# Sphere Operators and Their Applicability for Constrained Parameter Optimization Problems

Marc Schoenauer[1] and Zbigniew Michalewicz[2]

[1] CMAP – URA CNRS 756, Ecole Polytechnique, Palaiseau 91128, France,
*marc.schoenauer@polytechnique.fr*
[2] Department of Computer Science, University of North Carolina, Charlotte, NC 28223, USA, *and* Institute of Computer Science, Polish Academy of Sciences, ul. Ordona 21, 01-237 Warsaw, Poland, *zbyszek@uncc.edu*

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**Abstract.** In this paper we continue our earlier studies [13, 14] on boundary operators for constrained parameter optimization problems. The significance of this line of research is based on the observation that usually the global solution for many optimization problems lies on the boundary of the feasible region. Thus, for many constrained numerical optimization problems it might be beneficial to search just the boundary of the search space defined by a set of constraints (some other algorithm might be used for searching the interior of the search space, if activity of a constraint is not certain). We discuss one particular class of boundary operators — sphere operators — and discuss their applicability to some constrained problems (with convex feasible search spaces) through a mapping between the boundary of the feasible region of the search space and a sphere. We provide also with some experimental evaluation of these transformations.

## 1 INTRODUCTION

Evolutionary techniques (whether genetic algorithms, evolution strategies, or evolutionary programming) usually have difficulties in solving constrained numerical optimization problems (for a recent survey on constraint-handling techniques for parameter optimization problems, see [9]). Various search operators have been proposed and investigated; several different constraint handling techniques have been experimented with. However, for many test cases, the results of experiments were far from being satisfactory. As already stated in our recent paper [13], one of the main reasons behind this failure is the inability of evolutionary systems to search precisely the boundary area between feasible and infeasible regions of the search space. Note that in the case of optimization problems with active constraints, such ability is essential.

In [14] we have explored the potential of evolutionary computation techniques with special operators which search the boundary of feasible and infeasible regions in an efficient way; we discussed some possible boundary operators for

such a search and we illustrateed this approach on a few test cases. This paper examines further one particular class of boundary operators, namely sphere operators (sphere crossover and sphere mutation), and discusses their applicability (through a mapping between a sphere and the boundary of the feasible region) to some constrained problems (with convex feasible search spaces).

The paper is organized as follows. Section 2 introduces the problem and explains briefly the approach for numerical constrained optimization based on the idea of searching only the boundary of the feasible search space. Section 3 defines sphere operators and discusses a mapping between a sphere and the boundary of the feasible region. The sphere operators (together with the mapping) are experimented with on a few test cases; the test cases and the results of experiments are reported in section 4. Section 5 concludes the paper.

## 2 THE PROBLEM

It is a common situation for many constrained optimization problems that some constraints are active at the target global optimum. Thus the optimum lies on the boundary of the feasible space. On the other hand, it is commonly acknowledged that restricting the size of the search space in evolutionary algorithms (as in most other search algorithms) is generally beneficial. Hence, it seems natural in the context of constrained optimization to restrict the search of the solution to some part of the boundary of the feasible part of the space.

Let us consider the following constrained numerical optimization problem:

Find $x \in \mathcal{S} \subset R^n$ such that

$$\begin{cases} f(x) = \min\{f(y);\ y \in \mathcal{S}\}, \\ g_i(x) \leq 0, \quad for \ \ i = 1, \ldots, q, \\ g_i(x) = 0, \quad for \ \ i = q+1, \ldots, m, \end{cases} \tag{1}$$

where $f$ and $g_i$ are real-valued functions on $\mathcal{S}$. The set of feasible points (i.e., points satisfying the constraints $g_i(x)$ for $i = 1, \ldots, m$) is denoted $\mathcal{F}$. Restricting the search to the feasible region seems an elegant way to treat constrained problems: in [8], the described algorithm maintains feasibility of all linear constraints using a set of closed operators, which convert a feasible solution (feasible in terms of linear constraints only) into another feasible solution.

