $\mu : P(\Gamma) \to [0,1]$ such that for $A, B \subseteq \Gamma$ and for every $\gamma \in \Gamma$

- $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$ (*finite additivity*)
- $\mu(\gamma A) = \mu(A)$ (Γ*–invariance*)

- $\mu(\Gamma) = 1$ (*normalization*).

Theorem 3.2.2 (Følner) *A group Γ is amenable if and only if for each finite subset $F \subseteq \Gamma$ and each $\varepsilon > 0$ there exists a finite subset $K \subseteq \Gamma$ such that*

$$\frac{|KF \backslash K|}{|K|} < \varepsilon.$$

This characterization is equivalent to the following one.

For each pair of finite subsets $F, H \subseteq \Gamma$ with $1 \in H$ and each $\varepsilon > 0$ there exists a finite subset $K \supseteq H$ such that

$$\frac{|KF \backslash K|}{|K|} < \varepsilon.$$

Indeed, suppose that there exists $\bar{K}$ such that

$$\frac{|\bar{K}HF \backslash \bar{K}|}{|\bar{K}|} < \varepsilon.$$

We have that $\bar{K} \subseteq \bar{K}H$ and hence

$$\frac{|\bar{K}HF \backslash \bar{K}H|}{|\bar{K}H|} \leq \frac{|\bar{K}HF \backslash \bar{K}|}{|\bar{K}|} < \varepsilon;$$

it suffices to set $K := \bar{K}H$.

An analogous equivalent form of Følner's characterization is given by Namioka in [N].

Theorem 3.2.3 *Let Γ be an amenable group; then there exists a sequence of finite sets $(E_n)_{n \geq 1}$ such that:*

- $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n \subseteq \ldots$
- $\bigcup_{n \geq 1} E_n = \Gamma$,
- $\lim_{n \to \infty} \frac{|\partial_M E_n|}{|E_n|} = 0$.

Proof First, notice that in Følner condition there is no loss of generality if we suppose $1 \in K$. Now we construct, by induction, a nested sequence $1 \in K_1 \subseteq \ldots \subseteq K_n \subseteq \ldots$ such that, for each $n \geq 1$

$$\frac{|K_n(D_n^{+M}) \backslash K_n|}{|K_n|} < \frac{1}{n}.$$

Let K_1 be a finite subset $1 \in K_1 \subseteq \Gamma$ such that

$$\frac{|K_1(D_1^{+M}) \backslash K_1|}{|K_1|} < 1$$

whose existence is guaranteed by Theorem 3.2.2. Suppose to have found K_n, there exists $K_{n+1} \supseteq K_n$ such that

$$\frac{|K_{n+1}(D_{n+1}^{+M})\backslash K_{n+1}|}{|K_{n+1}|} < \frac{1}{n+1}.$$

Observe that

- $K_n(D_n^{+M}) = (K_n D_n)^{+M}$
- $K_n \subseteq K_n(D_n^{-M}) \subseteq (K_n D_n)^{-M}$
- $K_n \subseteq K_n D_n$

hence

$$\frac{|(K_n D_n)^{+M}\backslash (K_n D_n)^{-M}|}{|K_n D_n|} \leq \frac{|K_n(D_n^{+M})\backslash K_n|}{|K_n|} < \frac{1}{n}.$$

Setting $E_n := K_n D_n$ we have the stated properties because $D_n \subseteq K_n$. □

A sequence as in Theorem 3.2.3 is called *amenable* (or *Følner sequence*); from now on we fix the amenable sequence $(E_n)_{n\geq 1}$ found above and the entropy of a shift will be defined with respect to $(E_n)_{n\geq 1}$. As we have seen in Section 1.7, if the group Γ has non–exponential growth, the sequence $(E_n)_{n\geq 1}$ can be replaced by a suitable sequence of disks centered at 1.

Corollary 3.2.4 *If $(E_n)_{n\geq 1}$ is an amenable sequence, then*

$$\lim_{n\to\infty} \frac{|\partial_M^+ E_n|}{|E_n|} = 0$$

and

$$\lim_{n\to\infty} \frac{|\partial_M^- E_n|}{|E_n|} = 0.$$

Using the existence of an amenable sequence in the amenable group Γ, Ceccherini–Silberstein, Machì and Scarabotti have proved the following theorem for the full shift A^Γ.

Theorem 3.2.5 [CeMaSca, Corollary of Theorem 3] *Let Γ be a finitely generated amenable group and $\tau : A^\Gamma \to A^\Gamma$ be a transition function. Then τ is surjective if and only if there are no τ–mutually erasable patterns.*

As a consequence of this theorem we have the following statement.

Corollary 3.2.6 (MM–property for the full shift) *For an amenable group Γ, the full shift A^Γ has the MM–property.*

3.3 The Moore–Myhill Property for Subshifts of $A^{\mathbf{Z}}$

As far as irreducible shifts of finite type are concerned, we have the following result that stems from the works of Hedlund [H] and Coven and Paul [CovP].

Theorem 3.3.1 [LinMar, Theorem 8.1.16] *Let X be an irreducible shift of finite type, Y a shift and $\tau : X \to Y$ a local function. Then τ is pre–injective if and only if* $\mathrm{ent}(X) = \mathrm{ent}(\tau(X))$.

Corollary 3.3.2 (MM–property for irreducible subshifts of finite type of $A^{\mathbf{Z}}$) *An irreducible subshift of finite type of $A^{\mathbf{Z}}$ has the MM–property.*

Proof   If τ is pre–injective, then by Theorem 3.3.1 we have $\mathrm{ent}(X) = \mathrm{ent}(\tau(X))$. By Theorem 1.7.4, there does not exist a proper subshift of X whose entropy equals that of X. Thus $\tau(X) = X$ and τ is surjective. Conversely, if τ is surjective we have $\mathrm{ent}(X) = \mathrm{ent}(\tau(X))$ and Theorem 3.3.1 applies. □

Now we show that the irreducibility condition in the above theorem cannot be dropped.

Counterexample 3.3.3 Myhill–property no longer holds for a subshift of finite type of $A^{\mathbf{Z}}$ but not irreducible.

Let X be the full shift over the alphabet $A = \{0, 1\}$; clearly X is irreducible and of finite type. Consider the set $\bar{X} \subseteq \{0, 1, 2\}^{\mathbf{Z}}$ given by the union $X \cup \{\bar{2}\}$, where $\bar{2}$ is the bi–infinite word constant in 2. The set $\bar{X}$ is a shift of finite type over the alphabet $\bar{A} = \{0, 1, 2\}$ with set of forbidden blocks:

$$\{02, 20, 12, 21\}.$$

Moreover $\bar{X}$ is not irreducible; indeed we have $1, 2 \in L(\bar{X})$ but for no word $w \in L(\bar{X})$ the word $1w2$ belongs to $L(\bar{X})$.

Consider the transition function $\tau : \bar{X} \to \bar{X}$ defined by:

$$\tau(c) = \begin{cases} c & \text{if } c \in X \\ \bar{0} & \text{if } c = \bar{2}. \end{cases}$$

Clearly τ is 1–local where the local rule is given by $\delta(a) = a$ if $a \neq 2$ and $\delta(2) = 0$. This function is not surjective because the word $\bar{2}$ has no pre–images, but it is pre–injective. Actually, if c_1 and c_2 are different configurations which only differ on a finite subset of $\mathbf{Z}$, then they must belong to X and so their images under τ are different. □

Counterexample 3.3.4 Moore–property no longer holds for a shift of finite type but not irreducible.

Let X be the shift over the alphabet $A = \{0, 1, 2\}$ with set of forbidden blocks $\{01, 02\}$. The shift X is not irreducible; indeed for no word $u \in L(X)$ the word $0u1$ belongs to $L(X)$.

Consider the transition function $\tau : X \to X$ defined by the local rule:

$$\delta(a_1 a_2 a_3) = \begin{cases} a_2 & \text{if } a_3 \neq 0 \\ 0 & \text{if } a_3 = 0. \end{cases}$$

The function τ is surjective because a generic word of X, for example,

$$\ldots 1211122121212212121\ 0\ 0000000000000\ldots$$

has two pre–images:

$$\ldots 1211122121212212121\ \mathbf{1}\ 0000000000000\ldots$$

and

$$\ldots 1211122121212212121\ \mathbf{2}\ 0000000000000\ldots$$

This also shows that τ is not pre–injective. □

Using the generalizations to dimension 2 of Counterexamples 3.3.3 and 3.3.4, we obtain two irreducible shifts of finite type of $A^{\mathbf{Z}^2}$ which give the following counterexamples.

Counterexample 3.3.5 MM–property no longer holds for an irreducible shift of finite type contained in $A^{\mathbf{Z}^2}$.

Consider the cellular automaton of Counterexample 3.3.3; it is clear that the 1–local function τ can be extended to a 1–local function $\tau_2 : \bar{X}^2 \to \bar{X}^2$ with the same local rule ($\delta(a) = a$ if $a \neq 2$ and $\delta(2) = 0$). The function τ_2 is, as τ, pre–injective and non–surjective.

For the cellular automaton of Counterexample 3.3.4, we have similarly that the extended function τ_2 is surjective and not pre–injective □.

We now prove that a result similar to Theorem 3.3.1 holds for irreducible sofic shifts.

Theorem 3.3.6 *Let X be an irreducible sofic shift, Y a shift and $\tau : X \to Y$ a local function. Let $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ be the minimal deterministic presentation of X. Then $\tau \circ \mathcal{L}$ is pre–injective if and only if* $\mathrm{ent}(X) = \mathrm{ent}(\tau(X))$.

Proof The labeled graph $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ being a presentation of X, we have $X = X_{\mathcal{G}} = \mathcal{L}(X_{\mathbf{G}})$. By Corollary 1.5.8, $X_{\mathbf{G}}$ is an irreducible shift of finite type. Moreover $\tau \circ \mathcal{L} : X_{\mathbf{G}} \to Y$ is a local function; thus, by Theorem 3.3.1, $\tau \circ \mathcal{L}$ is pre–injective if and only if $\mathrm{ent}(X_{\mathbf{G}}) = \mathrm{ent}(\tau(\mathcal{L}(X_{\mathbf{G}}))) = \mathrm{ent}(\tau(X))$. By Proposition 1.7.2, $\mathrm{ent}(X_{\mathbf{G}}) = \mathrm{ent}(X)$ and the claim is proved. □

Corollary 3.3.7 (Myhill–property for irreducible sofic shifts) *Let X be an irreducible sofic shift and $\tau : X \to X$ a transition function. Then τ pre–injective implies τ surjective.*

Proof Let $\mathcal{G} = (\mathbf{G}, \mathcal{L})$ be the minimal deterministic presentation of X; we prove that if $\tau \circ \mathcal{L}$ is not pre–injective, then τ is not pre–injective either. Suppose that there exist two bi–infinite paths $c_1, c_2 \in X_{\mathbf{G}}$ which are different only on a finite path and such that $\tau(\mathcal{L}(c_1))) = \tau(\mathcal{L}(c_2))$. Then one can write c_1 and c_2, respectively, as:

$$c_1: \qquad \dots \xrightarrow{e_{-2}} i_{-1} \xrightarrow{e_{-1}} i_0 \xrightarrow{e_0} i_1 \xrightarrow{e_1} \dots \xrightarrow{e_{n-1}} i_n \xrightarrow{e_n} i_{n+1} \xrightarrow{e_{n+1}} \dots$$

and

$$c_2: \qquad \dots \xrightarrow{e_{-2}} i_{-1} \xrightarrow{e_{-1}} i_0 \xrightarrow{f_0} j_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} j_n \xrightarrow{f_n} i_{n+1} \xrightarrow{e_{n+1}} \dots,$$

with $e_0 \neq f_0$. Setting $a_i := \mathcal{L}(e_i)$ and $b_i := \mathcal{L}(f_i)$, the graph $\mathcal{G}$ being deterministic we have $a_0 \neq b_0$ and hence

$$\mathcal{L}(c_1) = a_{-2}a_{-1}\ a_0a_1 \dots a_{n-1}a_n\ a_{n+1} \dots$$

and

$$\mathcal{L}(c_2) = a_{-2}a_{-1}\ b_0b_1 \dots b_{n-1}b_n\ a_{n+1} \dots$$

are two configurations in X which differ only on a finite (non empty) set and whose images under τ are equal. Therefore τ is not pre–injective.

