Oriented vertex and arc-colorings of partial 2-trees

(Extended Abstract)

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Abstract

A homomorphism from an oriented graph G to an oriented graph H is an arc-preserving mapping φ from $V(G)$ to $V(H)$, that is $\varphi(x)\varphi(y)$ is an arc in H whenever xy is an arc in G. The oriented chromatic number of G is the minimum order of an oriented graph H such that G has a homomorphism to H. The oriented chromatic index of G is the minimum order of an oriented graph H such that the line-digraph of G has a homomorphism to H. In this paper, we determine the oriented chromatic number (resp. the oriented chromatic index) of the class of partial 2-trees for every girth $g \geq 3$ (resp. for every girth $g \geq 3$ excepted three).

Introduction We consider finite simple *oriented graphs*, that is digraphs with no opposite arcs. For an oriented graph G, we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. The number of vertices of G is the *order* of G. The *girth* of a graph G is the size of a smallest cycle in G. We denote by $\mathcal{T}_g$ the class of partial 2-tree with girth at least g.

The notion of oriented vertex-coloring was introduced by Courcelle [4] as follows: an *oriented k-vertex-coloring* of an oriented graph G is a mapping φ from $V(G)$ to a set of k colors such that (i) $\varphi(u) \neq \varphi(v)$ whenever $uv \in A(G)$ and (ii) $\varphi(v) \neq \varphi(x)$ whenever $uv, xy \in A(G)$ and $\varphi(u) = \varphi(y)$. The *oriented chromatic number* of G, denoted by $\chi_o(G)$, is defined as the smallest k such that G admits an oriented k-vertex-coloring. The oriented chromatic number $\chi_o(\mathcal{F})$ of a class of oriented graphs $\mathcal{F}$ is defined as the maximum of $\chi_o(G)$ taken over all graphs G in $\mathcal{F}$.

Let G and H be two oriented graphs. A *homomorphism* from G to H is a mapping φ from $V(G)$ to $V(H)$ that preserves the arcs: $\varphi(u)\varphi(v) \in A(H)$ whenever $uv \in A(G)$. An oriented k-vertex-coloring of an oriented graph G can be equivalently defined as a homomorphism φ from G to H, where H is an oriented graph of order k; such a homomorphism is called a *H-vertex-coloring* of G or simply a *vertex-coloring* of G.

The existence of such a homomorphism from G to H is denoted by $G \to H$. The vertices of H are called *colors*, and we say that G is H-vertex-colorable. The oriented chromatic number of G can then be equivalently defined as the smallest order of an oriented graph H such that $G \to H$. Links between colorings and homomorphisms are presented in more details in the monograph [5] by Hell and Nešetřil.

Oriented vertex-colorings have been studied by several authors in the last decade and the problem of bounding the oriented chromatic number has been investigated for various graph classes (see *e.g.* [3, 10, 11, 12]).

Concerning partial 2-trees, Sopena proved [12] that their oriented chromatic number is at most 7 (this bound was shown to be tight). In [10], the authors obtained tight bounds for the oriented chromatic number of outerplanar graphs with given girth (which is a graph class strictly included in the class of partial 2-trees). Moreover, they proved that $\chi_o(\mathcal{T}_g) = 7$ for every g, $3 \leq g \leq 4$. In this paper, we complete the characterization of the oriented chromatic numbers of partial 2-trees with given girth:

Theorem 1

(1) $\chi_o(\mathcal{T}_g) = 6$ *for every girth g,* $5 \leq g \leq 6$*;*

(2) $\chi_o(\mathcal{T}_g) = 5$ *for every girth g,* $g \geq 7$*;*

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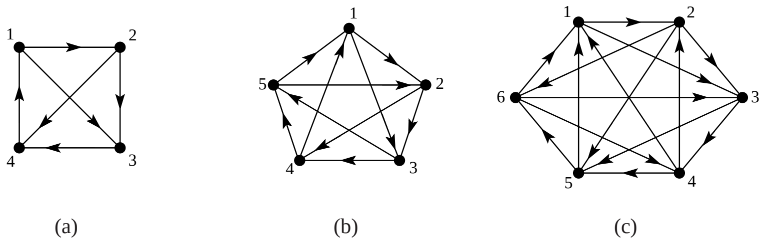


Figure 1: The three target tournaments.