However, for nonlinear constraints this ideal situation is generally out of reach, and during the last few years, several methods have been proposed for handling such constraints by evolutionary algorithms (for an experimental comparison of some of these methods, see [9]). In [14] we returned to the idea of maintaining feasibility of solution by a set of closed operators; however, the differences are that (1) constraints are nonlinear, and (2) the closed operators transform points from the boundary of the feasible region into other points from this boundary.

In general, any evolutionary system for searching the boundary between feasible and infeasible parts of the search space should initialize the population accordingly (all generated individuals should lie on the boundary) and the

boundary operators should transform one or more boundary points into a new boundary point. It seems that one of the "easiest" boundaries is a sphere: it is relatively easy to define such initialization process and closed operators [14]. It is also possible (for some classes of problems) to establish a one-to-one mapping between the boundary of the feasible part of the search space and a sphere. Here we assume that the set of constraints consists of inequalities $g_i(\mathbf{x}) \leq 0$ only; all equations $g_i(\mathbf{x}) = 0$ $(i = q+1, \ldots, m)$ can be replaced by a set of inequalities $g_i(\mathbf{x}) \leq \delta$ and $g_i(\mathbf{x}) \geq -\delta$ for some small $\delta > 0$.

All these three elements (initialization, closed operators, and a mapping) constitute another approach for searching boundaries of the feasible region; in the next section we examine this approach in detail.

## 3 A SPHERE: INITIALIZATION, OPERATORS AND A MAPPING

Let us consider a surface of $n$-dimensional unit-sphere, i.e., the set of points such that $\sum_{i=1}^{n} {x_i}^2 = 1$. In that case the initialization process is conceptually quite simple: a sphere should be sampled as uniformly as possible according to the distance at hand. For example, if a (randomly generated) solution is represented as $n$-dimensional vector of Euclidean coordinates $(a_1, a_2, \ldots, a_n)$, then the initialization procedure may divide the coordinates of the vector by its Euclidian norm:

$$p = (a_1/s, a_2/s, \ldots, a_n/s), \tag{2}$$

where $s = \sqrt{\sum_{i=1}^{n} a_i^2}$.

Assuming that the initial population consists of points on the sphere, we need to design *closed* operators which transform point(s) of the sphere into point(s) of the sphere. One possibility (see [14] for a survey of other possibilities) is to design a specialized operators which, by themselves—without any repair procedure, create a new sphere point. The *sphere crossover* is an example of such approach: it produces one offspring $(z_i)$ from two parents $(x_i)$ and $(y_i)$ by:

$$z_i = \sqrt{\alpha x_i^2 + (1-\alpha) y_i^2},\ i \in [1, n], \text{ with } \alpha \text{ randomly chosen in } [0, 1].$$

Note, that this operator can be generalized easily into $k$-parent sphere crossover (parents: $x^1, x^2, \ldots, x^k$) by

$$x_i^{k+1} = \sqrt{\alpha_1 (x_i^1)^2 + \ldots + \alpha_k (x_i^k)^2},$$

for non-negative $\alpha_i$ such that $\alpha_1 + \ldots + \alpha_k = 1$.

Similarly, the following sphere-specific mutation transforms $(x_i)$ by selecting two indices $i \neq j$ and a random number $p$ in $\langle 0, 1 \rangle$, and setting:

$$x_i \to p \cdot x_i \text{ and } x_j \to q \cdot x_j,$$
$$\text{where } q = \sqrt{(\tfrac{x_i}{x_j})^2 (1 - p^2) + 1}.$$

It is clear that these sphere operators respect some experimentally and empirically derived properties [11]:

– recombination should be able to generate all points "between" the parents;
– mutation should be *ergodic*, having non-zero probability to reach any point within a finite number of application, and should respect the principle of *strong causality* [12], i.e., small mutations must result in small changes in the fitness function.