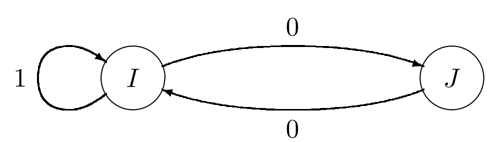
Thus, if τ is pre–injective, then $\tau \circ \mathcal{L}$ is also pre–injective; by Theorem 3.3.6, we have $\mathrm{ent}(X) = \mathrm{ent}(\tau(X))$. X being irreducible and by Theorem 1.7.4, $\tau(X)$ cannot be a proper subshift of X. Hence $\tau(X) = X$, i.e. τ is surjective. □

3.3.1 A counterexample to Moore–property for a sofic subshift of $A^{\mathbf{Z}}$

We now give an example of an irreducible sofic shift not of finite type for which the transition function is surjective but not pre–injective (that is, Moore–property no longer holds in general if the *finite type* condition is dropped). Our example will be the *even shift* X_e, that is the subshift of $\{0,1\}^{\mathbf{Z}}$ with forbidden blocks

$$\{10^{2n+1}1 \mid n \geq 0\}.$$

The shift X_e is sofic, indeed it is accepted by the following labeled graph.



We define a transition function τ as follows:

$$\tau(c)_{|z} = \delta(c_{|z-2}, c_{|z-1}, c_{|z}, c_{|z+1}, c_{|z+2})$$

where δ is the local rule:

$$\delta(a_1a_2a_3a_4a_5) = \begin{cases} 1 & \text{if } a_1a_2a_3 = 000 \text{ or } a_1a_2a_3 = 111 \text{ or } a_1a_2a_3a_4a_5 = 00100, \\ 0 & \text{otherwise.} \end{cases}$$

First we prove a Lemma.

Lemma 3.3.8 *If a block 0^n1 with $n \geq 3$, has a pre–image under τ of length $n+5$ in the language of X_e*

a_1	a_2	a_3	a_4	$\dots$	a_{n+1}	a_{n+2}	a_{n+3}	a_{n+4}	a_{n+5}
		0	0	$\dots$	0	0	1		

,

then this pre–image can be only of type

1. (*i*) $a_1a_2\ xx\ (1-x)(1-x)\dots 11\ 00\ 11\ 000a_{n+4}a_{n+5}$,
   (*ii*) $a_1a_2\ xx\ (1-x)(1-x)\dots 11\ 00\ 11\ 00100$,
   (*iii*) $a_1a_2\ (1-x)(1-x)\ xx\dots 00\ 11\ 00\ 111a_{n+4}a_{n+5}$,
   when n is even and for a suitable $x \in \{0,1\}$;
2. (*i*) $a_1a_2\ (1-x)\ xx\dots 11\ 00\ 11\ 000a_{n+4}a_{n+5}$,
   (*ii*) $a_1a_2\ (1-x)\ xx\dots 11\ 00\ 11\ 00100$
   (*iii*) $a_1a_2\ x\ (1-x)(1-x)\dots 00\ 11\ 00\ 111a_{n+4}a_{n+5}$
   when n is odd and for a suitable $x \in \{0,1\}$.

Proof We prove the statement by induction on $n \geq 3$. Assume that $\tau(a_1a_2a_3a_4a_5a_6a_7a_8) = 0001$; we distinguish three cases.

- $a_4a_5a_6 = 000$

a_1	a_2	a_3	0	0	0	a_7	a_8
		0	0	0	1		

.

Then $a_3 = 1$ otherwise $\delta(a_3a_4a_5a_6a_7) = \delta(0000a_7) = 1 \neq 0$.

- $a_4a_5a_6a_7a_8 = 00100$

a_1	a_2	a_3	0	0	1	0	0
		0	0	0	1		

.

Then, for the same reasons as above, $a_3 = 1$.

- $a_4a_5a_6 = 111$

a_1	a_2	a_3	1	1	1	a_7	a_8
		0	0	0	1		

.

Then $a_3 = 0$ otherwise $\delta(a_3a_4a_5a_6a_7) = \delta(1111a_7) = 1 \neq 0$.

Now let us suppose that the statement is true for n and that we have $\tau(a_1 \dots a_{n+6}) = 0^{n+1}1$:

a_1	a_2	a_3	a_4	$\dots$	a_{n+2}	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}
		0	0	$\dots$	0	0	1		

.

If n is even, by the inductive hypothesis one has either

$$a_4 \dots a_{n+4} = xx\ (1-x)(1-x) \dots 11\ 000$$

or

$$a_4 \dots a_{n+6} = xx\ (1-x)(1-x) \dots 11\ 00100$$

or

$$a_4 \dots a_{n+4} = (1-x)(1-x)\ xx \dots 00\ 111$$

for a suitable $x \in \{0, 1\}$.

In any case we have $a_4 = a_5$. If $a_3 = a_4$, then $\delta(a_3a_4a_5a_6a_7) = \delta(a_4a_4a_4a_6a_7) = 1 \neq 0$. Thus $a_3 \neq a_4$.

It follows, in the three cases, that either

$$a_1 \dots a_{n+6} = a_1a_2\ (1-x)\ xx\ (1-x)(1-x) \dots 11\ 000a_{n+5}a_{n+6}$$

or

$$a_1 \dots a_{n+6} = a_1a_2\ (1-x)\ xx\ (1-x)(1-x) \dots 11\ 00100$$

or

$$a_1 \dots a_{n+6} = a_1a_2\ x\ (1-x)(1-x)\ xx \dots 00\ 111a_{n+5}a_{n+6}.$$

If n is odd, by the inductive hypothesis we have either

$$a_4 \dots a_{n+4} = (1-x)\ xx \dots 11\ 000$$

or

$$a_4 \dots a_{n+6} = (1-x)\ xx \dots 11\ 00100$$

or

$$a_4 \dots a_{n+4} = x\ (1-x)(1-x) \dots 00\ 111$$

for a suitable $x \in \{0, 1\}$.

In any case $a_4 \neq a_5 = a_6$. If $a_3 \neq a_4$, then $a_3a_4a_5 = a_5a_4a_5$ so that $a_4 = 1$ (otherwise we had a forbidden block). For the same reason, $a_2 = a_3 = 0$. This implies $\delta(a_2a_3a_4a_5a_6) = \delta(00100) = 1 \neq 0$. Thus $a_3 = a_4$.

It follows, in the three cases, that either

$$a_1 \dots a_{n+6} = a_1a_2\ (1-x)(1-x)\ xx \dots 11\ 000a_{n+5}a_{n+6}$$

or

$$a_1 \dots a_{n+6} = a_1 a_2 \ (1-x)(1-x) \ xx \dots 11 \ 00100$$

or

$$a_1 \dots a_{n+6} = a_1 a_2 \ xx \ (1-x)(1-x) \dots 00 \ 111 a_{n+5} a_{n+6}.$$

Then the statement is still true for $0^{n+1}1$. □

Proposition 3.3.9 *The local function τ is a transition function, that is $\tau(X_e) \subseteq X_e$.*

PROOF $\tau(X_e)$ being a subshift of $\{0,1\}^{\mathbf{Z}}$, it suffices to prove that no forbidden block 10^n1 with n odd, has a pre–image of length $n+6$ in the language of X_e. First we prove that there is no block $a_1a_2a_3a_4a_5a_6a_7$ of length 7 such that $\tau(a_1a_2a_3a_4a_5a_6a_7) = 101$:

a_1	a_2	a_3	a_4	a_5	a_6	a_7
		1	0	1		

.

We distinguish two cases.

- $a_3a_4 = 00$

a_1	a_2	0	0	a_5	a_6	a_7
		1	0	1		

Then $a_2 = 1$ otherwise $\delta(a_2a_3a_4a_5a_6) = \delta(000a_5a_6) = 1 \neq 0$. Then $\delta(a_1a_2a_3a_4a_5)$ $= \delta(a_1100a_5a_6) = 0 \neq 1$.

- $a_3a_4a_5 = 111$

a_1	a_2	1	1	1	a_6	a_7
		1	0	1		

Then $a_2 = 0$ otherwise $\delta(a_2a_3a_4a_5a_6) = \delta(1111a_6) = 1 \neq 0$. Thus $\delta(a_1a_2a_3a_4a_5)$ $= \delta(a_10111a_6) = 0 \neq 1$. We have proved that no block of length 7 goes to 101 under τ.

Let us now prove that no block $a_1 \dots a_{n+6}$ of length $n+6$ has 10^n1 as image under τ, where $n \in \mathbf{N}$ is odd and strictly greater than 1. If

a_1	a_2	a_3	a_4	a_5	...	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}
		1	0	0	...	0	1		

,

by Lemma 3.3.8 we have $a_4a_5a_6 \dots = x(1-x)(1-x)\dots$, and being $\delta(a_1a_2a_3a_4a_5)$ $= 1$, we distinguish two cases:

- $x = 0$

a_1	a_2	a_3	0	1	1	...	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}
		1	0	0	0	...	0	1		

Then $a_3 = 0$ (otherwise we had a forbidden block) and $a_2 = 1$ because $\delta(a_2a_3a_4a_5a_6) = \delta(a_20011) = 0$ and $\delta(00011) = 1$. Then $\delta(a_1a_2a_3a_4a_5) = \delta(a_11001) = 0 \neq 1$.

- $x = 1$

a_1	a_2	a_3	1	0	0	...	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}
		1	0	0	0	...	0	1		

If $a_3 = 0$ then $a_2 = 0$ and $\delta(a_2a_3a_4a_5a_6) = \delta(00100) = 1 \neq 0$. Thus $a_3 = 1$. Then $\delta(a_2a_3100) = \delta(a_21100)$ and $\delta(a_21100) = 0$ implies $a_2 = 0$. Thus $\delta(a_1a_2a_310) = \delta(a_10110) = 0 \neq 1$. Hence 10^n1 has no pre–image under τ. □

Proposition 3.3.10 *The transition function $\tau : X_e \to X_e$ is surjective.*

Proof To prove the surjectivity of τ it suffices, as we have seen, to prove the non–existence of GOE patterns or, equivalently, the non–existence of GOE words. To this aim, as it can be easily seen, it is enough to prove that each block of kind $10^{n_1}10^{n_2}\dots10^{n_k}1$ (where $n_1, \dots n_k$ are even integers), has a pre–image block. Indeed each word in $L(X_e)$ is contained in such a special word.

First we prove that every block of the type 10^n1 where n is even, has a pre–image $a_1 \dots a_{n+6}$ in the language of X_e of length $n + 6$

a_1	a_2	a_3	a_4	a_5	...	a_{n+2}	a_{n+3}	a_{n+4}	a_{n+5}	a_{n+6}
		1	0	0	...	0	0	1		

,

in each of the three cases in which $a_{n+4} \mapsto 1$.
If $n = 0$

-

0	0	0	0	a_5	a_6
		1	1		

,

-

0	0	0	1	0	0
		1	1		

,

and

-

1	1	1	1	a_5	a_6
		1	1		

.

If $n = 2$

-

a_1	a_1	1	0	0	0	a_5	a_6
		1	0	0	1		

,

-

a_1	a_1	1	0	0	1	0	0
		1	0	0	1		

,

and

-

0	0	0	1	1	1	a_5	a_6
		1	0	0	1		

.

