Saclay, July 8, 2014
DataSense Research day

# Statistical properties of topological information inferred from data

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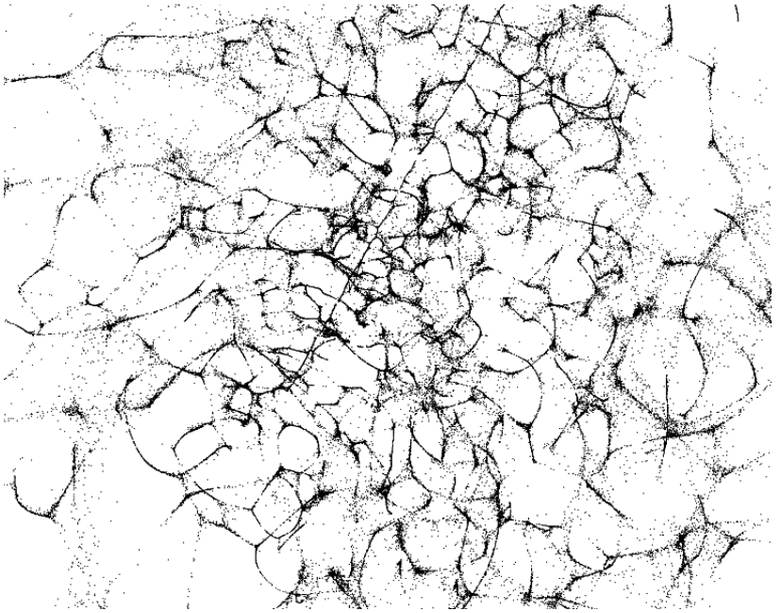
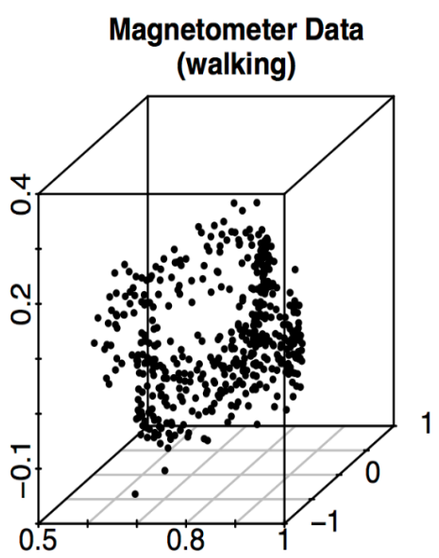
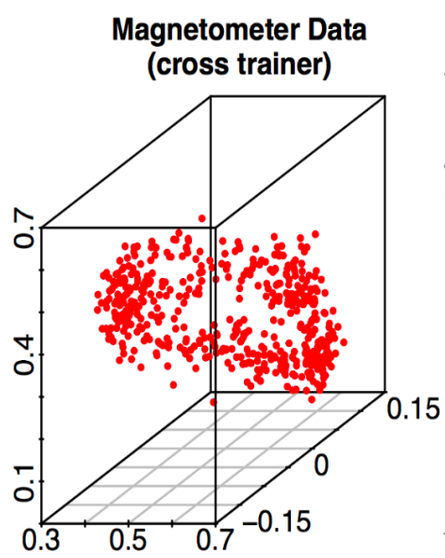
# Introduction

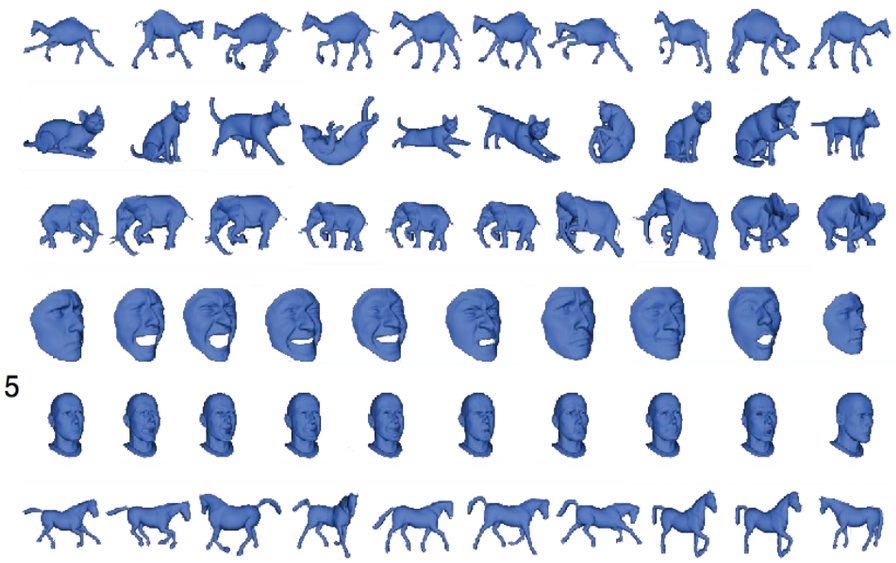
[distribution of galaxies]