One can define *oriented arc-colorings* of oriented graphs in a natural way by saying that, as in the undirected case, an oriented arc-coloring of an oriented graph G is an oriented vertex-coloring of its line digraph $LD(G)$ (recall that $LD(G)$ is given by $V(LD(G)) = A(G)$ and $ab \in A(LD(G))$ whenever $a = uv$ and $b = vw$). We say that an oriented graph G is *H-arc-colorable* if there exists a homomorphism φ from $LD(G)$ to H and φ is then an *H-arc-coloring* or simply an *arc-coloring* of G. Therefore, an oriented arc-coloring φ of G must satisfy (i) $\varphi(uv) \neq \varphi(vw)$ whenever uv and vw are two consecutive arcs in G, and (ii) $\varphi(vw) \neq \varphi(xy)$ whenever $uv, vw, xy, yz \in A(G)$ with $\varphi(uv) = \varphi(yz)$. Note that two incident but non-consecutive arcs (i.e. two arcs incoming into a same vertex or two arcs outgoing from a same vertex) can get the same color since the two corresponding vertices in $LD(G)$ are not adjacent and does not belong to a directed path of length two. The *oriented chromatic index* of G, denoted by $\chi'_o(G)$, is defined as the smallest order of an oriented graph H such that $LD(G) \rightarrow H$. The oriented chromatic index $\chi'_o(\mathcal{F})$ of a class of oriented graphs $\mathcal{F}$ is defined as the maximum of $\chi'_o(G)$ taken over all graphs G in $\mathcal{F}$.

The oriented chromatic index of oriented graphs was recently studied and several upper and lower bounds are known (see [8, 9, 10]).

Upper bounds for the oriented chromatic index can be easily derivated from oriented chromatic number:

Observation 2 *[8] For every oriented graph G, $\chi'_o(G) \leq \chi_o(G)$.*

Our second result gives estimates of the oriented chromatic indexes of partial 2-trees with girth 4, 5 and 6, and a characterization for all other girths:

Theorem 3

(1) $\chi'_o(\mathcal{T}_3) = 7$*;*

(2) $6 \leq \chi'_o(\mathcal{T}_4) \leq 7$*;*

(3) $5 \leq \chi'_o(\mathcal{T}_g) \leq 6$ *for every girth g, $5 \leq g \leq 6$;*

(4) $\chi'_o(\mathcal{T}_g) = 5$ *for every girth g, $7 \leq g \leq 17$;*

(5) $\chi'_o(\mathcal{T}_g) = 4$ *for every girth g, $g \geq 18$;*

Notation In the rest of the paper, we will use the following notation. A vertex of degree k (resp. at least k) will be called a k-vertex (resp. ${}^{\geq}k$-vertex). We denote by $\delta(G)$ (resp. $\Delta(G)$) the minimum (resp. maximum) degree of the graph G.

A *k-path* in a graph G is a path $P = [u, v_1, v_2, \ldots, v_{k-1}, w]$ of length k (*i.e.* a path with k arcs). The vertices u and w are the *endpoints* of P. Note that a 1-path is an arc. A (k,d)-path is a k-path such that all internal vertices v_i have degree d.

A *2-vertex contraction* is the contraction of an edge incident to a 2-vertex.

Preliminary results The upper bounds of Theorems 1 and 3 will be obtained by proving that the considered partial 2-trees admit a T-vertex- or a T-arc-coloring, for some tournament T. We will use the tournaments T_4, T_5, and T_6 depicted on Figure 1, whose properties, given below, have already been used in the literature to bound oriented chromatic number and oriented chromatic index of graphs.

The tournament T_4 is the only tournament on four vertices containing a directed 4-cycle.

Note that the tournament T_5 is a circular tournament and thus is vertex-transitive.

Proposition 4 *[3] For every pair of (not necessarily distinct) vertices $u, v \in V(T_5)$, there exists an oriented 4-path connecting u with v for any of the 16 possible orientations of such an oriented 4-path.*

Proposition 5 *[10] For every pair of (not necessarily distinct) vertices $u, v \in V(T_6)$, there exists an oriented 3-path connecting u with v for any of the 8 possible orientations of such an oriented 3-path.*

Lemma 6 *Let $\mathcal{C}$ be a graph class closed under 2-vertex contraction such that every non-empty graph $G \in \mathcal{C}$ with girth at least g contains either a 1-vertex or a $(k,2)$-path, for some $k \geq 2$. Then, for every $n \geq 0$, every non-empty graph $G' \in \mathcal{C}$ with girth at least $g + n \left\lfloor \frac{g-1}{k-1} \right\rfloor$ contains either a 1-vertex or a $(k+n,2)$-path.*

For a given graph G and a vertex $v \in V(G)$, we denote:

$$D^G(v) \quad = \quad |\{u \in V(G),\ d(u) \geq 3,\ \text{such that } uv \in A(G) \text{ or } \exists w \in V(G),\ d_G(w) = 2,\ uw, wv \in A(G)\}|.$$