If $n \geq 4$, for a suitable $x \in \{0,1\}$,

-

$1-x$	$1-x$	$1-x$	x	x	...	0	0	0	a_{n+5}	a_{n+6}
		1	0	0	...	0	0	1		

.

Similarly

-

$1-x$	$1-x$	$1-x$	x	x	...	0	0	1	0	0
		1	0	0	...	0	0	1		

and, finally,

-

x	x	x	$1-x$	$1-x$	...	1	1	1	a_{n+5}	a_{n+6}
		1	0	0	...	0	0	1		

.

Now, fix a word of kind $10^{n_1}10^{n_2}\dots10^{n_k}1$; we can construct a pre–image of this word starting from the first on the right block $10^{n_k}1$ (over the first on the right 1 we can write, arbitrarily, 000**, 111** or 00100). In this way we get a word $a_1 \dots a_5$ over the second on the left 1 and we can start from this word over 1 to construct a pre–image for the second on the right block $10^{n_{k-1}}1$, and so on:

							$\longleftarrow$							$\longleftarrow$			
...	b_1	b_2	b_3	b_4	b_5	...	a_1	a_2	a_3	a_4	a_5	...	*	*	*	*	*
...	0	0	1	0	0	...	0	0	1	0	0	...	0	0	1		
				$\underbrace{\qquad}_{n_{k-1}}$						$\underbrace{\qquad}_{n_k}$							

In each of the possible choices we can find a block whose image under τ is our fixed word.

For what we have stated before, τ is surjective. $\square$

Proposition 3.3.11 *The transition function $\tau : X_e \to X_e$ is not pre–injective.*

Proof Let us consider the configuration c_1:

...	0	0	0	0	0	**1**	**0**	**0**	1	0	0	0	0	0	...

and the configuration c_2:

...	0	0	0	0	0	**0**	**1**	**1**	1	0	0	0	0	0	...

.

These configurations are different only on a finite subset of $\mathbf{Z}$, but they have the same image under τ, that is the configuration

...	1	1	1	1	1	1	0	0	1	0	0	1	1	1	...

.

Thus τ is not pre–injective. $\square$

3.4 Gromov's Theorem

Recently, Gromov has proved a GOE–like theorem in a setting of graphs much more general than Cayley graphs, for alphabets not necessarily finite and for subset of the "universe" not necessarily invariant under translation. Because of the weakness of these hypotheses, in his theorem are needed properties that are stronger than ours (as we will see in next section), for example the *bounded propagation* of the spaces. In this section we apply Gromov's theorem to our cellular automata proving that all the properties required in the hypotheses of this theorem are satisfied.

Definition 3.4.1 A closed subset $X \subseteq A^\Gamma$ is of *bounded propagation* $\leq M$ if for each pattern $p \in A^F$ with support F one has

$$p_{|F\cap D(\alpha,M)} \in X_{F\cap D(\alpha,M)} \text{ for each } \alpha \in F \ \Rightarrow\ p \in X_F.$$

If $\gamma \in \Gamma$, the left translation $i_\gamma : \Gamma \to \Gamma$ defined by $i_\gamma(\alpha) = \gamma\alpha$ is an isometry. Indeed

$$\text{dist}(\gamma\alpha, \gamma\beta) = \|\alpha^{-1}\gamma^{-1}\gamma\beta\| = \|\alpha^{-1}\beta\| = \text{dist}(\alpha, \beta).$$

It is clear that not all the isometries are of this kind; for example consider $\mathbf{Z}^2 = \langle a, b \mid ab = ba\rangle$ and the unique homomorphism extending the function $i : \mathbf{Z}^2 \to \mathbf{Z}^2$ defined by

$$\begin{cases} i(a) = b \\ i(b) = a \end{cases};$$

clearly $\|i(\alpha)\| = \|\alpha\|$. Then

$$\text{dist}(i(\alpha), i(\beta)) = \|i(\alpha)^{-1}i(\beta)\| = \|i(\alpha^{-1}\beta)\| = \text{dist}(\alpha, \beta).$$

But i is not a translation.

Consider a subgroup $\bar\Gamma \leq \Gamma$ and the set $\mathcal{I}(\bar\Gamma)$ consisting of all restriction to each finite subset F of Γ of the left translations by an element of $\bar\Gamma$; a generic element of $\mathcal{I}(\bar\Gamma)$ is $i_{\gamma|F} : F \to \gamma F$. The set $\mathcal{I}(\bar\Gamma)$ is a *pseudogroup of partial isometries* that is, following Gromov's definition:

1. $\text{id}_F : F \to F$ is an element of $\mathcal{I}(\bar\Gamma)$, indeed $\text{id}_F = i_{1|F}$;

2. if $\gamma \in \bar{\Gamma}$ and $i_{\gamma_{|F}} : F \to \gamma F$ is an element of $\mathcal{I}(\bar{\Gamma})$, then $(i_{\gamma_{|F}})^{-1} : \gamma F \to F$ is still in $\mathcal{I}(\bar{\Gamma})$ because $(i_{\gamma_{|F}})^{-1} = i_{\gamma^{-1}_{|\gamma F}}$;

3. if $i_{\gamma_1{}_{|F}} : F \to \gamma_1 F$ and $i_{\gamma_2{}_{|\gamma_1 F}} : \gamma_1 F \to \gamma_2\gamma_1 F$ are two elements of $\mathcal{I}(\bar{\Gamma})$, then their composition is still in $\mathcal{I}(\bar{\Gamma})$ because $i_{\gamma_2{}_{|\gamma_1 F}} \circ i_{\gamma_1{}_{|F}} = i_{\gamma_2\gamma_1{}_{|F}}$;

4. the restriction of each element of $\mathcal{I}(\bar{\Gamma})$ defined on F to a (finite) subset $E \subseteq F$, is still in $\mathcal{I}(\bar{\Gamma})$.

Two elements $\alpha, \beta\ in \Gamma$ are $\bar{\Gamma}$–*equivalent* if there exists $\gamma \in \bar{\Gamma}$ such that $\gamma\alpha = \beta$, that is $\bar{\Gamma}\alpha = \bar{\Gamma}\beta$. Then the equivalence classes are the right cosets $\{\bar{\Gamma}\alpha \mid \alpha \in \Gamma\}$. The pseudogroup $\mathcal{I}(\bar{\Gamma})$ is *dense* if *each* equivalence class $\bar{\Gamma}\alpha$ is such that for some $R = R(\alpha) \in \mathbf{N}$ one has

$$\bigcup_{\gamma \in \bar{\Gamma}} D(\gamma\alpha, R) = \Gamma.$$

We prove that this property is equivalent to the existence of $R \in \mathbf{N}$ such that

$$\bigcup_{\gamma \in \bar{\Gamma}} D(\gamma, R) = \Gamma \tag{3.1}$$

Indeed if (3.1) holds and $\beta \in \Gamma$ there exists $\gamma \in \bar{\Gamma}$ such that $\mathrm{dist}(\beta, \gamma) \leq R$ hence

$$\mathrm{dist}(\beta, \gamma\alpha) \leq R + \mathrm{dist}(\gamma, \gamma\alpha) = R + \|\alpha\|$$

and then, fixed a right coset representative α of $\bar{\Gamma}\alpha$, we have $\bigcup_{\gamma \in \bar{\Gamma}} D(\gamma\alpha, R + \|\alpha\|) = \Gamma$. Moreover, we can prove that (3.1) holds if and only if $\bar{\Gamma}$ has finite index. Indeed, suppose that (3.1) holds; consider the right cosets $\bar{\Gamma}\alpha$ with $\alpha \in D_R$. It is clear that these are finitely many; furthermore $\bigcup_{\alpha \in D_R} \bar{\Gamma}\alpha = \Gamma$ because if $\beta \in \Gamma$ by (3.1), we have $\beta \in \gamma D_R$ with $\gamma \in \bar{\Gamma}$. Hence $\bar{\Gamma}$ has finite index.

Conversely, if $\bar{\Gamma}$ has finite index, fix a set $\{\alpha_1, \ldots, \alpha_n\}$ of cosets representatives. Let $R := \max_i \|\alpha_i\|$. If $\alpha \in \Gamma$, we have that $\alpha = \gamma\alpha_i$ for some $\gamma \in \bar{\Gamma}$ and some i. Hence $\mathrm{dist}(\alpha, \gamma) = \|\alpha_i\| \leq R$, that is $\alpha \in D(\gamma, R)$.

Now consider a *stable* (i.e. closed) and $\bar{\Gamma}$–invariant space $X \subseteq A^{\Gamma}$; if we consider the finite subsets of Γ and the elements of $\bar{\Gamma}$, a family of functions

$$H_{F,\gamma} : X_F \to X_{\gamma F} = X_{i_\gamma(F)}$$

which commute with the restriction (i.e. $(H_{F,\gamma}(c_{|F}))_{|\gamma E} = H_{E,\gamma}(c_{|E})$ or, in other words, the following diagram

$$\begin{array}{ccc} X_F & \xrightarrow{H_{F,\gamma}} & X_{\gamma F} \\ \downarrow & & \downarrow \\ X_E & \xrightarrow{H_{E,\gamma}} & X_{\gamma E} \end{array}$$

commutes), gives rise to a set of *holonomy maps*. In particular, we have a set of holonomy maps $\mathcal{H}(\bar{\Gamma})$ defining

$$H_{F,\gamma}(c_{|F}) := c^{\gamma^{-1}}{}_{|\gamma F};$$

indeed $(H_{F,\gamma}(c_{|F}))_{|\gamma E} = (c^{\gamma^{-1}}{}_{|\gamma F})_{|\gamma E} = c^{\gamma^{-1}}{}_{|\gamma E} = H_{E,\gamma}(c_{|E})$.

The set $\mathcal{H}(\bar{\Gamma})$ is a *pseudogroup of holonomies*, that is

1. $\mathrm{id}_{X_F} : X_F \to X_F$ is an element of $\mathcal{H}(\bar{\Gamma})$, indeed $\mathrm{id}_{X_F} = H_{F,1}$;
2. if $\gamma \in \bar{\Gamma}$ and $H_{F,\gamma} : X_F \to X_{\gamma F}$ is an element of $\mathcal{H}(\bar{\Gamma})$, then $(H_{F,\gamma})^{-1} : X_{\gamma F} \to X_F$ is still in $\mathcal{H}(\bar{\Gamma})$ because $(H_{F,\gamma})^{-1} = H_{\gamma F,\gamma^{-1}}$;
3. if $H_{F,\gamma_1} : X_F \to X_{\gamma_1 F}$ and $H_{\gamma_1 F,\gamma_2} : X_{\gamma_1 F} \to X_{\gamma_2\gamma_1 F}$ are two elements of $\mathcal{H}(\bar{\Gamma})$, then their composition is still in $\mathcal{H}(\bar{\Gamma})$ because $H_{\gamma_1 F,\gamma_2} \circ H_{F,\gamma_1} = H_{F,\gamma_2\gamma_1}$;
4. the restriction of each element of $\mathcal{H}(\bar{\Gamma})$ defined on X_F to X_E (where E is a finite subset $E \subseteq F$), is still in $\mathcal{H}(\bar{\Gamma})$.

Finally, if $\mathcal{I}(\bar{\Gamma})$ is dense (that is, if $\bar{\Gamma}$ has finite index), we have defined a *dense pseudogroup of holonomies*.

If $Y \subseteq A^{\Gamma}$ is another stable and $\bar{\Gamma}$–invariant space, a function $\tau : X \to Y$ is *of bounded propagation* $\leq M$ if it is the limit of a family of functions $\tau_F : X_F \to Y_{F^{-M}}$ that commute with the restrictions; then a function of bounded propagation is such that $\tau(c)_{|\alpha} = \tau_{D(\alpha,M)}(c_{|D(\alpha,M)})_{|\alpha}$ and, in general, $\tau_F(c_{|F}) = \tau(c)_{|F^{-M}}$.