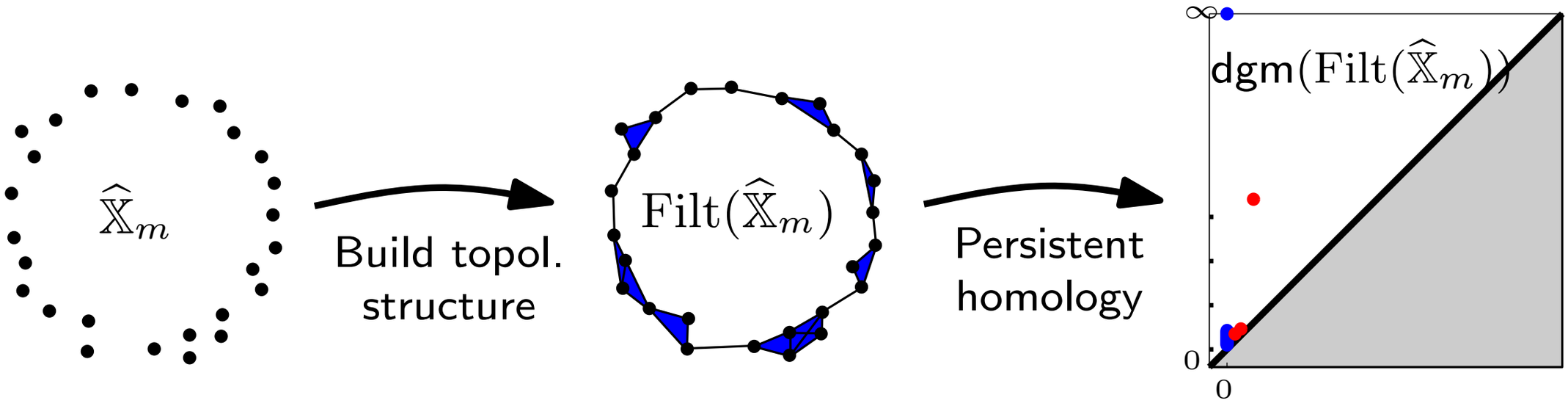




[Non-rigid shape database]