Sphere operators gave excellent performance on several test cases where the boundary of the feasible search space was a sphere [14]. However, it was unclear what might be the usefulness of these operators in cases where such boundary is not a sphere? One possibility (computationally expensive) would be to approximate (locally) an arbitrary surface by a sphere. Another possibility (pursued in this paper) is to establish a one-to-one mapping between the searched surface and a sphere (in a similar way as proposed in [6]): such a mapping maintains a feasibility of solutions. Assume that the feasible search space $\mathcal{F}$ is star-shaped around one of its points $c$, i.e., any line which originates in $c$ intersects the boundary of the feasible region in exactly one point. It is then easy to establish one-to-one mapping between the boundary of the feasible region and the unit sphere (2 dimensional case is illustrated in Figure 1): For any point $b$ on the boundary of $\mathcal{F}$, the line $bc$ intersects the unit sphere centered in $c$ exactly once, at point $s$, and vice-versa: for any boundary point $s$ of the sphere, the line $cs$ intersects the boundary of $\mathcal{F}$ (which is convex) in precisely one point $b$.

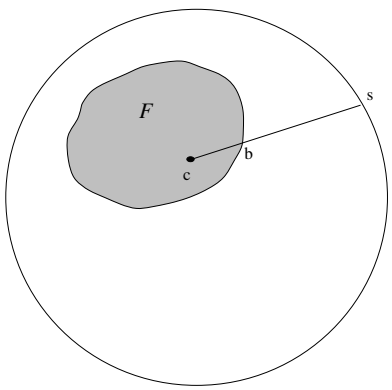


**Fig. 1.** A mapping between a boundary of feasible region $\mathcal{F}$ and a sphere

Having established a mapping between the boundary of the feasible part of the search space and a sphere, we can search the sphere: after initialization, each individual $s$ in the population (i.e., a point from the sphere) is decoded into a boundary point $b$ of $\mathcal{F}$ before evaluation. The sphere operators would do the rest.

In the next section we provide with experimental results of this approach.

## 4 TEST CASES, EXPERIMENTS AND RESULTS

This section presents experimental comparative results involving sphere operators and the mapping discussed in the previous section.

### 4.1 EXPERIMENTAL CONDITIONS

The results presented in this paper use an evolutionary algorithm based on standard modules: deterministic tournament selection of size 2, generational replacement with elitism, fixed population size of 30, fixed crossover and mutation rates. Note that the present work did not investigate yet some other evolution schemes (e.g., some $(\mu+\lambda)$-ES scheme), which sometimes give better results [14]. Moreover, only the sphere operators described originally in [10] were considered, though some other operators also working on the sphere have been defined and experimented later [14]. Clearly, further research along these lines is necessary; the complete study on boundary operators is being prepared [15].

### 4.2 TEST CASES

The first test case considered in this paper [5] is to maximize

$$f1(\mathbf{x}) = |\frac{\sum_{i=1}^{n} cos^4(x_i) - 2\prod_{i=1}^{n} cos^2(x_i)}{\sqrt{\sum_{i=1}^{n} ix_i^2}}|,$$

subject to

$$\prod_{i=1}^{n} x_i \geq 0.75 \quad \text{and bounds} \quad 0 \leq x_i \leq 10 \quad \text{for} \quad 1 \leq i \leq n.$$

Function $f1$ is nonlinear and its global maximum is unknown, lying somewhere near the origin. The problem has one nonlinear constraint and one linear constraint; the latter one is inactive around the origin and will be forgotten in the following.

Some potential difficulties of solving this test case are illustrated in figure 2, where infeasible points were assigned a value of zero. The interesting boundary between feasible and infeasible regions is defined by the equation $\Pi x_i = 0.75$. It is a difficult problem, on which no standard method (be it deterministic or evolutionary) gave satisfactory results.

The second test case considered in this paper [2] is to minimize

$$f2(\overline{X}, \overline{Y}) = 5x_1 + 5x_2 + 5x_3 + 5x_4 - 5\sum_{i=1}^{4} x_i^2 - \sum_{i=1}^{9} y_i,$$

subject to:

$$\begin{array}{ll}
2x_1 + 2x_2 + y_6 + y_7 \leq 10, & 2x_1 + 2x_3 + y_6 + y_8 \leq 10, \\
2x_2 + 2x_3 + y_7 + y_8 \leq 10, & -8x_1 + y_6 \leq 0, \\
-8x_2 + y_7 \leq 0, & -8x_3 + y_8 \leq 0, \\
-2x_4 - y_1 + y_6 \leq 0, & -2y_2 - y_3 + y_7 \leq 0, \\
-2y_4 - y_5 + y_8 \leq 0, & 0 \leq x_i \leq 1,\ i = 1, 2, 3, 4, \\
0 \leq y_i \leq 1,\ i = 1, 2, 3, 4, 5, 9, & 0 \leq y_i,\ i = 6, 7, 8.
\end{array}$$