If τ is a function of bounded propagation, we have that the holonomies in $\mathcal{H}(\bar{\Gamma})$ commute with τ if τ commutes with the $\bar{\Gamma}$–action:

$$\begin{array}{ccc} X_F & \xrightarrow{H_{F,\gamma}} & X_{\gamma F} \\ \tau_F \Big\downarrow & & \Big\downarrow \tau_{\gamma F} \\ X_{F^{-M}} & \xrightarrow{H_{F^{-M},\gamma}} & X_{\gamma F^{-M}} \end{array}$$

Indeed (notice that $\gamma(F^{-M}) = (\gamma F)^{-M}$),

$$\tau_{\gamma F}(H_{F,\gamma}(c_{|F})) = \tau_{\gamma F}((c^{\gamma^{-1}})_{|\gamma F})) = \tau(c^{\gamma^{-1}})_{|(\gamma F)^{-M}}$$

and

$$H_{F^{-M},\gamma}(\tau_F(c_{|F})) = H_{F^{-M},\gamma}(\tau(c)_{|F^{-M}}) = (\tau(c)^{\gamma^{-1}})_{|\gamma F^{-M}}.$$

In this case, provided that $\mathcal{I}(\bar{\Gamma})$ is dense, we say that the function τ *admits a dense holonomy*.

Under these hypotheses and supposing that Γ is amenable, we have the following theorem.

Theorem 3.4.2 *Let $X, Y \subseteq A^\Gamma$ be stable spaces of bounded propagation and $\tau : X \to Y$ a map of bounded propagation admitting a dense holonomy, then* $\mathrm{ent}(X) = \mathrm{ent}(Y)$ *implies that τ is surjective if and only if it is pre–injective.*

Suppose that τ is a bounded propagation $\leq M$ function between two $\bar{\Gamma}$–invariant stable spaces and τ commutes with the $\bar{\Gamma}$–action, if the pseudogroup $\mathcal{I}(\bar{\Gamma})$ is dense, we can write each $\alpha \in \Gamma$ as $\alpha = \gamma d$ ($\gamma \in \bar{\Gamma}$, $d \in D_R$) and

$$\tau(c)_{|\alpha} = \tau(c)_{|\gamma d} = (\tau(c^\gamma))_{|d} = \tau_{D(d,M)}(c^\gamma{}_{|D(d,M)})_{|d} = \tau_{D_{M+R}}(c^\gamma{}_{|D_{M+R}})_{|d}.$$

This means that in order to know the function τ it is sufficient to know how the image under τ of a configuration in X acts on D_R. In other words, it is sufficient to know the function $\tau_{D_{M+R}} : X_{D_{M+R}} \to \tau(X)_{D_R}$.

On the other hand, if τ is M–local between two shift spaces, we have

$$\tau(c)_{|\alpha} = \tau(c^\alpha)_{|1} = \tau_{D_M}(c^\alpha{}_{|D_M})_{|1}$$

that is it suffices to know how the image under τ of a configuration in X acts on the identity of Γ, i.e. the local rule δ.

For this reasons, the notion of bounded propagation is a generalization of the notion of local function as far as stable spaces not necessarily Γ–invariant are concerned.

Hence, if Γ is amenable, the next two theorems follow from Theorem 3.4.2.

Corollary 3.4.3 (GOE–theorem for shifts of bounded propagation) *Let $X, Y \subseteq A^\Gamma$ shift spaces of bounded propagation and $\tau : X \to Y$ a local function, then* $\mathrm{ent}(X) = \mathrm{ent}(Y)$ *implies τ surjective $\iff$ τ pre–injective.*

Corollary 3.4.4 (MM–property for shifts of bounded propagation) *If $X \subseteq A^\Gamma$ is a shift space of bounded propagation then X has the MM–property.*

3.5 Strongly Irreducible Shifts

In this section we give the definition of *strong irreducibility* for a shift. In general, as we have seen at the end of Section 1.2, it is possible to give a definition of irreducibility that generalizes the one–dimensional one. But although we can prove the MM–property for irreducible shifts of finite type of $A^{\mathbf{Z}}$, this irreducibility is too weak in the general case of subshifts of finite type of A^Γ (see Counterexample 3.3.5). We prove the MM–property for the strongly irreducible shifts of finite type of A^Γ. On the other hand, we will see that a shift of bounded propagation (that has, by Gromov's theorem, the MM–property), is strongly irreducible and of finite type, but the converse does not hold.

Definition 3.5.1 A shift X is called *M–irreducible* if for each pair of finite sets $E, F \subseteq \Gamma$ such that $\mathrm{dist}(E, F) > M$ and for each pair of patterns $p_1 \in X_E$ and $p_2 \in X_F$, there exists a configuration $c \in X$ that satisfies $c = p_1$ in E and $c = p_2$ in F. The shift X is called *strongly irreducible* if it is M–irreducible for some $M \in \mathbf{N}$.

In the particular case $\Gamma = \mathbf{Z}$, it can be easily seen that a shift $X \subseteq A^{\mathbf{Z}}$ is M–irreducible if for each $n \geq M$ and for each pair of words $u, v \in L(X)$, there exists a word $w \in L(X)$ with $|w| = n$, such that $uwv \in L(X)$.

The following theorem will be proved in the next section in the case of groups with non–exponential growth and semi–strongly irreducible shift.

Proposition 3.5.2 *Let Γ be an amenable group. Let X be a strongly irreducible shift of finite type and let $\tau : X \to A^\Gamma$ be a local and pre–injective function. Then* $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$.

Proof Suppose that the memory of X is M, that X is M–irreducible and that τ is M–local. Set $Y := \tau(X)$ and fix an amenable sequence $(E_n)_n$; we have

$$|Y_{E_n^{+2M}}| \leq |Y_{E_n}||A|^{|\partial^+_{2M} E_n|}$$

and then

$$\frac{\log |Y_{E_n^{+2M}}|}{|E_n|} \leq \frac{\log |Y_{E_n}|}{|E_n|} + \frac{|\partial^+_{2M} E_n| \log |A|}{|E_n|}.$$

Taking the maximum limit and being

$$\lim_{n\to\infty} \frac{|\partial^+_{2M} E_n|}{|E_n|} = 0,$$

we have

$$\limsup_{n\to\infty} \frac{\log |Y_{E_n^{+2M}}|}{|E_n|} \leq \mathrm{ent}(Y).$$

Suppose that $\mathrm{ent}(Y) < \mathrm{ent}(X)$; then

$$\limsup_{n\to\infty} \frac{\log |Y_{E_n^{+2M}}|}{|E_n|} < \limsup_{n\to\infty} \frac{\log |X_{E_n}|}{|E_n|};$$

so that there exists $n \in \mathbf{N}$ such that

$$\frac{\log |Y_{E_n^{+2M}}|}{|E_n|} < \frac{\log |X_{E_n}|}{|E_n|}$$

that is

$$|Y_{E_n^{+2M}}| < |X_{E_n}|.$$

Fix $v \in X_{\partial^+_{2M} E_n^{+M}}$; since $\mathrm{dist}(\partial^+_{2M} E_n^{+M}, E_n) = M + 1 > M$ for each $u \in X_{E_n}$ there exists a pattern $p \in X_{E_n^{+3M}}$ that coincides with u on E_n and with v on $\partial^+_{2M} E_n^{+M}$. Then

$$|\{p \in X_{E_n^{+3M}} \mid p_{|\partial^+_{2M} E_n^{+M}} = v\}| \geq |X_{E_n}| > |Y_{E_n^{+2M}}|.$$

Since $\tau_{E_n^{+3M}} : X_{E_n^{+3M}} \to Y_{E_n^{+2M}}$ is surjective, there exist two patterns $p_1, p_2 \in X_{E_n^{+3M}}$ such that $p_1 \neq p_2$ but $p_1 = v = p_2$ on $\partial^+_{2M} E_n^{+M}$ and $\tau_{E_n^{+3M}}(p_1) =$

$\tau_{E_n^{+3M}}(p_2)$. By Corollary 1.3.4, there exist two configurations $c_1, c_2 \in X$ which extend p_1 and p_2 and which coincide outside E_n^{+M}. We prove that $\tau(c_1) = \tau(c_2)$, and hence that τ is not pre–injective. If $\gamma \in E_n^{+2M}$ we have $\gamma D_M \subseteq E_n^{+3M}$ and hence, if $D_M = \{\alpha_1, \ldots, \alpha_m\}$, $\tau(c_1)_{|\gamma} = \delta(c_{1|\gamma\alpha_1}, \ldots, c_{1|\gamma\alpha_m}) = \delta(p_{1|\gamma\alpha_1}, \ldots, p_{1|\gamma\alpha_m}) = \tau_{E_n^{+3M}}(p_1)_{|\gamma} = \tau_{E_n^{+3M}}(p_2)_{|\gamma} = \delta(p_{2|\gamma\alpha_1}, \ldots, p_{2|\gamma\alpha_m}) = \delta(c_{2|\gamma\alpha_1}, \ldots, c_{2|\gamma\alpha_m}) = \tau(c_2)_{|\gamma}$. If $\gamma \notin E_n^{+2M}$, we have $\gamma D_M \subseteq \complement(E_n^{+M})$ and hence $\tau(c_1)_{|\gamma} = \tau(c_2)_{|\gamma}$, since c_1 coincide with c_2 on $\complement(E_n^{+M})$. $\square$

Lemma 3.5.3 *If Γ is a finitely generated group, there exists a sequence of disks $(F_j)_{j\in\mathbf{N}}$ obtained by translation of a disk D and at distance $> M$ such that $\bigcup_{j\in\mathbf{N}} F_j^{+R} = \Gamma$ for a suitable $R > 0$.*

Proof Let D be the disk centered at 1 and of radius ρ; define the following sequence of finite subsets of Γ:

$$\Gamma_0 := \{1\},$$

$$\Gamma_1 := \{\gamma \in \Gamma \mid \|\gamma\| = 2\rho + M + 1\}$$

and, in general,

$$\Gamma_n := \{\gamma \in \Gamma \mid \|\gamma\| = n(2\rho + M + 1)\}.$$

It is clear that for each n, $\mathrm{dist}(\Gamma_n, \Gamma_{n+1}) = 2\rho + M + 1$. Inside the set Γ_n, fix $\gamma_{n,1}$ and eliminate all the points in Γ_n whose distance from $\gamma_{n,1}$ is less than $2\rho + M + 1$.

Next, fix $\gamma_{n,2}$ among the remaining points and eliminate all the points whose distance from $\gamma_{n,2}$ is less than $2\rho + M + 1$. In this way, we will get a set $\bar{\Gamma}_n$ whose elements have mutual distance $\geq 2\rho + M + 1$ and such that for each element γ_n of Γ_n there exists an element of $\bar{\Gamma}_n$ whose distance from γ_n is less than $2\rho + M + 1$.

We now prove that, denoting by $(\beta_j)_{j\in\mathbf{N}}$ the sequence of the elements of $\bigcup_{n\in\mathbf{N}} \bar{\Gamma}_n$, the sequence $(\beta_j D)_{j\in\mathbf{N}}$ is a (D, M, R)–net with $R := 2\rho + 2M$; so that we can set $F_j := \beta_j D$.

Let then $\gamma \in \Gamma$; there exists $\gamma_n \in \Gamma_n$ such that $\mathrm{dist}(\gamma, \gamma_n) \leq \rho + M$. Since γ_n belongs to Γ_n, there is $\bar{\gamma}_n \in \bar{\Gamma}_n$ such that $\mathrm{dist}(\gamma_n, \bar{\gamma}_n) \leq 2\rho + M$ and hence $\mathrm{dist}(\gamma, \bar{\gamma}_n) \leq 3\rho + 2M$; then $\gamma \in (\bar{\gamma}_n D)^{+(2\rho+2M)}$. $\square$

We call the above sequence a (D, M, R)–net.