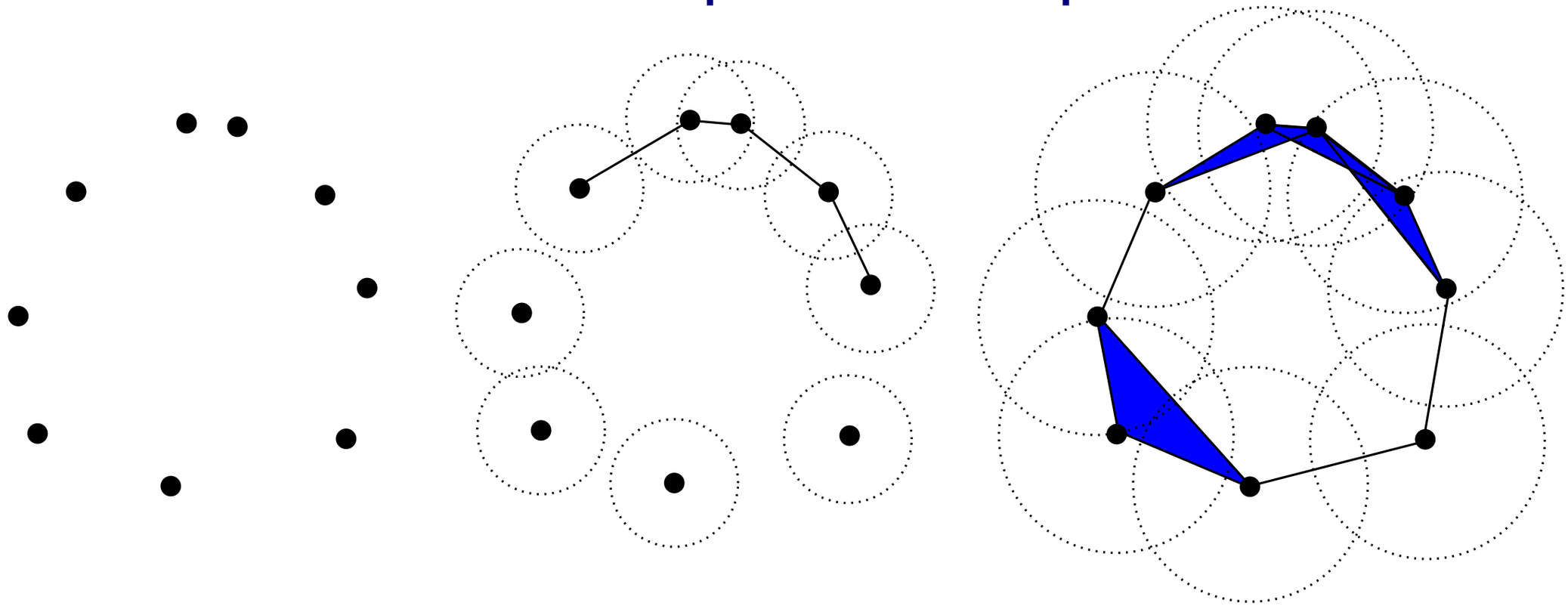
- Data often come as (sampling of) metric spaces or sets/spaces endowed with a similarity measure with, possibly complex, topological/geometric structure.
- Topological Data Analysis (TDA):
  - infer relevant topological and geometric features of these spaces.
  - take advantage of topol./geom. information for further processing of data (classification, recognition, learning, clestering,...).

# Introduction




- Build a geometric filtered simplicial complex on top of $\widehat{\mathbb{X}}_m \rightarrow$ multiscale topol. structure.
- Compute the persistent homology of the complex $\rightarrow$ multiscale topol. signature.
- Compare the signatures of "close" data sets $\rightarrow$ robustness and stability results.
- Statistical properties of signatures

# Filtered simplicial complexes



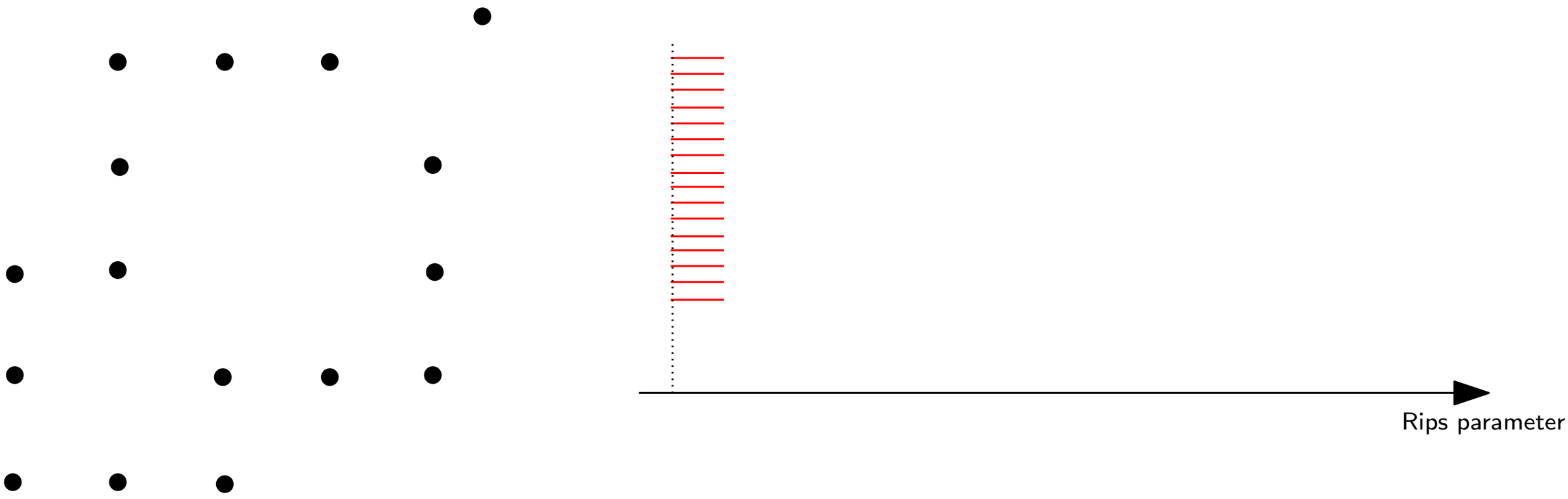
- simplicial complex: generalizes the notion of neighboring graph to higher dimensions by adding triangles,tetrahedra,...