Lih, Wang and Zhu [6] proved the following structural lemma for partial 2-trees:

Lemma 7 *[6] Let G be a partial 2-tree such that $\delta(G) \geq 2$. Then, one the following holds:*

1. *there exists a (3,2)-path (*i.e. *two adjacent 2-vertices);*
2. *there exists a $^{\geq}3$-vertex v such that $D^G(v) \leq 2$.*

We generalize the previous lemma to partial 2-trees with given girth. For a graph G with girth at least g and a vertex $v \in V(G)$, we denote:

$$S_g^G(v) \quad = \quad \{u \in V(G),\ d(u) \geq 3,\ \text{such that there exists a unique path of 2-vertices linking } u \text{ and } v \text{ or } u \text{ and } v \text{ are the endpoints of at least a } (\lceil \tfrac{g}{2} \rceil, 2)\text{-path}\}.$$

Then, we denote $D_g^G(v) = |S_g^G(v)|$. Note that $D_3^G(v) = D^G(v)$ for every $v \in V(G)$.

Lemma 8 *Let G be a partial 2-tree with girth g such that $\delta(G) \geq 2$. Then, one the following holds:*

1. *there exists a $(\lceil \frac{g}{2} \rceil + 1, 2)$-path;*
2. *there exists a $^{\geq}3$-vertex v such that $D_g^G(v) \leq 2$.*

Proof. Let $H \in \mathcal{T}_g$ with $\delta(H) \geq 2$ such that it contains no $(\lceil \frac{g}{2} \rceil + 1, 2)$-path. Note that in this case H is not a cycle and thus contains vertices of degree at least 3. Then, consider the graph H' obtained from H by removing all the 2-vertices and adding an arc between every pair of the remaining vertices which was linked by at least a $(k,2)$-path in H, for some k. Since H contains 3-vertices, H' is not reduced to a unique vertex.

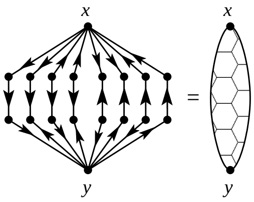
Let v be any vertex of H' and let $N_{H'}(v)$ be the set of v's neighbors in H'. It is clear that, for every $w \in N_{H'}(v)$, there exists at least one path of 2-vertices linking v and w (possibly of length one, i.e. an arc) in H. In addition, if there exists more than one path of 2-vertices linking v and w in H, then at most one of these paths is a $(\lfloor \frac{g}{2} \rfloor, 2)$-path and the others are $(\lceil \frac{g}{2} \rceil, 2)$-paths since H has girth g. This shows that for every $v \in H'$, we have $d_{H'}(v) = D_g^H(v)$.

By construction, H' is clearly a partial 2-tree, and thus contains a vertex v such that $d_{H'}(v) \leq 2$ and thus $D_g^H(v) \leq 2$ (note that $d_H(v) \geq 3$ since this vertex remains in H'). That completes the proof. □

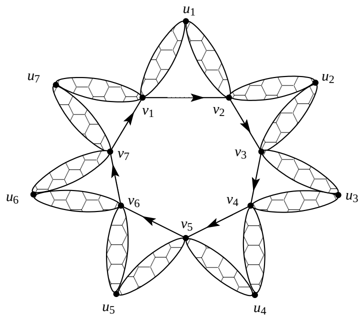
The oriented chromatic number of partial 2-trees

Scketch of proof of Theorem 1(1). We first prove that $\chi_o(\mathcal{T}_5) \leq 6$ (note that it is sufficient to consider the case $g = 5$); more precisely we show that every partial 2-tree with girth at least 5 admits a homomorphism to the tournament T_6 depicted in Figure 1(c). Let H be a minimal (w.r.t. the number vertices) partial 2-tree with girth at least 5 having no homomorphism to T_6. By minimality and Proposition 5, we prove that H contains neither a 1-vertex nor a $(3,2)$-path. It is well known that partial 2-trees are 2-degenerated, and we thus get a contradiction thanks to Lemma 6.

The graph G_6 depicted in Figure 2 is a partial 2-tree with girth 6 and has oriented chromatic number 6. This shows that $\chi_o(\mathcal{T}_6) = 6$, that completes the proof.