Lemma 3.5.4 *Let Γ be an amenable group and let $(E_n)_n$ be a fixed amenable sequence of Γ. Let $(F_j)_{j\in\mathbf{N}}$ be a $(D_r, 2M, R)$–net, let X be an M–irreducible shift and let Y be a subset of X such that $Y_{F_j} \subset X_{F_j}$ for every $j \in \mathbf{N}$. Then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

Proof Let $(p_j)_{j\in\mathbf{N}}$ be a sequence of patterns such that $p_j \in X_{F_j} \backslash Y_{F_j}$; let $N(n)$ be the number of F_j's such that $F_j^{+M} \subseteq E_n$ and denote by $F_{j_1}, \ldots, F_{j_N}$

these disks. Set $\xi := |X_{D^{+M}}|$ and denote by $\pi_{j_i} : X_{E_n} \to X_{F_{j_i}}$ the restriction to F_{j_i} of the patterns of X_{E_n}. We prove that

$$|X_{E_n} \backslash \bigcup_{i=1}^{N} \pi_{j_i}^{-1}(p_{j_i})| \leq (1-\xi^{-1})^N |X_{E_n}| \tag{3.2}$$

by induction on $m \in \{1, \ldots, N\}$. We have

$$|X_{E_n}| \leq |X_{F_{j_1}^{+M}}||X_{E_n \backslash F_{j_1}^{+M}}|$$

then

$$|X_{E_n}| \leq \xi |X_{E_n \backslash F_{j_1}^{+M}}|.$$

Since X is an M–irreducible shift and since $\mathrm{dist}(F_{j_1}, E_n \backslash F_{j_1}^{+M}) > M$, given a pattern $p \in X_{E_n \backslash F_{j_1}^{+M}}$, there exists a pattern $\bar{p}$ defined on all E_n that coincides with p on $E_n \backslash F_{j_1}^{+M}$ and with p_{j_1} on F_{j_1}; then

$$|X_{E_n \backslash F_{j_1}^{+M}}| \leq |\pi_{j_1}^{-1}(p_{j_1})|.$$

Hence

$$\frac{1}{\xi}|X_{E_n}| \leq |\pi_{j_1}^{-1}(p_{j_1})|$$

and

$$|X_{E_n} \backslash \pi_{j_1}^{-1}(p_{j_1})| \leq |X_{E_n}| - \frac{1}{\xi}|X_{E_n}| = (1-\xi^{-1})|X_{E_n}|.$$

Suppose

$$|X_{E_n} \backslash \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})| \leq (1-\xi^{-1})^{m-1}|X_{E_n}|.$$

Since

$$|X_{E_n} \backslash \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})| \leq |X_{F_{j_m}^{+M}}||\{p_{|E_n \backslash F_{j_m}^{+M}} \mid p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})\}|,$$

and, being $|X_{F_{j_m}^{+M}}| = \xi$,

$$|X_{E_n} \backslash \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})| \leq \xi |\{p_{|E_n \backslash F_{j_m}^{+M}} \mid p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})\}|.$$

Moreover, since X is M–irreducible,

$$|\{p_{|E_n \backslash F_{j_m}^{+M}} \mid p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})\}| \leq$$

$$\leq |\{p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i}) \mid p_{|F_{j_m}} = p_{j_m}\}|.$$

Hence

$$\frac{1}{\xi}|X_{E_n}\setminus\bigcup_{i=1}^{m-1}\pi_{j_i}^{-1}(p_{j_i})| \le |\{p\in X_{E_n}\setminus\bigcup_{i=1}^{m-1}\pi_{j_i}^{-1}(p_{j_i}) \mid p_{|F_{j_m}} = p_{j_m}\}|$$

and then

$$|X_{E_n}\setminus\bigcup_{i=1}^{m}\pi_{j_i}^{-1}(p_{j_i})| = |\Big(X_{E_n}\setminus\bigcup_{i=1}^{m-1}\pi_{j_i}^{-1}(p_{j_i})\Big)\setminus\pi_{j_m}^{-1}(p_{j_m})| \le$$

$$\le |\Big(X_{E_n}\setminus\bigcup_{i=1}^{m-1}\pi_{j_i}^{-1}(p_{j_i})\Big)\setminus\{p\in X_{E_n}\setminus\bigcup_{i=1}^{m-1}\pi_{j_i}^{-1}(p_{j_i}) \mid p_{|F_{j_m}} = p_{j_m}\}| \le$$

$$\le |X_{E_n}\setminus\bigcup_{i=1}^{m-1}\pi_{j_i}^{-1}(p_{j_i})| - \frac{1}{\xi}|X_{E_n}\setminus\bigcup_{i=1}^{m-1}\pi_{j_i}^{-1}(p_{j_i})| \le$$

$$\le (1-\frac{1}{\xi})(1-\xi^{-1})^{m-1}|X_{E_n}| = (1-\xi^{-1})^{m}|X_{E_n}|.$$

Hence (3.2) holds, and since $|Y_{E_n}| \le |X_{E_n}\setminus\bigcup_{i=1}^{N}\pi_{j_i}^{-1}(p_{j_i})|$, we have

$$\frac{\log|Y_{E_n}|}{|E_n|} \le \frac{N(n)\log(1-\xi^{-1})}{|E_n|} + \frac{\log|X_{E_n}|}{|E_n|}. \tag{3.3}$$

Observe that

$$E_n \subseteq \bigcup_{i=1}^{N} F_{j_i}^{+R}\cup(E_n\setminus E_n^{-(R+2r+M)}). \tag{3.4}$$

Indeed suppose that $\gamma\in E_n$ and $\gamma\notin\bigcup_{i=1}^{N}F_{j_i}^{+R}$; $(F_j)_j$ being a $(D_r,2M,R)$–net, we have that $\gamma\in F_k^{+R}$ for some k, that is $\gamma\in\beta D^{+R}$ with β such that $\beta D^{+M}\nsubseteq E_n$. Hence $\mathrm{dist}(\gamma,\beta)\le r+R$ so that $\beta\in\gamma D^{+R}$. If $\gamma\in E_n^{-(R+2r+M)}$, then $\beta\in\gamma D^{+R}\subseteq E_n^{-(r+M)}$ so that $F_k^{+M}=\beta D^{+M}\subseteq E_n$ which is excluded.

From (3.4), we have

$$|E_n| \le N(n)|D^{+R}| + |E_n\setminus E_n^{-(R+2r+M)}|$$

so that

$$1 \le \frac{N(n)}{|E_n|}|D^{+R}| + \frac{|\partial^-_{R+2r+M}E_n|}{|E_n|};$$

taking the minimum limit and being $\lim_{n\to\infty}\frac{|\partial^-_{R+2r+M}E_n|}{|E_n|}=0$,

$$\zeta := \liminf_{n\to\infty}\frac{N(n)}{|E_n|} > 0$$

and then from (3.3) it follows

$$\mathrm{ent}(Y) \le \zeta\log(1-\xi^{-1}) + \mathrm{ent}(X) < \mathrm{ent}(X). \quad \square$$

Proposition 3.5.5 *Let X be a strongly irreducible shift of finite type, let $\tau : X \to A^{\mathbf{Z}}$ be a local function such that* $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$*. Then τ is pre–injective.*

PROOF Suppose that X has memory M, that is X is M–irreducible and that τ is M–local. Moreover suppose that τ is not pre–injective; then there exist $c_1, c_2 \in X$ and a disk D contained in Γ, such that $c_1 \neq c_2$ on D, $c_1 = c_2$ out of D and $\tau(c_1) = \tau(c_2)$. Set $(F_j)_{j\in\mathbf{N}} = (\beta_j D^{+2M})_{j\in\mathbf{N}}$ a $(D^{+2M}, 2M, R)$–net and denote by Y the subset of X defined by

$$Y := \{c \in X \mid (c^{\beta_j})_{|D^{+2M}} \neq c_{2|D^{+2M}} \text{ for every } j \in \mathbf{N}\},$$

that is the subset of X avoiding the pattern $c_{2|D^{+2M}}$ on the disk D^{+2M} and on the translated disks $F_j = \beta_j D^{+2M}$. The set Y is a subset of X such that $Y_{F_j} \subset X_{F_j}$; we prove that $\tau(Y) = \tau(X)$. Indeed if $c \in X \backslash Y$, there exists a subset $J \subseteq \mathbf{N}$ such that for every $j \in J$, we have $(c^{\beta_j})_{|D^{+2M}} = c_{2|D^{+2M}}$. Define $\bar{c} \in X$ in the following way:

- $\bar{c} = {c_1}^{\beta_j^{-1}}$ on F_j for every $j \in J$,
- $\bar{c} = c$ out of the union $\bigcup_{j\in J} F_j$.

That is, $\bar{c}$ is obtained from c substituting all the occurences of $c_{2|D^{+2M}}$ with $c_{1|D^{+2M}}$.

By Proposition 1.3.3, we have $\bar{c} \in X$ and moreover $\bar{c} \in Y$; we prove that $\tau(\bar{c}) = \tau(c)$.

If $\gamma \in \beta_j D^{+M}$ for some $j \in J$, we have $\gamma D_M \subseteq F_j$ and then $\tau(\bar{c})_{|\gamma} = \tau({c_1}^{\beta_j^{-1}})_{|\gamma} = \tau(c_1)_{|\beta_j^{-1}\gamma} = \tau(c_2)_{|\beta_j^{-1}\gamma} = \tau({c_2}^{\beta_j^{-1}})_{|\gamma} = \tau(c)_{|\gamma}$.

Suppose that $\gamma \notin \beta_j D^{+M}$ for every $j \in J$; then $\gamma D_M \subseteq \complement(\beta_j D)$ and hence $\tau(\bar{c})_{|\gamma} = \tau(c)_{|\gamma}$. Indeed $\bar{c}$ and c coincide on $\bigcup_{j\in J} \complement(\beta_j D)$: if $j \in J$ and $\gamma \in \partial^+_{2M}\beta_j D = F_j \backslash \beta_j D$, we have $\bar{c}_{|\gamma} = ({c_1}^{\beta_j^{-1}})_{|\gamma} = c_{1|\beta_j^{-1}\gamma}$. Since $\beta_j^{-1}\gamma \in \partial^+_{2M} D$ one has $c_{1|\beta_j^{-1}\gamma} = c_{2|\beta_j^{-1}\gamma} = ({c_2}^{\beta_j^{-1}})_{|\gamma} = c_{|\gamma}$.

Then, by Lemma 3.5.4,

$$\mathrm{ent}(\tau(X)) = \mathrm{ent}(\tau(Y)) \leq \mathrm{ent}(Y) < \mathrm{ent}(X). \quad \Box$$

Proposition 3.5.6 *Let Γ be an amenable group. Let X be a shift, let Y be a strongly irreducible shift and let $\tau : X \to Y$ be a local function such that* $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y)$*. Then τ is surjective.*

PROOF Let X and Y be as in the hypotheses and let $\tau : X \to Y$ be a local function. We prove that if $\tau(X) \subset Y$, then $\mathrm{ent}(\tau(X)) < \mathrm{ent}(Y)$. Indeed if $\tau(X) \subset Y$, there exists a configuration $c \in Y$ which does not belong to $\tau(X)$ and then there exists a disk D such that $c_{|D} \in Y_D \backslash (\tau(X))_D$. Let $(F_j)_{j\in\mathbf{N}}$ be a $(D, 2M, R)$–net; then $(\tau(X))_{F_j} \subset Y_{F_j}$; by Lemma 3.5.4, $\mathrm{ent}(\tau(X)) < \mathrm{ent}(Y)$. $\Box$

Theorem 3.5.7 *Let Γ be an amenable group. Let X be a strongly irreducible shift of finite type, let Y be a strongly irreducible shift and let $\tau : X \to Y$ be a local function with* $\mathrm{ent}(X) = \mathrm{ent}(Y)$. *Then τ is pre–injective if and only if is surjective.*

PROOF If τ is pre–injective we have, by Proposition 3.5.2, that $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$. Then $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y)$ so that, by Proposition 3.5.6, τ is surjective.