- filtered simplicial complex $=$ nested family of simplicial complexes indexed by $\mathbf{R}$

**Example (used all along the talk):** Let $(X, d_X)$ be a metric space.
The Vietoris-Rips complex $\mathbb{R}\mathrm{ips}(X)$ is defined by: for $a \in \mathbf{R}$,

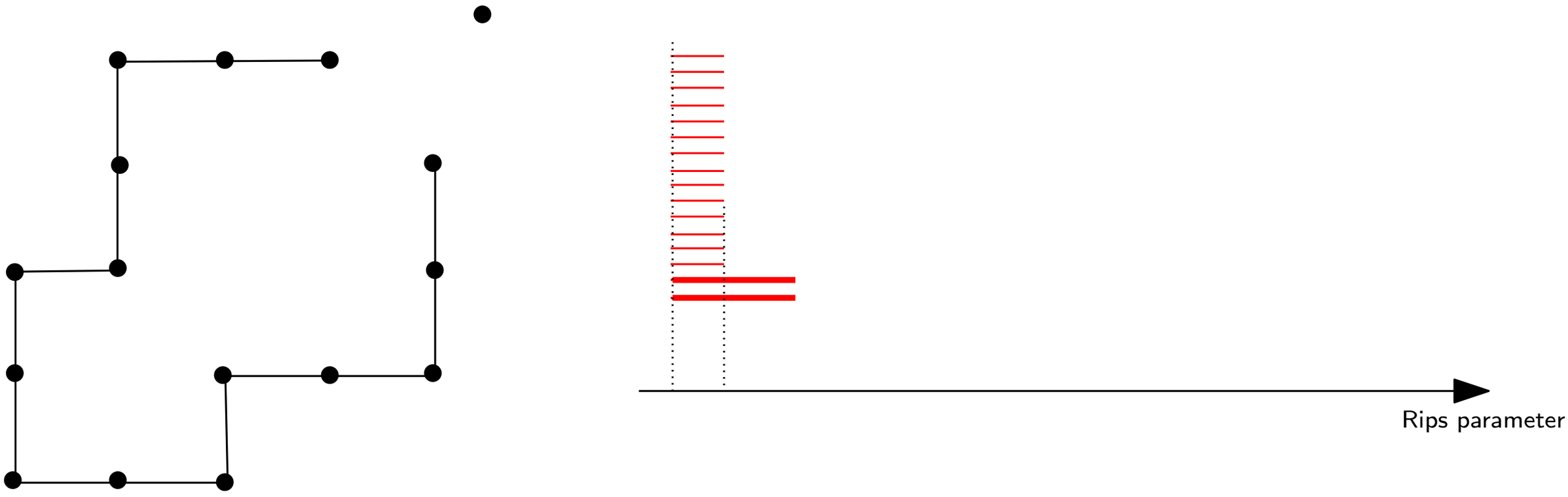
$$[x_0, x_1, \cdots, x_k] \in \mathrm{Rips}(X, a) \Leftrightarrow d_X(x_i, x_j) \leq a, \quad \text{for all } i, j$$

# Persistent homology of filtered complexes




- An efficient way to encode the evolution of the topology (homology) of families of nested spaces (filtered complex, sublevel sets,...).
- Multiscale topological information.
- Barcodes/persistence diagrams can be efficiently computed.
- Stability properties

