Numeric Programs

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MPRI 2-36-1 “Preuve de Programme”

January 23rd, 2013

Reminder of Previous Lectures

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  Internal clock ticks every 0.1 second.
  Time is tracked by fixed-point arith.: $0.1 \simeq 209715 \cdot 2^{-24}$.
  Cumulated skew after 24h: -0.08s, distance: 160m.
  System was supposed to be rebooted periodically.

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  Bug in binary/decimal conversion.
  Failing inputs: 12 FP numbers.
  Probability to uncover them by random testing: 10^{-18}.

Outline

Binary Search

Exercise: Find appropriate precondition, postcondition, loop invariant, and variant, for this program:

```
function binary_search(a:map int, n v:int): int
body
  try
    let ref l = 0 in
    let ref u = n - 1 in
    while l ≤ u do
      let m = div (l + u) 2 in
      if a[m] < v then
        l := m + 1
      else if a[m] > v then
        u := m - 1
      else
        raise (Break m)
    done;
    raise Not_found
  with Break i → i
```

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- Otherwise, an overflow occurs.
  Behavior depends on language and environment:
  modulo arith, saturated arith, abrupt termination, etc.

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A program is safe if no overflow occurs.

Safety Checking

Idea: replace all arithmetic operations by abstract functions with preconditions. $x - y$ becomes int32_sub(x, y).

```
function int32_sub(x: int, y: int): int
  requires -2^31 ≤ x - y < 2^31
  ensures result = x - y
```

Safety Checking, Try 2

Idea: replace

- type *int* with an abstract type coercible to it,
- all operations by abstract functions with preconditions,

and add an axiom about the range of *int*32.

```
type int32
function of_int32(x: int32): int
axiom bounded_int32:
  forall x: int32. -2^31 ≤ of_int32(x) < 2^31

function int32_sub(x: int32, y: int32): int32
  requires -2^31 ≤ of_int32(x) + of_int32(y) < 2^31
  ensures of_int32(result) = of_int32(x) - of_int32(y)
```

Exercises

1. How to handle int32 constants in programs?
2. How to specify saturating arithmetic?

Outline

Floating-Point Arithmetic

- Limited range $\Rightarrow$ exceptional behaviors.
- Limited precision $\Rightarrow$ inaccurate results.

Floating-Point Data

IEEE-754 Binary Floating-Point Arithmetic.
Width: $1 + w_e + w_m = 32$, or 64, or 128.
Bias: $2^{w_e - 1} - 1$. Precision: $p = w_m + 1$.

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- if $e' = 0$,
  - ± 0 if $m' = 0$, zeros
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- if $e' = 2^{w_e} - 1$,
  - $(-1)^s \cdot \infty$ if $m' = 0$, infinity
  - *Not-a-Number* otherwise. NaN

Floating-Point Data

$$\begin{array}{ccccc} \boxed{1} & & \boxed{11000110} & & \boxed{10010011110000111000000} \\ s & & e & & f \\ \downarrow & & \downarrow & & \downarrow \\ (-1)^{s} & \times & 2^{e-B} & \times & 1.f \end{array}$$

$$(-1)^{1} \times 2^{198-127} \times 1.10010011110000111000000_{2}$$

$$-2^{54} \times 206727 \approx -3.7 \times 10^{21}$$

Semantics for the Finite Case

A floating-point operator shall behave as if it was first computing the infinitely-precise value and then rounding it so that it fits in the destination floating-point format.

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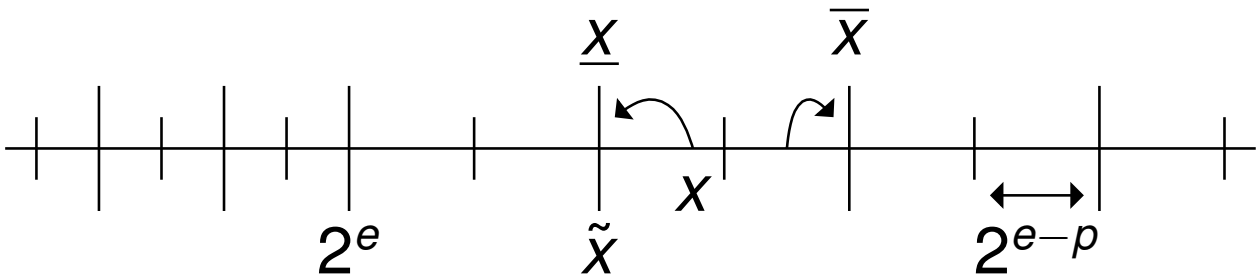
Rounding of a real number x:



Overflows are not considered when defining rounding: exponents are supposed to have no upper bound!