If, conversely, τ is surjective then $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y)$ that is $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$. By Proposition 3.5.5, τ is pre–injective. □

Corollary 3.5.8 (MM–property for strongly irreducible shifts of finite type) *If Γ is an amenable group, a strongly irreducible subshift of finite type of A^Γ has the MM–property.*

We conclude this section proving that the property of bounded propagation for a shift is strictly stronger than the union of strong irreducibility and finite type condition.

Lemma 3.5.9 *A shift X is of finite type with memory M if and only if each configuration $c \in A^\Gamma$ such that $c_{|D(\alpha,M)} \in X_{D(\alpha,M)}$ for every $\alpha \in \Gamma$, belongs to X.*

PROOF Let X be a shift of finite type with memory M, let $\mathcal{F}$ a finite set of forbidden blocks each one with support D_M and let $c \in A^\Gamma$ be a configuration such that $c_{|D(\alpha,M)} \in X_{D(\alpha,M)}$ for every $\alpha \in \Gamma$.

We prove that $c \in X$; it is clear that for each α, ${c^\alpha}_{|D_M} \in X_{D_M}$ and hence we have ${c^\alpha}_{|D_M} \notin \mathcal{F}$, that is $c \in X_\mathcal{F} = X$.

For the converse, suppose that each configuration $c \in A^\Gamma$ such that $c_{|D(\alpha,M)} \in X_{D(\alpha,M)}$ for every $\alpha \in \Gamma$, belongs to X. Define

$$\mathcal{F} := \{c_{|D_M} \mid c_{|D_M} \notin X_{D_M}\};$$

if $c \in X$ we have that for each α, ${c^\alpha}_{|D_M} \in X_{D_M} \Rightarrow {c^\alpha}_{|D_M} \notin \mathcal{F}$ and $c \in X_\mathcal{F}$. If $c \in X_\mathcal{F}$ we have for each α that ${c^\alpha}_{|D_M} \in X_{D_M} \Rightarrow c_{|\alpha D_M} \in X_{\alpha D_M}$ and $c \in X$. □

Now we can prove the following statement.

Proposition 3.5.10 *If $X \subseteq A^\Gamma$ is a shift of bounded propagation, then X is strongly irreducible and of finite type.*

PROOF Suppose that X has bounded propagation $\leq M$; if $E, F \subseteq \Gamma$ are such that $\mathrm{dist}(E,F) > M$ and $p_1 \in X_E$, $p_2 \in X_F$ are two patterns of X, consider the pattern p with support $E \cup F$ given by the union of the functions p_1 and p_2. Clearly $p \in X_{E\cup F}$ because if $\alpha \in E \cup F$ and, for instance $\alpha \in E$, we have $(E \cup F) \cap \alpha D_M \subseteq E$ and hence $p_{|(E\cup F)\cap\alpha D_M} \in X_{(E\cup F)\cap\alpha D_M}$. A configuration in X extending p is such that $c_{|E} = p_1$ and $c_{|F} = p_2$. Hence X is M–irreducible.

Now suppose that $c \in A^\Gamma$ is such that $c_{|D(\alpha,M)} \in X_{D(\alpha,M)}$ for every $\alpha \in \Gamma$. Then if $n \geq M$ and $\alpha \in D_n$ we have

$$c_{|D_n \cap D(\alpha,M)} = (c_{|D(\alpha,M)})_{|D_n \cap D(\alpha,M)} \in X_{D_n \cap D(\alpha,M)};$$

X being of bounded propagation we have $c_{|D_n} \in X_{D_n}$. X being closed we have $c \in X$. □

If $\Gamma = \mathbf{Z}$ and X is an edge shift, then also the converse of this theorem holds.

Proposition 3.5.11 *If $X \subseteq A^{\mathbf{Z}}$ is a is strongly irreducible edge shift, then it is of bounded propagation.*

Proof Let $\mathbf{G}$ be a graph such that $X = X_{\mathbf{G}}$; notice that if $uv, vw \in L(X)$, then $uvw \in L(X)$ for every word $v \in L(X)$ such that $|v| \geq 1$. Indeed if $e_1, \dots, e_n,\ f_1, \dots, f_m,\ g_1, \dots, g_l \in \mathcal{E}(\mathbf{G})$ are edges of $\mathbf{G}$ such that $u = e_1 \dots e_n,\ v = f_1 \dots f_m$ and $w = g_1 \dots g_l$, we have:

$$uv: \quad i_1 \xrightarrow{e_1} i_2 \xrightarrow{e_2} \cdots \xrightarrow{e_n} i_{n+1} \xrightarrow{f_1} i_{n+2} \xrightarrow{f_2} \cdots \xrightarrow{f_m} i_{n+m+1}$$

$$vw: \quad i_{n+1} \xrightarrow{f_1} j_{n+2} \xrightarrow{f_2} \cdots \xrightarrow{f_m} i_{n+m+1} \xrightarrow{g_1} i_{n+m+2} \xrightarrow{g_2} \cdots \xrightarrow{g_l} i_{n+m+l+1}.$$

Then it is clear that the word

$$uvw: \quad i_1 \xrightarrow{e_1} \cdots \xrightarrow{e_n} i_{n+1} \xrightarrow{f_1} \cdots \xrightarrow{f_m} i_{n+m+1} \xrightarrow{g_1} \cdots \xrightarrow{g_l} i_{n+m+l+1}$$

belongs to $L(X)$.

Suppose that X is M–irreducible; we prove that X has bounded propagation $\leq M$.

Let F be the interval $[1, L]$ and let $p \in A^F$ a pattern such that for each $\alpha \in F$ we have $p_{|F \cap D(\alpha,M)} \in X_{F \cap D(\alpha,M)}$; then there exist $q \geq 0$ and $0 \leq r < M$ for which $L = qM + r + 1$. Set $\alpha_1 := M + 1$, then $F \cap D(\alpha_1, M) = [1, 2M + 1]$ and hence $p_{|[1,2M+1]} \in X_{[1,2M+1]}$. Set $\alpha_2 := 2M + 1$, then $F \cap D(\alpha_2, M) = [M+1, 3M+1]$ and hence $p_{|[M+1,3M+1]} \in X_{[M+1,3M+1]}$. By the above property, we have that $p_{|[1,3M+1]} \in X_{[1,3M+1]}$. In this way we can prove that $p_{|[1,qM+1]} \in X_{[1,qM+1]}$. Set $\alpha_q := qM + 1$, then $F \cap D(\alpha_q, M) = [(q-1)M + 1, L]$ and hence $p = p_{|[1,L]} \in X_{[1,L]} = X_F$.

If F is the union of two disjoint intervals $F_1 = [1, L_1]$ and $F_2 = [L_1 + n, L_2]$ at distance $n \leq M$, we already have from the above case that $p_{|F_1} \in X_{F_1}$ and $p_{|F_2} \in X_{F_2}$. Set $\alpha := L_1$, then $F \cap D(\alpha, M) = F_1 \cup [L_1 + n, L_1 + M]$ and hence $p_{|F_1 \cup [L_1+n, L_1+M]} \in X_{F_1 \cup [L_1+n, L_1+M]}$. Then there exists a word u of length $n-2$ such that $p_{|F_1}\ u\ p_{|[L_1+n,L_1+M]} \in L(X)$ and the word $p_{|F_2} \in L(X)$. For the above property we have $p_{|F_1}\ u\ p_{|F_2} \in L(X)$ and hence $p \in X_F$.

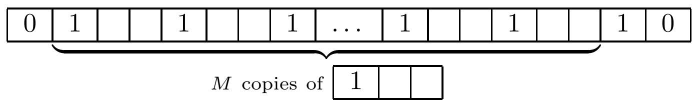
Finally, if F is the union of two disjoint intervals F_1 and F_2 at distance $> M$, we have that $p \in X_F$ for the M–irreducibility of X.

Because each finite subset F of $\mathbf{Z}$ is a finite union of intervals, we have that the property holds for every F. □

Now we prove that in general strong irreducibility and finite type condition do not imply the bounded propagation property. Consider the subshift $X \subseteq \{0,1\}^{\mathbf{Z}}$ with set of forbidden blocks:

$$\{010, 111\}.$$

Clearly X is a strongly irreducible (in fact 2–irreducible) shift of finite type; if $M \geq 1$ consider the following pattern p with $F := \mathrm{supp}(p)$



In this case we have $p_{|F \cap D(\alpha,M)} \in X_{F \cap D(\alpha,M)}$ but $p \notin X_F$; hence X is not of bounded propagation $\leq M$ for each $M \geq 1$.

3.6 Semi–Strongly Irreducible Shifts

As we have seen, the MM–property holds for irreducible subshifts of finite type of $A^{\mathbf{Z}}$. On the other hand we have that the MM–property holds, in general, for strongly irreducible subshifts of finite type of A^Γ (provided that Γ is amenable). In this section we define another form of irreducibility: the *semi–strong irreducibility*. For sofic shifts, this notion is equivalent to the general irreducibility in the one–dimensional case and, in all other cases, allows us to prove the Myhill–property for the subshifts of finite type (if Γ has non–exponential growth).

Definition 3.6.1 A shift X is called (M,k)*–irreducible* (where M, k are natural numbers such that $M \geq k$) if for each pair of finite sets $E, \alpha D \subseteq \Gamma$ (the second one is a ball centered at α) such that $\mathrm{dist}(E, \alpha D) > M$ and for each pair of patterns $p_1 \in X_E$ and $p_2 \in X_{\alpha D}$, there exists a configuration $c \in X$ that satisfies $c = p_1$ in E and $c = p_2$ in $\alpha\varepsilon D$, where $\varepsilon \in \Gamma$ is such that $\|\varepsilon\| \leq k$. The shift X is called *semi–strongly irreducible* if it is (M,k)–irreducible for some $M, k \in \mathbf{N}$.

Hence the difference between this new property and the strong irreducibility one, lies in the fact that the support of the second pattern must be a ball and the configuration c merging the two patterns moves this support "slightly". Notice that this move is a translation and hence it make sense to say that the configuration c restricted to $\alpha\varepsilon D$ coincides with $p_2 \in X_{\alpha D}$. Moreover, under the previous hypotheses, the translated disk $\alpha\varepsilon D$ is still contained in $(\alpha D)^{+M}$; indeed if $D = D_r$ and $\gamma \in \alpha\varepsilon D_r$, then $\mathrm{dist}(\gamma, \alpha) \leq \mathrm{dist}(\gamma, \alpha\varepsilon) + \mathrm{dist}(\alpha\varepsilon, \alpha) \leq r + \|\varepsilon^{-1}\| \leq r + k$. In particular we have that $E \cap \alpha\varepsilon D = \emptyset$.

In Definition 3.6.1 is in fact essential that, given a finite set $F \subseteq \Gamma$, it exists $\alpha \in \Gamma$ such that the translated set αF is still contained in F^{+M}. If the group is not abelian, the set αF could be quite far from F. On the other hand the set $F\alpha$ is α–near to F, but it is not, in general, obtained from F under translation.