# Persistent homology of filtered complexes

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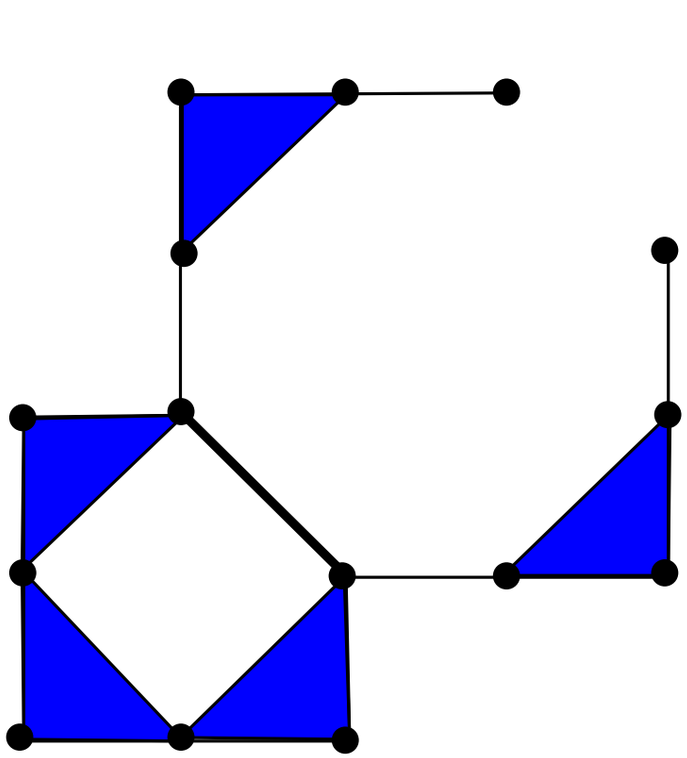


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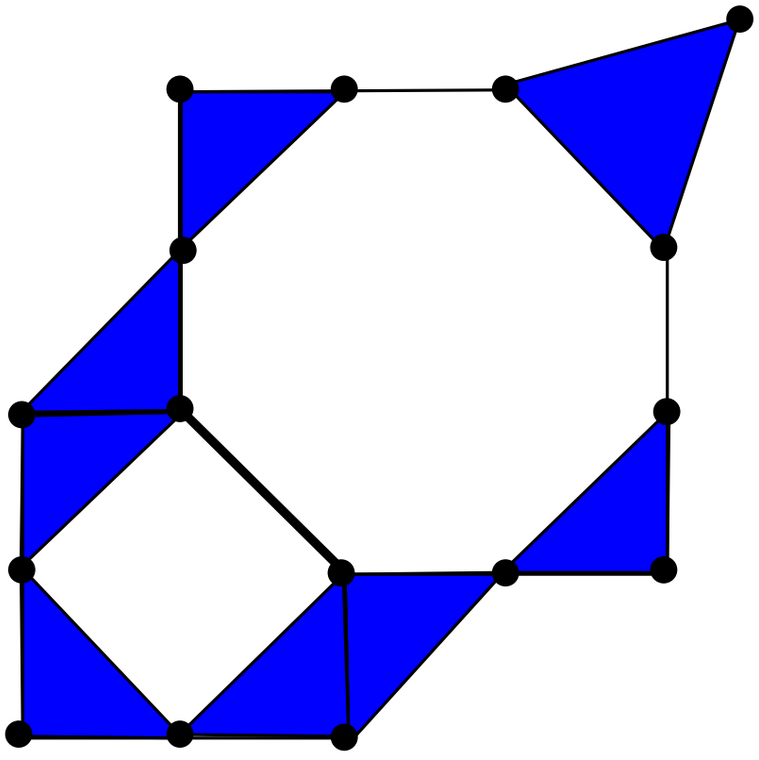