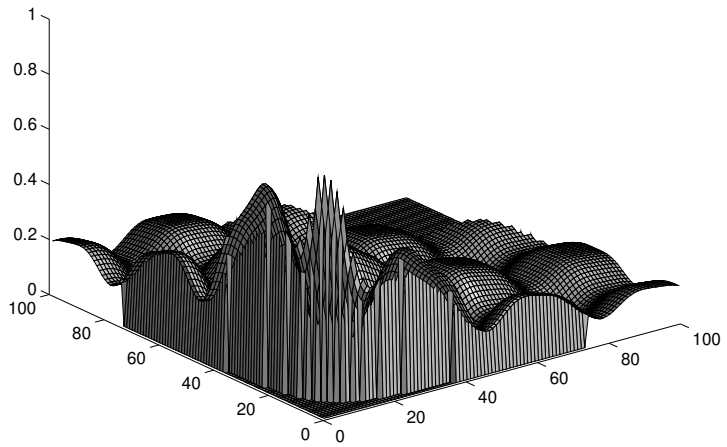


**Fig. 2.** The graph of function $f1$ for $n = 2$. Infeasible solutions were assigned value zero

The global solution is $(\overline{X}^*, \overline{Y}^*) = (1,1,1,1,1,1,1,1,1,3,3,3,1)$, and $f2(\overline{X}^*, \overline{Y}^*) = -15$.

The third test case considered in this paper [2] is to minimize

$$f3(x, \overline{Y}) = 6.5x - 0.5x^2 - y_1 - 2y_2 - 3y_3 - 2y_4 - y_5,$$

subject to:

$$\begin{aligned}
&x + 2y_1 + 8y_2 + y_3 + 3y_4 + 5y_5 \leq 16,\\
&-8x - 4y_1 - 2y_2 + 2y_3 + 4y_4 - y_5 \leq -1,\\
&2x + 0.5y_1 + 0.2y_2 - 3y_3 - y_4 - 4y_5 \leq 24,\\
&0.2x + 2y_1 + 0.1y_2 - 4y_3 + 2y_4 + 2y_5 \leq 12,\\
&-0.1x - 0.5y_1 + 2y_2 + 5y_3 - 5y_4 + 3y_5 \leq 3,\\
&y_3 \leq 1,\ y_4 \leq 1, \text{ and } y_5 \leq 2,\\
&x \geq 0,\ y_i \geq 0, \text{ for } 1 \leq i \leq 5.
\end{aligned}$$

The global solution is $(x, \overline{Y}^*) = (0,6,0,1,1,0)$, and $f3(x, \overline{Y}^*) = -11$.

### 4.3 RESULTS

The problem $f1$ has already been addressed successfully by a specific evolutionary algorithm based on some *geometrical* operators working on the hyperboloid boundary of the feasible region defined by $\prod_{i=1}^{n} x_i = 0.75$. The geometrical crossover takes two parents, $(x_i)$ and $(y_i)$ and produces a single offspring:

$$z_i = x_i^{\alpha} \cdot y_i^{1-\alpha}, \text{ for } 0 \leq \alpha \leq 1,$$

which also lies on the boundary of feasible region: $\prod_{i=1}^{n} z_i = 0.75$ (in [10], the above geometrical crossover was generalized into one with multiple parents). Also the task of designing a feasibility-preserving mutation is relatively simple: the $i$-th and $j$-th components are modified in a such way that the product of $x_i$ and $x_j$ remains constant before and after mutation.

Though the hyperboloid does not define the whole boundary of the actual feasible region, the search was limited to this part of the boundary. The reason is that the hyperboloid represents an active constraint (actually, one can prove—before solving this problem—that this is the case). Consequently, only a part of the sphere was considered (with 'positive coordinates') and mapped onto the hyperboloid, as the other constraints are not active at the solution. Then the results of this approach were compared to the ones which use the specialized geometrical operator.