Consider, for example, the free group $\mathbf{F}_2$ generated by the elements a and b. If $F = \{a^n, b^n\}$ with $n > M$, we have that does not exist an $\alpha \neq 1$ such that

$$\alpha F = \{\alpha a^n, \alpha b^n\} \subseteq F^{+M} = D(a^n, M) \cup D(b^n, M).$$

Indeed, if the reduced form of α is $\bar{\alpha} b$ or $\bar{\alpha} b^{-1}$ then $\mathrm{dist}(\alpha a^n, a^n) = \|a^{-n} \alpha a^n\| = \|a^{-n} \bar{\alpha} b^{\pm 1} a^n\| \geq n + 1$ and $\mathrm{dist}(\alpha a^n, b^n) = \|b^{-n} \alpha a^n\| = \|b^{-n} \bar{\alpha} b^{\pm 1} a^n\| \geq n$, that is $\alpha a^n \notin F^{+M}$. If, otherwise, the reduced form of α is $\bar{\alpha} a^{\pm 1}$ then $\alpha b^n \notin F^{+M}$.

To have a counterexample also in the amenable case, consider the infinite dihedral group $\mathbf{C}_2 * \mathbf{C}_2$ with the presentation $\langle a, b \mid a^2 = b^2 = 1\rangle$; we have that if $M = 1$ there is no $\alpha \neq 1$ such that $\alpha\{a, b\} \subseteq D(a, 1) \cup D(b, 1)$. Indeed if $\alpha = (ab)^n$ with $n > 0$ then $\alpha a = (ab)^n a = a(ba)^n$, hence

$$\mathrm{dist}(\alpha a, a) = \|a \alpha a\| = \|(ba)^n\| = 2n > 1$$

and

$$\mathrm{dist}(\alpha a, b) = \|b \alpha a\| = \|ba(ba)^n\| = \|(ba)^{n+1}\| = 2(n + 1) > 1.$$

In both cases, we have that $\alpha a \notin D(a, 1) \cup D(b, 1)$. If $\alpha = (ab)^n a$ with $n > 0$ then $\alpha a = (ab)^n$, hence

$$\mathrm{dist}(\alpha a, a) = \|a(ab)^n\| = \|(ba)^{n-1} b\| = 2(n - 1) + 1 = 2n - 1$$

and

$$\mathrm{dist}(\alpha a, b) = \|b(ab)^n\| = \|(ba)^n b\| = 2n + 1 > 1.$$

To have $\alpha a \in D(a, 1) \cup D(b, 1)$, it must be $n = 1$. Hence $\alpha = aba$ and $\alpha b = abab$. But, in this case, we have

$$\mathrm{dist}(\alpha b, a) = \|bab\| = 3 \quad \text{and} \quad \mathrm{dist}(\alpha b, b) = \|babab\| = 5.$$

For the symmetry between a and b, we have an analogous result if $\alpha = (ba)^n b$ or $\alpha = (ba)^n$.

This is the reason why, to avoid this problem, we require that the second set in Definition 3.6.1 is a ball centered at α. Then we consider the new center $\alpha\varepsilon$ (which is ε–near to α). The ball $\alpha\varepsilon D$ having the same radius as αD, is obtained by translation of it. As we have seen, if Γ has non–exponential growth the sets in the amenable sequence $(E_n)_n$ are balls centered at 1. Hence if M is large enough we have that $\varepsilon E_n \subseteq E_n^{+M}$.

In the particular case $\Gamma = \mathbf{Z}$, it can be easily seen that a shift $X \subseteq A^{\mathbf{Z}}$ is (M, k)–irreducible if for each $n \geq M$ and for each pair of words $u, v \in L(X)$, there exists a word $w \in L(X)$ with $n - k \leq |w| \leq n + k$, such that $u\, w\, v \in L(X)$.

In order to see that in the one–dimensional case irreducibility and semi–strong irreducibility are equivalent, we state the well–known Pumping Lemma as follows.

Lemma 3.6.2 (Pumping Lemma) *Let L be a regular language. There exists $M \geq 1$ such that if $uwv \in L$ and $|w| \geq M$, there exists a decomposition*

$$w = xyz$$

with $0 < |y| \leq M$ so that for each $n \in \mathbf{N}$ we have $uxy^n zv \in L$.

Moreover, one can take as M the number of vertices of a graph accepting L.

Corollary 3.6.3 *If $X \subseteq A^{\mathbf{Z}}$ is a sofic shift, then*

$$X \text{ irreducible} \iff X \text{ semi–strongly irreducible.}$$

Proof If X is irreducible, we claim that X is (M, M)–irreducible, where M is given by the Pumping Lemma.

If $n \geq M$ and $u, v \in L(X)$, there exists $w \in L(X)$ such that $uwv \in L(X)$. We distinguish two cases.

If $|w| > n + M$, then $w = x_1 y_1 z_1$ with $0 < |y_1| \leq M$ and if $w_1 := x_1 z_1$, then $uw_1 v \in L(X)$ and $|w| - M \leq |w_1| \leq |w| - 1$. If $|w_1| \leq n + M$, moreover we have $|w_1| \geq |w| - M > n > n - M$. If $|w_1| > n + M$, we repeat the above construction to obtain, for some $i \geq 1$, a string of elements $w_1, \ldots, w_i, w_{i+1}$ such that for each $j = 1, \ldots, i + 1$

1. $uw_j v \in L(X)$
2. $w_j = x_j y_j z_j$ with $0 < |y_j| \leq M$
3. $w_{j+1} = x_j z_j$
4. $|w_j| > n + M$ for each $j = 1, \ldots, i$
5. $|w_{i+1}| \leq n + M$.

Then $|w_{i+1}| \geq |w_i| - M > n > n - M$ so that we can set $w := w_{i+1}$.

In the second case, suppose that $|w| < n - M$; there exists $w_1 \in L(X)$ such that $uwv\ w_1\ uwv \in L(X)$ and $|wv\ w_1\ uw| > |w|$. In this way we obtain a word $w_i \in L(X)$ such that $uw_i v \in L(X)$ and $|w_i| \geq n - M$. If, moreover, $|w_i| > n + M$, we can apply the former case. □

Now we prove a fundamental result that in the amenable case has been proved in Proposition 3.5.2 of previous section. We refer to that proof for the details.

Proposition 3.6.4 *Let Γ be a group of non–exponential growth. Let X be a semi–strongly irreducible shift of finite type and let $\tau : X \to A^\Gamma$ be a local and pre–injective function. Then $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$.*

PROOF Suppose that the memory of X is M, that X is (M,k)–irreducible and that τ is M–local. Set $Y := \tau(X)$ and fix an amenable sequence of disks $(E_n)_n$; as we have seen in the proof of Proposition 3.5.2 we have

$$\limsup_{n\to\infty} \frac{\log |Y_{E_n^{+2M}}|}{|E_n|} \leq \mathrm{ent}(Y).$$

Let $l = l(k)$ be the number of ε's such that $\|\varepsilon\| \leq k$ and suppose that $\mathrm{ent}(Y) < \mathrm{ent}(X)$; then

$$\limsup_{n\to\infty} \frac{\log |Y_{E_n^{+2M}}|}{|E_n|} < \limsup_{n\to\infty} \frac{\log |X_{E_n}|}{|E_n|} = \limsup_{n\to\infty} \frac{\log(\frac{|X_{E_n}|}{l})}{|E_n|};$$

so that there exists $n \in \mathbf{N}$ such that

$$\frac{\log |Y_{E_n^{+2M}}|}{|E_n|} < \frac{\log(\frac{|X_{E_n}|}{l})}{|E_n|}$$

that is

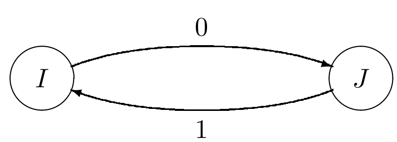
$$|Y_{E_n^{+2M}}| < \frac{|X_{E_n}|}{l}.$$

Fix $v \in X_{\partial^+_{2M} E_n^{+M}}$; since $\mathrm{dist}(\partial^+_{2M} E_n^{+M}, E_n) = M + 1 > M$ for each $u \in X_{E_n}$ there exists an $\varepsilon \in D_k$ and a pattern $p \in X_{E_n^{+3M}}$ that coincides with u on εE_n and with v on $\partial^+_{2M} E_n^{+M}$. Then

$$|\{p \in X_{E_n^{+3M}} \mid p_{|\partial^+_{2M} E_n^{+M}} = v\}| \geq \frac{|X_{E_n}|}{l} > |Y_{E_n^{+2M}}|.$$

Since $\tau_{E_n^{+3M}} : X_{E_n^{+3M}} \to Y_{E_n^{+2M}}$ is surjective, there exist two patterns $p_1, p_2 \in X_{E_n^{+3M}}$ such that $p_1 \neq p_2$ but $p_1 = v = p_2$ on $\partial^+_{2M} E_n^{+M}$ and $\tau_{E_n^{+3M}}(p_1) = \tau_{E_n^{+3M}}(p_2)$. By Corollary 1.3.4, there exist two configurations $c_1, c_2 \in X$ which extend p_1 and p_2 and which coincide outside E_n^{+M}. As we have seen in the proof of Proposition 3.5.2, one has $\tau(c_1) = \tau(c_2)$; hence τ is not pre–injective. □

Observe that for semi–strongly irreducible shifts, Lemma 3.5.4 does not necessarily hold. Indeed consider the subshift $X = \{\overline{01}, \overline{10}\} \subseteq A^{\mathbf{Z}}$; this shift is of finite type. Being accepted by the graph



the shift X is $(2,2)$–irreducible. Now $(\{5j\})_{j\in\mathbf{Z}}$ is a $(\{0\},4,10)$–net. If c is the configuration $=0$ on the even numbers and $=1$ on the odd ones, set $Y := \{c\}$; we have $Y_{\{5j\}} \subset Y_{\{5j\}}$ but $\mathrm{ent}(Y) = \mathrm{ent}(X) = 0$.

The following lemma is very similar to Lemma 3.5.4 but as one can see the hypotheses are quite stronger.

Lemma 3.6.5 *Let Γ be a group with non–exponential growth and let $(E_n)_n$ be a fixed amenable sequence of disks. Let $(F_j)_{j\in\mathbf{N}} = (D(\beta_j, r))_{j\in\mathbf{N}}$ be a $(D_r, 2M, R)$–net, let X be an (M,k)–irreducible shift and let Y be a subset of X such that for each $j \in \mathbf{N}$, there exists a pattern $p_j \in X_{F_j}$ for which $p_j \notin Y_{D(\beta_j\varepsilon,r)}$ whenever $\varepsilon \in D_k$. Then $\mathrm{ent}(Y) < \mathrm{ent}(X)$.*

Proof Let $N(n)$ be the number of F_j's such that $F_j^{+M} \subseteq E_n$ and denote by $F_{j_1}, \ldots, F_{j_N}$ these disks. Set $\xi := |X_{D^{+M}}|$ and denote by $P_{j_m} \subseteq X_{E_n}$ the set of the blocks p of X_{E_n} such that $p_{|D(\beta_{j_m}\varepsilon,r)} = p_{j_m}$ for some $\varepsilon \in D_k$; we prove that

$$|X_{E_n} \backslash \bigcup_{i=1}^{N} P_{j_i}| \leq (1-\xi^{-1})^N |X_{E_n}| \tag{3.5}$$

by induction on $m \in \{1,\ldots,N\}$. We have

$$|X_{E_n}| \leq |X_{F_{j_1}^{+M}}||X_{E_n\backslash F_{j_1}^{+M}}|$$

then

$$|X_{E_n}| \leq \xi |X_{E_n\backslash F_{j_1}^{+M}}|.$$

Since X is (M,k)–irreducible and since $\mathrm{dist}(F_{j_1}, E_n\backslash F_{j_1}^{+M}) > M$, given a pattern $p \in X_{E_n\backslash F_{j_1}^{+M}}$ there exists a pattern $\bar{p}$ defined on all E_n that coincides with p on $E_n\backslash F_{j_1}^{+M}$ and with p_{j_1} on some $D(\beta_{j_1}\varepsilon, r)$; then

$$|X_{E_n\backslash F_{j_1}^{+M}}| \leq |P_{j_1}|.$$

Hence we have

$$\frac{1}{\xi}|X_{E_n}| \leq |P_{j_1}|$$

so that

$$|X_{E_n}\backslash P_{j_1}| \leq |X_{E_n}| - \frac{1}{\xi}|X_{E_n}| = (1-\frac{1}{\xi})|X_{E_n}|.$$