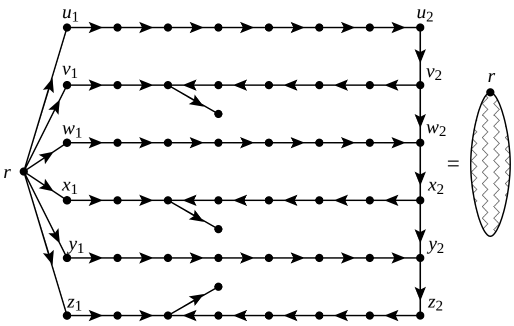


(a) The graph H

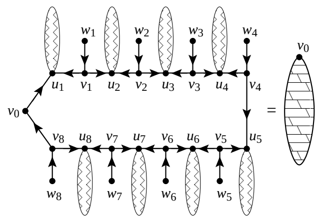


(b) The graph G_6

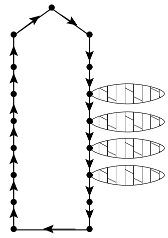
Figure 2: An oriented partial 2-tree with girth 6 and oriented chromatic number 6.



(a) The graph H_1.



(b) The graph H_2.



(c) The graph G_{17}.

Figure 3: Construction of an oriented partial 2-tree with girth 17 and oriented chromatic index 5.

□

Scketch of proof of Theorem 1(2). We first prove that $\chi_o(\mathcal{T}_7) \leq 5$ (note that it is sufficient to consider the case $g = 7$). To prove this upper bound, we use the same kind of arguments than in the previous proof.

In [7], Nešetřil *et al.* constructed for every $g \geq 3$, an oriented outerplanar graph with girth at least g which has oriented chromatic number 5. This completes the proof. □

The oriented chromatic index of partial 2-trees

We denote by $\mathcal{O}_g$ the class of outerplanar graph with girth at least g. The first three assumptions of Theorem 3 directly follow from Observation 2, Theorem 1(1) and some results of Pinlou and Sopena from [10], namely $\chi_o(\mathcal{T}_3) = 7$, $\chi'_o(\mathcal{O}_4) = 6$, and $\chi'_o(\mathcal{O}_6) = 5$.

It now remains to prove Theorems 3(4) and 3(5).

Scketch of proof of Theorem 3(4). The upper bound follows from Theorem 1(2) and Observation 2.

To complete this proof, we have to construct a partial 2-tree with girth 17 and oriented chromatic index 5. Let us consider the graph G_{17} depicted in Figure 3(c). This graph is a partial 2-tree with girth 17 and has oriented chromatic index 5.

□

Scketch of proof of Theorem 3(5). We first prove that $\chi_o(\mathcal{T}_{18}) \leq 4$ (note that it is sufficient to consider the case $g = 18$); more precisely, we prove that every partial 2-tree with girth at least 18 admits a homomorphism to the tournament T_4 depicted in Figure 1(a). Let H be a minimal (w.r.t. the number of vertices) partial 2-tree with girth at least 18 having no homomorphism to T_4. By minimality and a case to case analysis, we prove that H contains neither a 1-vertex, nor a $(10,2)$-path, nor a vertex v such that $D_{18}(v) \leq 2$. This leads us to a contradiction by Lemma 8.

To complete this proof, we have to construct a partial 2-tree for every girth $g \geq 18$ which needs 4 colors for any oriented arc-coloring. It not difficult to check that the partial 2-tree obtained from two vertex-disjoint circuits, the first one of size g and the second one of size $k \geq g$ with $k \not\equiv 0 \pmod 3$ has girth g and oriented chromatic index 4. □

Conclusion In this paper, we characterized the oriented chromatic number of partial 2-trees for every girth $g \geq 3$. We also characterized the oriented chromatic index of partial 2-trees for every girth excepted three. Hence, finding the optimal bound of the oriented chromatic index of partial 2-trees still remains an open question for girths $4,5$ and 6.

Oriented vertex-colorings of planar graphs (which is a superclass of the class of partial 2-trees) with given girth have been considered in the literature and the following bounds are known:

Theorem 9 *[1, 2, 3] Let G be a planar graph with girth g. Then,* $\chi_o(G) \leq 5$ *(resp. 7, 11, 19, 47) if* $g \geq 14$ *(resp.* $g \geq 7$, $g \geq 6$, $g \geq 5$, $g \geq 4$*).*

We can remark that the techniques used in this paper can be applied to planar graphs and help us to improve the previous known bounds:

Theorem 10 *Let G be a planar graph with girth at least* 11*. Then* $\chi_o(G) \leq 6$.

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A Proofs of Theorems

Proof of Lemma 6. The proof is an induction on n. The case $n=0$ is trivial. We now prove the contraposition of the induction step. Suppose that the statement is false at step $n+1$, that is, there exists a graph H_{n+1} such that:

- $H_{n+1} \in \mathcal{C}$,
- $\delta(H_{n+1}) \geq 2$,
- H_{n+1} contains no $(k+n+1)$-path,
- H_{n+1} has girth at least $g+(n+1)\left\lfloor \frac{g-1}{k} \right\rfloor$.