As expected, the specialized algorithm performs better than the proposed (spherical plus mapping) approach. However, the results of both algorithms are of the same order of precision, and are in particular better than all other previously published results [5].

Figure 3 presents the best and median plots out of 21 independent runs: whereas the median result of the geometrical operators is significantly better that that of the spherical operators, the best runs are almost indistinguishable. It seems that the geometrical approach is far more robust than the spherical one, both being able to give the best results so far on this difficult problem (note that the replacement of the medians by means increases this tendency [3]).

Another interesting observation is as follows: the best results are obtained with the same mutation rate of 0.2 for both algorithms, while the best performing crossover rates are different: 0.8 gives the best results for the geometrical operators, and 1.0 outperforms all other settings for the spherical operators (statistically significant differences with a 5% Student test).

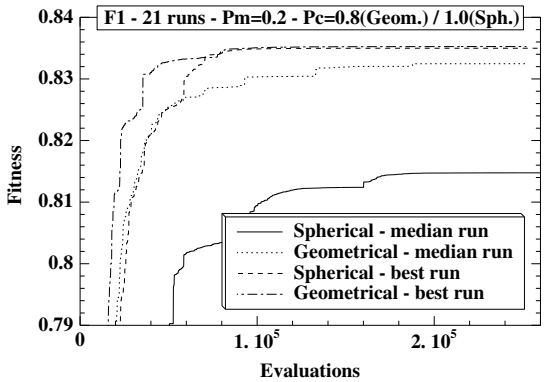


**Fig. 3.** Best and median comparative runs between the geometrical operators and the spherical operators on Keane's function $f1$.

Test cases $f2$ and $f3$ involve linear constraints only; because of that, the feasible part $\mathcal{F}$ of the search space is convex. These two test cases were taken from the suit of test cases of Genocop—an evolutionary system designed for numerical optimization problems with linear costraints. There are several closed operators used by Genocop system [8], which starts with and maintains a population of feasible individuals.

Again, as expected, the performance of the proposed (spherical plus mapping) approach was worse than this of Genocop in terms of the number of function evaluations. It is difficult to provide a meaningful comparisons between these two systems since Genocop uses 7 operators with various rates, whereas the proposed system has only two operators (sphere crossover and sphere mutation).

An additional interesting observation is as follows: 3-parents sphere crossover performs *much better* than ordinary 2-parents sphere crossover. For example, for test function $f3$, figure 4 displays the average fitness values with respect to the number of function evaluations, for $k$-sphere crossover ($1 \leq k \leq 5$) and geodesian mutation.

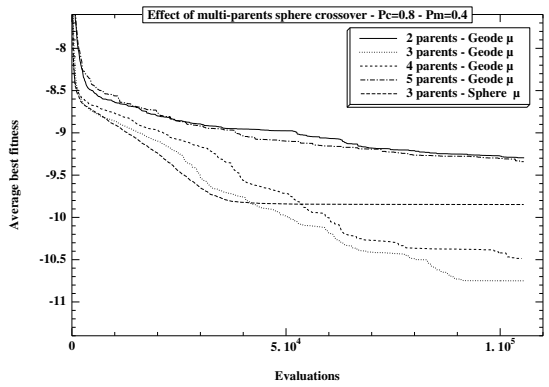


**Fig. 4.** Comparison of various sphere crossover operators with geodesian mutation

Clearly, these limited experiments raised more questions than answers; there are several unclear issues which are worth examining further.

## 5 DISCUSSION AND CONCLUSIONS

The results of our earlier experiments [14] indicated that different boundary operators and different rates of these operators gave different performance for different test functions. However, it seems that the proposed additional stage of mapping between the boundaries of a sphere and the feasible convex part of the search space $\mathcal{F}$ reduces greatly this variability. The proposed approach, which

– initializes the population by points from a sphere,

– evaluates them by mapping them first into boundary points of $\mathcal{F}$ before applying the evaluation function,
– applies closed operators, which transform sphere point(s) into sphere points,

is not that parameter sensitive. For all test cases we obtained satisfactory results of high precision.