Suppose

$$|X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}| \leq (1-\xi^{-1})^{m-1}|X_{E_n}|;$$

since

$$|X_{E_n} \backslash \bigcup_{i=1}^{m-1} P_{j_i}| \leq$$

$\leq |X_{F_{j_m}^{+M}}||\{p \in X_{E_n \setminus F_{j_m}^{+M}} \mid p_{|D(\beta_{j_i}\varepsilon,r)} \neq p_{j_i}$ for each $i = 1, \ldots, m-1$ and each $\varepsilon\}|$ and being $|X_{F_{j_m}^{+M}}| = \xi$, we have

$$|X_{E_n} \setminus \bigcup_{i=1}^{m-1} P_{j_i}| \leq$$

$$\leq \xi|\{p \in X_{E_n \setminus F_{j_m}^{+M}} \mid p_{|D(\beta_{j_i}\varepsilon,r)} \neq p_{j_i} \text{ for each } i = 1, \ldots, m-1 \text{ and each } \varepsilon|\}.$$

Moreover, since X is (M,k)–irreducible,

$$|\{p \in X_{E_n \setminus F_{j_m}^{+M}} \mid p_{|D(\beta_{j_i}\varepsilon,r)} \neq p_{j_i} \text{ for each } i = 1, \ldots, m-1 \text{ and each } \varepsilon\}| \leq$$

$$\leq |\{p \in X_{E_n} \mid p_{|D(\beta_{j_i}\varepsilon,r)} \neq p_{j_i} \text{ for each } i = 1, \ldots, m-1 \text{ and each } \varepsilon$$

$$\text{and } p_{|D(\beta_{j_m}\bar{\varepsilon},r)} = p_{j_m} \text{ for some } \bar{\varepsilon}\}|.$$

Hence

$$\frac{1}{\xi}|X_{E_n} \setminus \bigcup_{i=1}^{m-1} P_{j_i}| \leq |\{p \in X_{E_n} \setminus \bigcup_{i=1}^{m-1} P_{j_i} \mid p_{|D(\beta_{j_m}\varepsilon,r)} = p_{j_m} \text{ for some } \varepsilon\}|$$

and then

$$|X_{E_n} \setminus \bigcup_{i=1}^{m} P_{j_i}| = |\Big(X_{E_n} \setminus \bigcup_{i=1}^{m-1} P_{j_i}\Big) \setminus P_{j_m}| \leq$$

$$\leq |\Big(X_{E_n} \setminus \bigcup_{i=1}^{m-1} P_{j_i}\Big) \setminus \{p \in X_{E_n} \setminus \bigcup_{i=1}^{m-1} P_{j_i} \mid p_{|D(\beta_{j_m}\varepsilon,r)} = p_{j_m} \text{ for some } \varepsilon\}| \leq$$

$$\leq |X_{E_n} \setminus \bigcup_{i=1}^{m-1} P_{j_i}| - \frac{1}{\xi}|X_{E_n} \setminus \bigcup_{i=1}^{m-1} P_{j_i}| \leq$$

$$\leq (1 - \frac{1}{\xi})(1 - \xi^{-1})^{m-1}|X_{E_n}| = (1 - \xi^{-1})^m |X_{E_n}|.$$

Hence (3.5) holds, and since $|Y_{E_n}| \leq |X_{E_n} \setminus \bigcup_{m=1}^{N} P_{j_m}|$, we have

$$\frac{\log|Y_{E_n}|}{|E_n|} \leq \frac{N(n)\log(1 - \xi^{-1})}{|E_n|} + \frac{\log|X_{E_n}|}{|E_n|}. \tag{3.6}$$

As we have proved in Lemma 3.5.4,

$$\zeta := \liminf_{n\to\infty} \frac{N(n)}{|E_n|} > 0;$$

then taking the maximum limit in (3.6), it follows

$$\text{ent}(Y) \leq \zeta \log(1 - \xi^{-1}) + \text{ent}(X) < \text{ent}(X). \quad \square$$

The following statement is an easy consequence of Lemma 3.6.5 and generalizes the result given in Theorem 1.7.4.

Corollary 3.6.6 *Let Γ be a group with non–exponential growth and let X be a semi–strongly irreducible subshift of A^Γ. If Y is a proper subshift of X then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

Proof Let X be (M, k)–irreducible. If $Y \subset X$, there exists a configuration $c \in X$ which does not belong to Y and then there exists a disk D_r such that $c_{|D_r} \in X_{D_r} \backslash Y_{D_r}$. Let $(F_j)_{j\in\mathbf{N}} = (D(\beta_j, r))_{j\in\mathbf{N}}$ be a $(D_r, 2M, R)$–net; then $c_{|D_r} \notin Y_{D(\beta_j\varepsilon,r)}$ whenever $\varepsilon \in D_k$; by Lemma 3.6.5, $\mathrm{ent}(Y) < \mathrm{ent}(X)$. □

Proposition 3.6.7 *Let Γ be a group with non–exponential growth. Let X be a shift, let Y be a semi–strongly irreducible shift and let $\tau : X \to Y$ be a local function such that* $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y)$*. Then τ is surjective.*

Proof Let X and Y be as in the previous hypotheses and let $\tau : X \to Y$ be a local function. Clearly $\tau(X)$ is a subshift of Y. By Corollary 3.6.6, we have that if $\tau(X) \subset Y$, then $\mathrm{ent}(\tau(X)) < \mathrm{ent}(Y)$. □

Theorem 3.6.8 *Let Γ be a group of non–exponential growth, let X be a semi–strongly irreducible shift of finite type and let Y be a semi–strongly irreducible shift. If $\tau : X \to Y$ is a local function and* $\mathrm{ent}(X) = \mathrm{ent}(Y)$*, then τ pre–injective implies τ surjective.*

Proof If τ is pre–injective we have, by Proposition 3.6.4, that $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$. Then $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y)$ so that, by Proposition 3.6.7, τ is surjective. □

Hence we may conclude stating this (partial) generalization of Corollary 3.5.8.

Corollary 3.6.9 (Myhill–property for semi–strongly irreducible shifts of finite type) *Let Γ be a group of non–exponential growth. Let X be a semi–strongly irreducible subshift of finite type of A^Γ and let $\tau : X \to X$ be a transition function. Then τ pre–injective implies τ surjective.*

Bibliography

[AdlKoM] R. Adler, A. Konheim and M. McAndrew, *Topological Entropy*, Trans. Amer. Math. Soc. **114** (1965), 309-319.

[AP] S. Amoroso and Y. N. Patt, *Decision Procedures for Surjectivity and Injectivity of Parallel Maps for Tessellation Structures*, J. Comput. System Sci. **6** (1972), 448-464.

[Ax] J. Ax, *The Elementary Theory of Finite Fields*, Ann. of Math. (2) **88** (1968), 239-271.

[B] R. Berger, *The Undecidability of the Domino Problem*, Mem. AMS **66**, 1966.

[CeMaSca] T. G. Ceccherini–Silberstein, A. Machì and F. Scarabotti, *Amenable Groups and Cellular Automata*, Ann. Inst. Fourier (Grenoble) **49** 2 (1999), 673-685.

[CovP] E. Coven and M. Paul, *Endomorphisms of Irreducible Shifts of Finite Type*, Math. Systems Theory **8** (1974), 167-175.

[E] V. A. Efremovič, *The Proximity Geometry of Riemannian Manifolds* (Russian), Uspehi Matem. Nauk **8** (1953), 189

[F] E. Følner, *On Groups with full Banach Mean Value*, Math. Scand. **3** (1955), 243-254.

[Gott] W. Gottschalk, *Some General Dynamical Notions*, Recent Advances in Topological Dynamics, 120-125. Lecture Notes in Math. **318**, Springer, Berlin, 1973.

[Gr] F. P. Greenleaf, *Invariant Means on Topological Groups and their Applications*, van Nostrand Mathematical Studies **16**, van Nostrand, New York–Toronto–London, 1969.

[G] M. Gromov, *Endomorphisms of Symbolic Algebraic Varieties*, J. Eur. Math. Soc.**1** (1999), 109-197.

[H] G. A. Hedlund, *Endomorphisms and Automorphisms of the Shift Dynamical System*, Math. Systems Theory **3** (1969), 320-375.

[K1] J. Kari, *Reversibility of 2D Cellular Automata is Undecidable*, Phys. D **45** 1-3 (1990), 379-385.

[K2] J. Kari, *Reversibility and Surjectivity Problems of Cellular Automata*, J. Comput. System Sci. **48** 1 (1994), 149-182.

[KitS1] B. Kitchens and K. Schmidt, *Periodic Points, Decidability and Markov Subgroups*, Dynamical Systems, Proceedings of the special year (J. C. Alexander, ed.) Springer Lecture Notes in Math. **1342** (1988), 440-454.

[KitS2] B. Kitchens and K. Schmidt, *Automorphisms of Compact Groups*, Ergod. Th. and Dynam. Sys. **9** (1989), 691-735.

[LinMar] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1995.

[Moo] E. F. Moore, *Machine Models of Self-Reproduction*, Proc. Symp. Appl. Math., AMS Providence R. I. **14** (1963), 17-34.

[MaMi] A. Machì and F. Mignosi, *Garden of Eden Configurations for Cellular Automata on Cayley Graphs on Groups*, SIAM J. Disc. Math. **6** (1993), 44-56.

[Mil] J. Milnor, *A Note on Curvature and Fundamental Groups*, J. Differential Geometry **2** (1968), 1-7.

[Mu] G. Muratore, *Automi Cellulari*, Tesi di Laurea, University of Palermo, Italy 1990.

[My] J. Myhill, *The converse of Moore's Garden of Eden Theorem*, Proc. Amer. Math. Soc. **14** (1963), 685-686.

[N] I. Namioka, *Følner's Condition for Amenable Semi–Groups*, Math. Scand. **15** (1964), 18-28.

[R] D. Richardson, *Tessellation with Local Transformations*, J. Comput. Syst. Sci. **6** (1972), 373-388.

[Rot] J. J. Rotman, *An Introduction to the Theory of Groups*, Fourth Edition, GTM **148**, Springer–Verlag, New York, 1995.

[Sha] C. Shannon, *A Mathematical Theory of Communication*, Bell System Tech. J. **27** (1948), 379-423, 623-656.

[Š] A. S. Švarc, *A Volume Invariant of Coverings* (Russian), Dokl. Akad. Nauk SSSR **105** (1955), 32-34.

[U] S. Ulam, *Random Processes and Transformations*, Proc. Int. Congr. Mathem. **2** (1952), 264-275.

[vN] J. von Neumann, *The Theory of Self–Reproducing Automata* (A. Burks, ed.), University of Illinois Press, Urbana and London, 1966.

[Wa] H. Wang, *Proving Theorems by Pattern Recognition II*, Bell System Tech. J. **40** (1961), 1-41.

[Wei1] B. Weiss, *Subshifts of Finite Type and Sofic Systems*, Monats. Math. **77** (1973), 462-474.

[Wei2] B. Weiss, *Sofic Groups and Dynamical Systems*, preprint.

[Y] S. Yukita, *Dynamics of Cellular Automata on Groups*, IEICE Trans. Inf. & Syst., Vol. E82-D **10** (1999), 1316-1323.