Let H_n be the graph obtained from H_{n+1} by contracting every $(k+n,2)$-path into a $(k+n-1,2)$-path. Notice that this transformation is well-defined since H_{n+1} contains no $(k+n+1,2)$-path. Notice also that exactly one 2-vertex per $(k+n,2)$-path is contracted. Consider an induced cycle C of length l in H_{n+1}. The cycle C contains at most $\left\lfloor \frac{l}{k+n+1} \right\rfloor$ $(k+n,2)$-paths. The length of the corresponding cycle C' in H_n is thus at least $l - \left\lfloor \frac{l}{k+n+1} \right\rfloor$. Since this expression is an increasing function of l, the girth of H_n is at least

$$g+(n+1)\left\lfloor \frac{g-1}{k} \right\rfloor - \left\lfloor \frac{g+(n+1)\left\lfloor \frac{g-1}{k} \right\rfloor}{k+n+1} \right\rfloor .$$

Let us set $g-1 = \alpha k + \beta$ with $0 \leq \beta < k$.

$$\begin{aligned}
g+(n+1)\left\lfloor \frac{g-1}{k} \right\rfloor - \left\lfloor \frac{g+(n+1)\left\lfloor \frac{g-1}{k} \right\rfloor}{k+n+1} \right\rfloor &= g+(n+1)\alpha - \left\lfloor \frac{g+(n+1)\alpha}{k+n+1} \right\rfloor \\
&= g+(n+1)\alpha - \left\lfloor \frac{\alpha k+\beta+1+(n+1)\alpha}{k+n+1} \right\rfloor \\
&= g+(n+1)\alpha - \left\lfloor \alpha + \frac{\beta+1}{k+n+1} \right\rfloor \\
&= g+n\alpha - \left\lfloor \frac{\beta+1}{k+n+1} \right\rfloor \\
&= g+n\alpha \\
&= g+n\left\lfloor \frac{g-1}{k} \right\rfloor
\end{aligned}$$

So we have:

- $H_n \in \mathcal{C}$, since $k+n>1$ and $\mathcal{C}$ is closed under 2-vertex contraction,
- $\delta(H_n) \geq 2$, by construction,
- H_n contains no $(k+n,2)$-path, by construction,
- H_n has girth at least $g+n\left\lfloor \frac{g-1}{k} \right\rfloor$, as shown above.

This contradicts the statement at step n. □

Proof of Theorem 1(1). We first prove that $\chi_o(T_5) \leq 6$ (note that it is sufficient to consider the case $g=5$); more precisely, we prove that every partial 2-tree with girth at least 5 admits a homomorphism to the tournament T_6 depicted in Figure 1(c). Let H be a minimal (w.r.t. the number of vertices) partial 2-tree with girth at least 5 having no homomorphism to T_6. We show that H contains neither a 1-vertex nor a $(3,2)$-path.

1. Suppose that H contains a 1-vertex u. Then, due to the minimality of H, the partial 2-tree $H' = H \setminus u$ with girth at least 5 admits a T_6-vertex-coloring f. Since every vertex of T_6 has at least two successors and at least two predecessors, f can easily be extended to H.

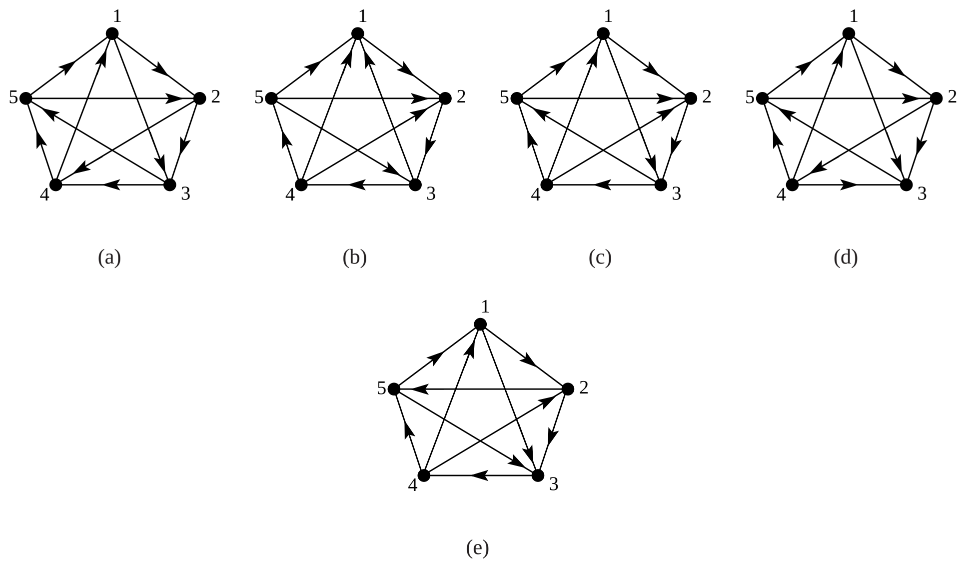

Figure 4: The five non isomorphic tournaments on 5 vertices without source nor sink.