Clearly, the specialized systems (whether Genocop for $f2$ and $f3$, where there are linear constraints only, or a system with geometrical crossover and mutation [10] for $f1$) outperform the proposed method in terms of the number of function evaluations or the variability of results over several runs. Note, however, that this was expected: after all, the proposed method is more general than any specialized system and can handle non-linear constraints without any need for development of specialized operators.

In our further research we will examine the significance of multi-parent sphere crossover as well as adaptiveness of the reference point $c$ used for defining the mapping. Note, that in all reported experiments the reference point $c$ was fixed for all runs of the evolutionary method. Clearly, adapting the position of $c$ (e.g., moving $c$ towards the best solution found so far) may provided with additional enhancements.

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## References

1. S. Baluja. An empirical comparison of seven iterative and evolutionary function optimization heuristics. Technical Report CMU-CS-95-193, School of Computer Science, Carnegie Mellon University, 1995.
2. C.A. Floudas and P.M. Pardalos. *Recent Advances in Global Optimization*. Princeton Series in Computer Science, Princeton University Press, Princeton, NJ, 1992.
3. C. M. Fonseca and P. J. Fleming. On the performance assessment and comparison of stochastic multiobjective optimizers. In H.-M. Voigt, W. Ebeling, I. Rechenberg, and H.-P. Schwefel, editors, *Proceedings of the* $4^{th}$ *Conference on Parallel Problems Solving from Nature*, number 1141 in LNCS, pages 584–593. Springer Verlag, Sept. 1996.
4. N. Hansen, A. Ostermeier, and A. Gawelczyk. On the adaptation of arbitrary normal mutation distributions in evolution strategies: The generating set adaptation. In L. J. Eshelman, editor, *Proceedings of the* $6^{th}$ *International Conference on Genetic Algorithms*, pages 57–64. Morgan Kaufmann, 1995.
5. A. Keane. Genetic Algorithms Digest. V8n16. (1994, May 19).
6. S. Koziel. Homomorphous mappings in numerical optimization with constraints using genetic algorithms. Technical University of Gdańsk, Technichal Report, 1997.
7. C. Margerin. Une introduction à la géométrie. Course at ENSTA – Palaiseau – France, 1990.

8. Z. Michalewicz, T.D. Logan, and S. Swaminathan. Evolutionary Operators for Continuous Convex Parameter Spaces. In *Proceedings of the $3^{rd}$ Annual Conference on Evolutionary Programming*, pages 84–97, 1994.
9. Z. Michalewicz and M. Schoenauer. Evolutionary Algorithms for Constrained Parameter Optimization Problems. *Evolutionary Computation*, Vol.4, No.1, 1996, pp.1–32.
10. Z. Michalewicz, G. Nazhiyath, and M. Michalewicz. A note on usefulness of geometrical crossover for numerical optimization problems. In L. J. Fogel, P. J. Angeline, and T. Bäck, editors, *Proceedings of the $5^{th}$ Annual Conference on Evolutionary Programming*, pages 305–312. MIT Press, 1996.
11. N. J. Radcliffe. Equivalence class analysis of genetic algorithms. *Complex Systems*, 5:183–20, 1991.
12. I. Rechenberg. *Evolutionstrategie: Optimierung Technisher Systeme nach Prinzipien des Biologischen Evolution.* Stuttgart: Fromman-Holzboog Verlag, 1973.
13. M. Schoenauer and Z. Michalewicz. Evolutionary Computation at the Edge of Feasibility. In H.-M. Voigt, W. Ebeling, I. Rechenberg, and H.-P. Schwefel, editors, *Proceedings of the 4th Parallel Problem Solving from Nature*, Springer-Verlag, LNCS 1141, pp.245–254, 1996.
14. M. Schoenauer and Z. Michalewicz. Boundary Operators for Constrained Parameter Optimization Problems. In T. Bäck, editor, *Proceedings of the $7^{th}$ International Conference on Genetic Algorithms*, pages 322–329. Morgan Kaufmann, 1997.
15. M. Schoenauer and Z. Michalewicz. Boundary Operators for Evolutionary Algorithms. In preparation.
16. H.-P. Schwefel. *Evolution and Optimum Seeking.* John Wiley, Chichester, UK, 1995.

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