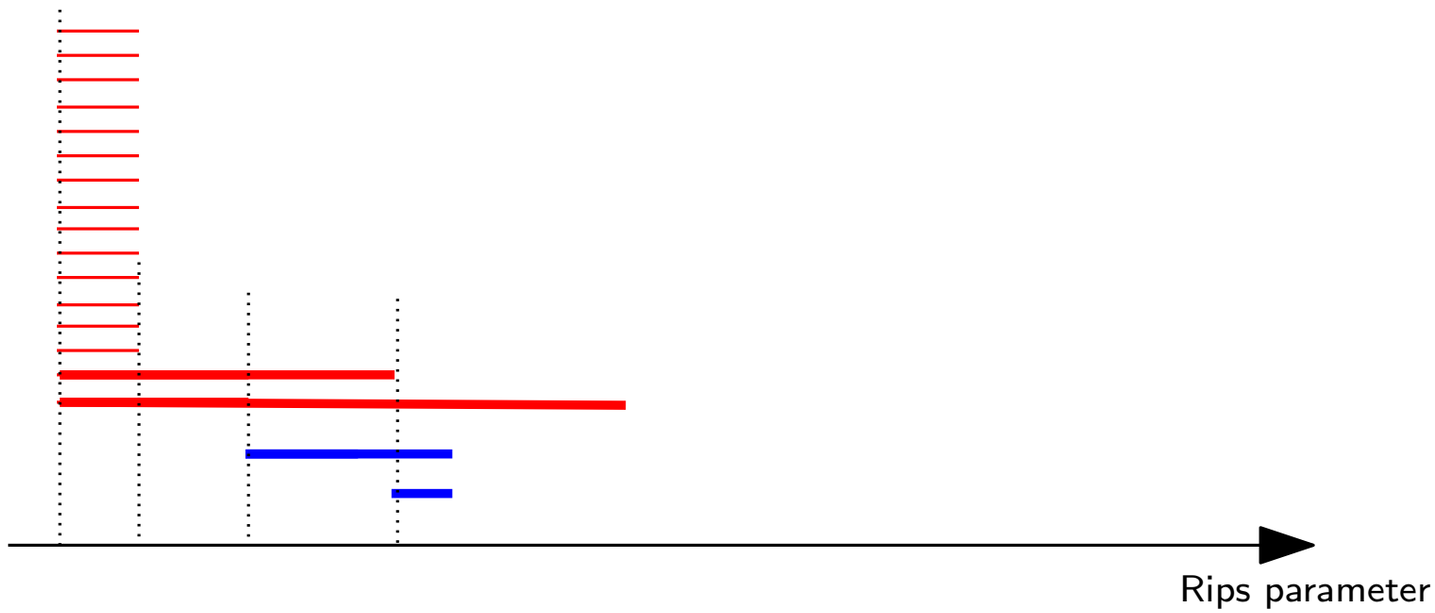


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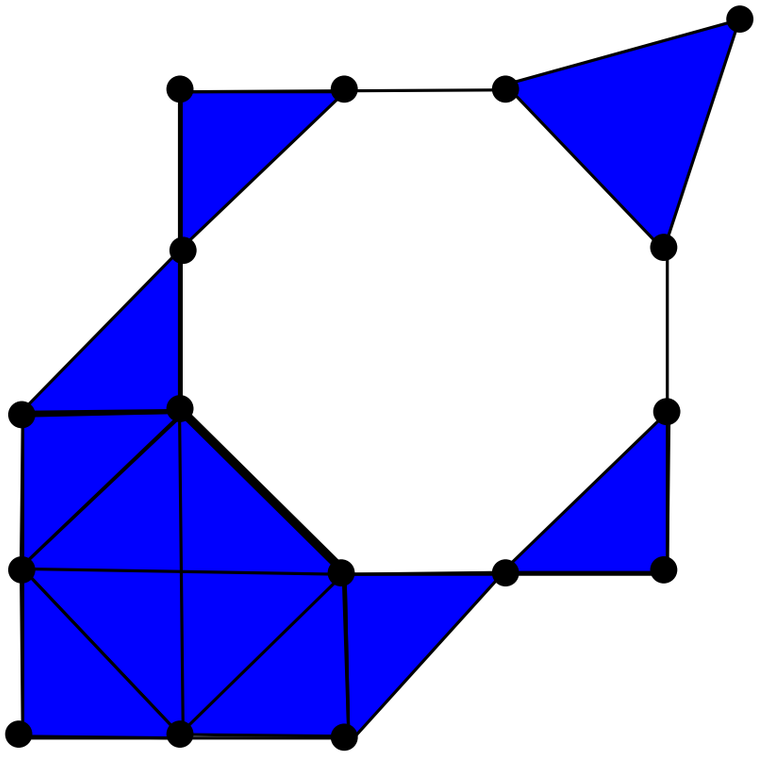