2. Suppose now that H contains a $(3,2)$-path $[u, v_1, v_2, w]$. Then, due to the minimality of H, the partial 2-tree $H' = H \setminus \{v_1, v_2\}$ admits a T_6-vertex-coloring f. By Proposition 5, f can be extended to H.

It is well known that partial 2-trees are closed under 2-vertex contraction. Moreover, they are 2-degenerated (*i.e.* every partial 2-tree contains either a 1-vertex or a $(2,2)$-path). Therefore, by Lemma 6, every partial 2-tree with girth at least 5 contains either a 1-vertex or a $(3,2)$-path, that is a contradiction.

To complete this proof, we have to construct a partial 2-tree with girth 6 and oriented chromatic number 6. Let us consider the graph G_6 constructed from a circuit $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1]$ of length seven and fourteen copies of the graph H (see Figure 2(a)) arranged as depicted on Figure 2(b). We can easily see that G_6 is a partial 2-tree with girth 6.

Suppose first that $\chi_o(G_6) \leq 4$, and therefore there exists a homomorphism $f : G_6 \to T$, where T is a tournament on 4 vertices. Since G_6 contains a circuit of length $7 \not\equiv 0 \pmod 3$, T must contain a circuit of length 4. There exist four non isomorphic tournaments on 4 vertices, but only one contains a circuit of length 4: the tournament T_4 depicted in Figure 1(a). However, we can check that there does not exist a pair of vertices $u, v \in V(T_4)$ such that u and v are the endpoint of the 8 possible 3-paths. Thus, $H \not\to T_4$, and thus $G_6 \not\to T_4$.

Therefore, $\chi_o(G_6) \geq 5$. Suppose that $\chi_o(G_6) = 5$. There exist twelve non isomorphic tournaments on 5 vertices. We will prove that none of these tournaments allows us to color G_6. We can first omit those containing a source or a sink. The five remaining tournaments are depicted in Figure 4.

A computer check shows that every T-vertex-coloring f of H, where T is one of the four tournaments in Figures 4(a), 4(b), 4(c), 4(d), implies that $f(x) = f(y)$. Hence, this would mean that $f(v_1) = f(u_1) = f(v_2)$ in G_6, which is a contradiction. It is then clear that if $\chi_o(G_6) = 5$, then $G_6 \to T$ where T is the tournament of Figure 4(e). A computer check shows that every T-vertex-coloring f of H implies that $f(x) \in \{1,2,5\}$ and $f(y) \in \{1,2,5\}$. However, the cycle $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1]$ in G_6 needs four colors, that is a contradiction. The graph G_6 has oriented chromatic number 6.

□

Proof of Theorem 1(2).

We first prove that $\chi_o(T_7) \leq 5$ (note that it is sufficient to consider the case $g = 7$). To prove this upper bound, we use the same kind of arguments than in the previous proof.

More precisely, we prove that every partial 2-tree with girth at least 7 admits a T_5-vertex-coloring, where T_5 is the tournament depicted in Figure 1(b). Let H be a minimal (w.r.t. the number of vertices) partial 2-tree with girth at least 7 having no homomorphism to T_5. By minimality and Proposition 4, we prove that H contains neither a 1-vertex nor a $(4,2)$-path. We thus get a contradiction thanks to Lemma 6.

To complete this proof, we have to construct, for all girth $g \geq 7$, a partial 2-tree with girth g and oriented chromatic number 5. In [7], Nešetřil *et al.* constructed for every g, $g \geq 3$, an oriented outerplanar graph with girth at least g which has oriented chromatic number 5. The class of outerplanar graphs is included in the class of partial 2-trees: that completes the proof. □

Proof of Theorem 3(1). Thanks to Observation 2 and the result of Pinlou and Sopena [10] ($\chi_o(\mathcal{T}_3) = 7$), we directly obtain that $\chi'_o(\mathcal{T}_3) \leq 7$. Moreover, Pinlou and Sopena [10] constructed an outerplanar graph with oriented chromatic index 7. This completes the proof. □

Proof of Theorem 3(2). The upper bound directly follows from Theorem 3(1) and the lower bound follows from the result of Pinlou and Sopena [10] ($\chi'_o(\mathcal{O}_4) = 6$, where $\mathcal{O}_4$ is the class of outerplanar graphs with girth at least 4). □

Proof of Theorem 3(3). The upper bound follows from Theorem 1(1) and Observation 2. The lower bound follows from the result of Pinlou and Sopena [10] ($\chi'_o(\mathcal{O}_6) = 5$, where $\mathcal{O}_6$ is the class of outerplanar graphs with girth at least 6). □

Proof of Theorem 3(4). The upper bound follows from Theorem 1(1) and Observation 2.