Partial Specification

Same as with integers, we specify FP operations so that no overflow occurs.

```
type bin32
function of_bin32(x: bin32): real
axiom finite_bin32: forall x: bin32. ???

function rnd...(x: real): real
axiom about_rnd...: ???

function bin32_sub(x: bin32, y: bin32): bin32
  requires abs(rnd...(of_bin32(x) - of_bin32(y))) ≤ ...
  ensures of_bin32(result) =
    rnd(of_bin32(x) - of_bin32(y))
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Canonical representation:

- either $2^{p-1} \leq |m| < 2^p$ and $e \geq e_{\min}$, normal
- or $|m| < 2^{p-1}$ and $e = e_{\min}$. subnormal

Usual Properties: Representation and Successors

Given a representable number $x = m_x \cdot 2^{e_x} \geq 0$,

1. $y = (m_x + 1) \cdot 2^{e_x} \in \mathbb{F}$,
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Proof:

1. Hyp: $0 \leq m_x < 2^p$ et $e_x \geq e_{\min}$.
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   Otherwise $m_x + 1 = 2^p$, so $y = 1 \cdot 2^{e_x + p} \in \mathbb{F}$.

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   Otherwise $m_x + 1 = 2^p$, so $y = 1 \cdot 2^{e_x + p} \in \mathbb{F}$.

2. Hyp: $2^{p-1} \leq m_x < 2^p$ or $e_x = e_{\min}$.
   If $m_x \cdot 2^{e_x} < m_z \cdot 2^{e_z} < (m_x + 1) \cdot 2^{e_x}$,
   then $e_z > e_x$ and $m_z > 2^{e_x - ez} m_x \geq 2m_x$.

Usual Properties: Rounding Modes

Faithful rounding:

- $\nabla(x) = \max\{y \in \mathbb{F} \mid y \leq x\}$,
- $\Delta(x) = \min\{y \in \mathbb{F} \mid y \geq x\}$,
- either $\mathrm{rnd}(x) = \nabla(x)$ or $\mathrm{rnd}(x) = \Delta(x)$.

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Idempotency:

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Local monotonicity:

$$\forall x, y \in \mathbb{R},\ y \in [\mathrm{rnd}(x), x] \Rightarrow \mathrm{rnd}(y) = \mathrm{rnd}(x)$$

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- If $\nabla(x) < \nabla(y)$,
  1. $x < \nabla(y)$ by definition of $\nabla(x)$,
  2. $\mathrm{rnd}(x) \leq \Delta(x) \leq \nabla(y) \leq \square(y)$.

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  3. if $\mathrm{rnd}(y) = \Delta(y)$, then $\mathrm{rnd}(x) \leq \mathrm{rnd}(y)$,

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Proof:

- If $\nabla(x) < \nabla(y)$,
  1. $x < \nabla(y)$ by definition of $\nabla(x)$,
  2. $\mathrm{rnd}(x) \leq \triangle(x) \leq \nabla(y) \leq \square(y)$.
- If $\nabla(x) \geq \nabla(y)$,
  1. $\nabla(x) = \nabla(y)$ by definition of $\nabla(y)$,
  2. $\triangle(x) = \triangle(y)$ by idempotency or successor,
  3. if $\mathrm{rnd}(y) = \triangle(y)$, then $\mathrm{rnd}(x) \leq \mathrm{rnd}(y)$,
  4. otherwise $\mathrm{rnd}(x) = \mathrm{rnd}(y)$ by local monotonicity.

Usual Properties: Monotonicity

Monotonicity:

$$\forall x, y \in \mathbb{R},\ x \leq y \Rightarrow \mathrm{rnd}(x) \leq \mathrm{rnd}(y)$$

Ordering with respect to representable numbers:

$$\forall x \in \mathbb{F},\ \forall y \in \mathbb{R},\ x \leq y \Rightarrow x \leq \mathrm{rnd}(y)$$