To complete the proof, we have to construct a partial 2-tree with girth 17 and oriented chromatic index 5.

Consider first the graph H_1 depicted in Figure 3(a). We show that H_1 does not admit a T_4-arc-coloring f such that $f(ru_1) = f(rv_1) = f(rw_1) = f(rx_1) = f(ry_1) = f(rz_1) = 3$. Suppose to the contrary and let f be a T_4-arc-coloring such that $f(ru_1) = f(rv_1) = f(rw_1) = f(rx_1) = f(ry_1) = f(rz_1) = 3$. Then, by a tedious but not difficult case to case analysis, we can show that we necessarily have $f(u_2v_2) \in \{1,4\}$, $f(w_2x_2) \in \{1,4\}$ and $f(y_2z_2) \in \{1,4\}$. However, any T_4-arc-coloring h of a directed 3-path $[w,x,y,z]$ such that $h(wx) = 4$ implies that $h(xy) = 1$ and $h(yz) \in \{2,3\}$. This shows that the directed 6-path $[u_2,v_2,w_2,x_2,y_2,z_2]$ does not admit a T_4-arc-coloring such that $f(u_2v_2) \in \{1,4\}$, $f(w_2x_2) \in \{1,4\}$ and $f(y_2z_2) \in \{1,4\}$.

Now, consider the graph H_2 constructed from 8 copies of H_1 arranged as shown on Figure 3(b). We show that any T_4-arc-coloring f of H_2 is such that $f(v_0u_1) \neq 1$. Suppose to the contrary and let f be a T_4-arc-coloring of H_2 is such that $f(v_0u_1) = 1$. Note that $f(v_iu_i) = 1$ for some $i \in [1,8]$ implies that $f(w_iv_i) = 4$ and therefore $f(v_iu_{i+1}) = 1$ since the vertex 4 of T_4 has a unique successor. In addition, we necessarily have $f(v_iu_i) \in \{1,2\}$ for any $i \in [1,8]$ such that $f(v_{i-1}u_i) = 1$ since the colors of the incoming arcs on u_i must have at least one common successor to color the six outgoing arcs of u_i. Moreover, it is clear that the arcs of the cycle $v_0,u_1,v_1,u_2,\dots,u_8,v_8,v_0$ are not all colored with color 1. It thus implies that there exists $j \in [1,8]$ such that $f(v_{j-1}u_j) = 1$ and $f(v_ju_j) = 2$. Therefore, the six outgoing arcs of u_j must be colored with the color 3, that is forbidden by the remark we made in the previous paragraph.

To conclude the proof, let us consider the graph G_{17} obtained from a circuit of length 17 and 4 copies of the graph H_2 arranged as shown in Figure 3(c). The graph G_{17} contains a circuit of size $17 \not\equiv 0 \pmod 3$. It then implies that any arc-coloring with 4 colors of G is a T_4-arc-coloring (T_4 is the only tournament on 4 vertices which contains a circuit of length 4). Moreover, any T_4-arc-coloring of a circuit is such that the color 4 appears at least once on four consecutive arcs. Thus, the v_0-vertex of one of the copies of H_2 has its incoming arc colored with the color 4. This implies that $f(v_0u_1) = 1$, that is forbidden by the remark we made in the previous paragraph. The graph G_{17}, which is a partial 2-tree with girth 17, has oriented chromatic index 5.

□

To prove Theorem 3(5), we need to introduced the following new notion. Let T_4 be the tournament on four vertices depicted in Figure 1(a). We say that a T_4-arc-coloring f of an oriented graph G is *good* if

1. $\forall\, u \in V(G)$, $C_f^+(u) \in \{\{1\},\{2\},\{3\},\{4\},\{2,3\},\{3,4\}\}$,
2. $\forall\, u \in V(G)$, $C_f^-(u) \in \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{2,3\}\}$.

In [8], Ochem *et al.* proved the following lemma:

Lemma 11 *[8] Let $P = [u,v_1,\dots,v_9,w]$ be an oriented (10,2)-path. Then, any good T_4-arc-coloring of $P' = P \setminus \{v_2,\dots,v_8\}$ can be extended to a good T_4-arc-coloring of P.*

A careful study of T_4 allow us to obtain the next lemma:

Lemma 12 *Let $P = [u, v_1, \ldots, v_8, w]$ be an oriented (9,2)-path and let h be a good T_4-arc-coloring of $P' = P \setminus \{v_2, \ldots, v_7\}$. Then, h can be extended a good T_4-arc-coloring of P whenever $uv_1 \in A(P')$ and $h(uv_1) \in \{1,2,4\}$, or $v_1u \in A(P')$ and $h(v_1u) \in \{1,3,4\}$.*

Proof of Theorem 3(5). We first prove that $\chi'_o(T_{18}) \leq 4$ (note that it is sufficient to consider the case $g = 18$); more precisely, we prove that every partial 2-tree with girth at least 18 admits a good T_4-arc-coloring. Let H be a minimal (w.r.t. the number of vertices) partial 2-tree with girth at least 18 having no good T_4-arc-coloring. We show that H contains neither a 1-vertex, nor a $(10,2)$-path, nor a vertex v such that $D^H_{18}(v) \leq 2$.

1. Suppose that H contains a 1-vertex u, let v be its neighbor and suppose that $uv \in A(H)$. Let $H' = H \setminus u$. Due to the minimality of H, H' admits a good T_4-arc-coloring f. Therefore, $C^+_f(v) \in \{\{1\},\{2\},\{3\},\{4\},\{2,3\},\{3,4\}\}$. For each possible case, there exists a predecessor in T_4 we can use to extend f to good T_4-arc-coloring of H. The proof of the case $vu \in A(H)$ is similar.

2. Suppose that H contains a (10,2)-path $[u, v_1, v_2, \ldots, v_9, w]$ and let $H' = H \setminus \{v_2, v_3, \ldots, v_8\}$. Due to the minimality of H, H' admits a good T_4-arc-coloring f. Lemma 11 insures that f can be extended to a good T_4-arc-coloring of H.

3. Suppose that there exists $v \in V(H)$ such that $d(v) \geq 3$ and $D^H_{18}(v) = 1$. Let $S^H_{18}(v) = \{w\}$. Therefore there exist at least three $(\lceil \frac{g}{2} \rceil, 2)$-path linking u and w. Let H' be the graph obtained from H be removing the vertex v and all the $(\lceil \frac{g}{2} \rceil, 2)$-paths linking v and w. Due to the minimality of H, H' admits a good T_4-arc-coloring f. Then, we give to each outgoing vertex from v the color 1 and from each incoming arc to v the color 4. Then, by Lemma 12, f can be extended to good T_4-arc-coloring of H.

4. Suppose that there exists $v \in V(H)$ such that $d(v) \geq 3$ and $D^H_{18}(v) = 2$. Let $S^H_{18}(v) = \{w_1, w_2\}$. Let H' be the graph obtained from H be removing the vertex v and all the $(\lceil \frac{g}{2} \rceil, 2)$-paths linking v and w_1, and v and w_2. Due to the minimality of H, H' admits a good T_4-arc-coloring f.

   Since $d(v) \geq 3$, w.l.o.g. we may assume that there exist at least two $(\lceil \frac{g}{2} \rceil, 2)$-paths linking u and w_1.

   - Suppose that there exists only one path $P = [x, y_1, \ldots, y_n, z]$ of 2-vertices linking v and w_2. We first color P such that the arc y_nz or zy_n takes its color in $\{1,3,4\}$ (it is easy to check that this is always possible). Then, in any case, we can color each outgoing vertex from v either with 1, or 2, or 4 and each incoming arc to v with either 1, or 3, or 4. Then, by Lemma 12, f can be extended to good T_4-arc-coloring of H.
   - Suppose now that there exist at least two paths of 2-vertices linking v and w_2. Then, these paths are $(\lceil \frac{g}{2} \rceil, 2)$-path. Then, we can color each outgoing vertex from v with 1 and each incoming arc to v with 4. Then, by Lemma 12, f can be extended to good T_4-arc-coloring of H.

By Lemma 8, every partial 2-tree G with minimum degree 2 and girth 18 contains either a $(10,2)$-path or a $^{\geq}$3-vertex v such that $D^G_{18}(v) \leq 2$, a contradiction.

To complete the proof, let us recall that Pinlou and Sopena [10] proved $\chi'_o(\mathcal{O}_g) = 4$ for every $g \geq 10$ (where $\mathcal{O}_g$ denotes the class of outerplanar graphs with girth g). □