Usual Properties: Round-Off Errors

Rounding to nearest:
For all $x \in$, there are ε and δ such that

$$\mathrm{rnd}(x) = x \cdot (1 + \varepsilon) + \delta \quad \text{and} \quad |\varepsilon| \leq 2^{-p} \quad \text{and} \quad |\delta| \leq 2^{e_{\min}-1}$$

Moreover, $\delta = 0$ or $\varepsilon = 0$.

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   $\nabla(x) = m \cdot 2^e$ and $\Delta(x) = (m+1) \cdot 2^{e+1}$.
2. $|\mathrm{rnd}(x) - x| \leq (\Delta(x) - \nabla(x))/2 = 2^{e-1}$.

Usual Properties: Round-Off Errors

Rounding to nearest:
For all $x \in$, there are ε and δ such that

$$\mathrm{rnd}(x) = x \cdot (1 + \varepsilon) + \delta \quad \text{and} \quad |\varepsilon| \leq 2^{-p} \quad \text{and} \quad |\delta| \leq 2^{e_{\min}-1}$$

Moreover, $\delta = 0$ or $\varepsilon = 0$.

Proof:

1. Hyp: $0 < x \notin \mathbb{F}$.
   $\nabla(x) = m \cdot 2^e$ and $\Delta(x) = (m+1) \cdot 2^{e+1}$.
2. $|\mathrm{rnd}(x) - x| \leq (\Delta(x) - \nabla(x))/2 = 2^{e-1}$.
   - If $\nabla(x)$ is subnormal, $e = e_{\min}$.
     $\varepsilon = 0$ and $\delta = \mathrm{rnd}(x) - x$ so $|\delta| \leq 2^{e_{\min}-1}$.

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     $\varepsilon = 0$ and $\delta = \mathrm{rnd}(x) - x$ so $|\delta| \leq 2^{e_{\min}-1}$.
   - If $\nabla(x)$ is normal, $2^{p-1} \leq m$.
     $\delta = 0$ and $\varepsilon = (\mathrm{rnd}(x) - x)/x$ so
     $|\varepsilon| \leq 2^{e-1}/(2^{p-1} \cdot 2^e) = 2^{-p}$.

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Directed rounding:
For all $x \in$, there are ε and δ such that

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Moreover, $\delta = 0$ or $\varepsilon = 0$.

Usual Properties: Subnormal Addition

Sums in the subnormal range are representable:

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1. $x = m_x \cdot 2^{e_x}$ and $y = m_y \cdot 2^{e_y}$.
2. $m = m_x \cdot 2^{e_x - e_{\min}} + m_y \cdot 2^{e_y - e_{\min}}$ and $x + y = m \cdot 2^{e_{\min}}$.
3. $|m| \leq 2^p$ so $x + y \in \mathbb{F}$.

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Round-off error for addition:

$$\forall x, y \in \mathbb{F},\ \exists \varepsilon,\ \circ(x + y) = (x + y) \cdot (1 + \varepsilon) \quad \text{and} \quad |\varepsilon| \leq 2^{-p}$$

Outline

Numerical Errors

Given two real numbers u and v,

- absolute error: $u - v$, $|u - v|$
- relative error: $u/v - 1$, ...

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Remark: $\mathrm{rnd}(u) - v = (\mathrm{rnd}(u) - u) + (u - v)$

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Notations:

- a mathematical function $f(x)$,
- a floating-point program $\tilde{f}(x)$,
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Remark: $\tilde{f}(x) - f(x) \simeq (\tilde{x} - x) \times \frac{\partial f}{\partial x}$.

In other words: *forward err* $\simeq$ *backward err* $\times$ *condition num*.

Numerical Analysis

Evaluating $\sum_i a_i \cdot x^i$:

```
function Horner
  (a:map binary32, n:int, x:binary32)
body
  let ref y := binary32_cst(0.) in
  let ref i := n in
  for i = 0 to n - 1 do
    y := binary32_add(binary32_mul(y, x), a[i]);
  done;
  y
```

Outline

Using Ghost Variables for Model Values

```
function det(a b c d: binary32, aM bM cM dM: real):
  (binary32, real)
body
  let t1 := binary32_mul(a, d) in
  let t1M := aM * dM in
  let t2 := binary32_mul(b, c) in
  let t2M := bM * cM in
  let t3 := binary32_sub(t1, t2) in
  let t3M := t1M - t2M in
  (t3, t3M)
```

Forward error: property about `t3 - t3M` or `t3/t3M - 1`.

Implicit Model Values

```
function of_bin32(x: binary32): real
function model_of(x: binary32): real

function binary32_add(x y: binary32): binary32
  requires
    abs(rnd...(of_bin32(x) + of_bin32(y))) ≤
      max_binary32
  ensures
    of_bin32(result) =
      rnd(of_bin32(x) + of_bin32(y)) ∧
    model_of(result) = model_of(x) + model_of(y)
```

Abstract Interpretation

Domains for floating-point variables:

- for the computed value x,
- for the infinitely-precise value $\hat{x}$,
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Naive domains:

- $[\underline{x}, \overline{x}]$ such that $x \in [\underline{x}, \overline{x}]$,
  ex: $\mathrm{rnd}(x + y) \in [\mathrm{rnd}(\underline{x} + \underline{y}), \mathrm{rnd}(\overline{x} + \overline{y})]$,
- no domain for $\hat{x}$,
- δ_x such that $|x - \hat{x}| \leq \delta_x$,
  ex: $\delta_{x+y} = \delta_x + \delta_y + 2^{-p} \max(\overline{x} + \overline{y}, -(\underline{x} + \underline{y}))$

Outline

Newton's Iterated Square Root

```
function fp_sqrt_init(x:binary64) : binary64
  requires 0.5 ≤ x ≤ 2;
  ensures abs(result - 1/sqrt(x)) ≤ 2^-6 * 1/sqrt(x);

function fp_sqrt(x:binary64) : binary64
  requires 0.5 ≤ x ≤ 2;
  ensures abs(result - sqrt(x)) ≤ 2^-43 * sqrt(x);
body
  let ref t := fp_sqrt_init(x) in
  for i = 1 to 3 do
    t := 0.5 * t * (3 - t * t * x);
  done;
  t * x
```

Quadratic Convergence

For all u and x:

$$0.5u(3 - u^2x)\sqrt{x} - 1 = -(1.5 + 0.5(u\sqrt{x} - 1)) \times (u\sqrt{x} - 1)^2$$

Loop iterations:

$$t_{n+1}\sqrt{x} - 1 \simeq 0.5t_n(3 - t_n^2x)\sqrt{x} - 1 \simeq -1.5(t_n\sqrt{x} - 1)^2$$

Round-off error at step n vanishes at step $n + 1$.

Accurate Summation

Computing $\sum_i x_i$:

```
s := x[0];
e := 0.;
for i = 1 to n - 1 do
  y := x[i];
  t := s + y;
  u := t - y;
  r := (s - u) + (y - (t - u));
  s := t;
  e := e + r;
done;
s' := s + e;
```

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Naive sum

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Error-free addition: $t + r = s + y$

Error-Free Transformations

- Sterbenz: $\forall x, y \in F,\ x/2 \leq y \leq 2x \Rightarrow \mathrm{rnd}(x - y) = x - y$

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- fast twosum: $\forall x, y \in F,\ |x| \geq |y| \Rightarrow s + e = x + y$
  with $s = \mathrm{rnd}(x + y)$ and $e = \mathrm{rnd}(y - \mathrm{rnd}(s - x))$

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- twosum: $\forall x, y \in F,\ s + e = x + y$
  with $s = \text{rnd}(x + y)$ and $u = \text{rnd}(s - y)$ and
  $e = \text{rnd}(\text{rnd}(x - u) + \text{rnd}(y - \text{rnd}(s - u)))$

Payne & Hanek's Argument Reduction

Reducing $x \geq 2^{31}$ to $0 \leq y \lesssim \pi/4$ for circular functions:

```
function reduce(x:binary32): (binary32, int)
  requires 2^31 ≤ x
  ensures exists l:int.
    abs((result + k * pi/4) - (x + l * 2*pi)) ≤ 2^-25
body
  let x' = binary64_of_binary32 x in
  let t = x' * 1.273239545... in
  let k = trunc(t) in
  let y = (t - k) * 0.785398163... in
  (binary32_of_binary64(y), k)
```

Note: computations are performed with `binary64`.

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body
  let x' = binary64_of_binary32 x in
  let t = x' * 0.02323954474... in
  let k = trunc(t) in
  let y = (t - k) * 0.785398163... in
  (binary32_of_binary64(y), k)
```

Note: computations are performed with `binary64`.