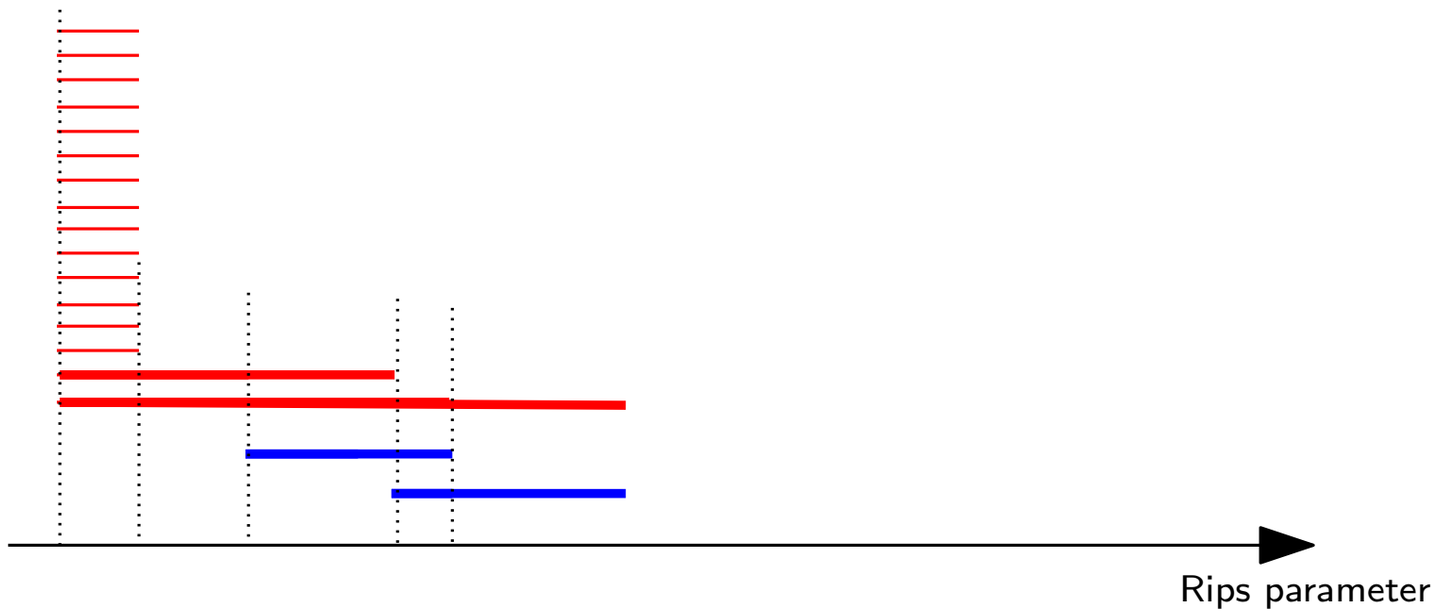


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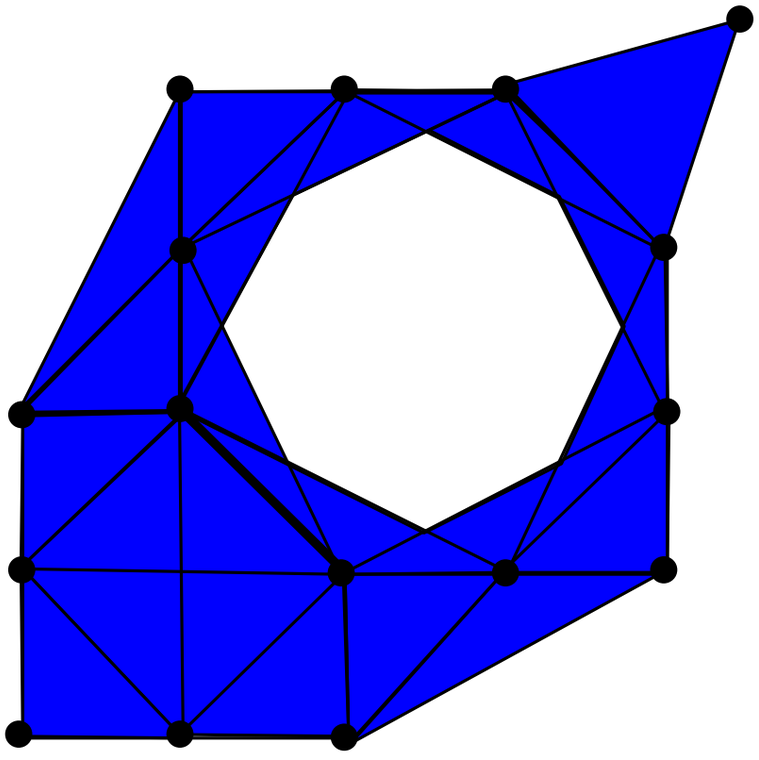

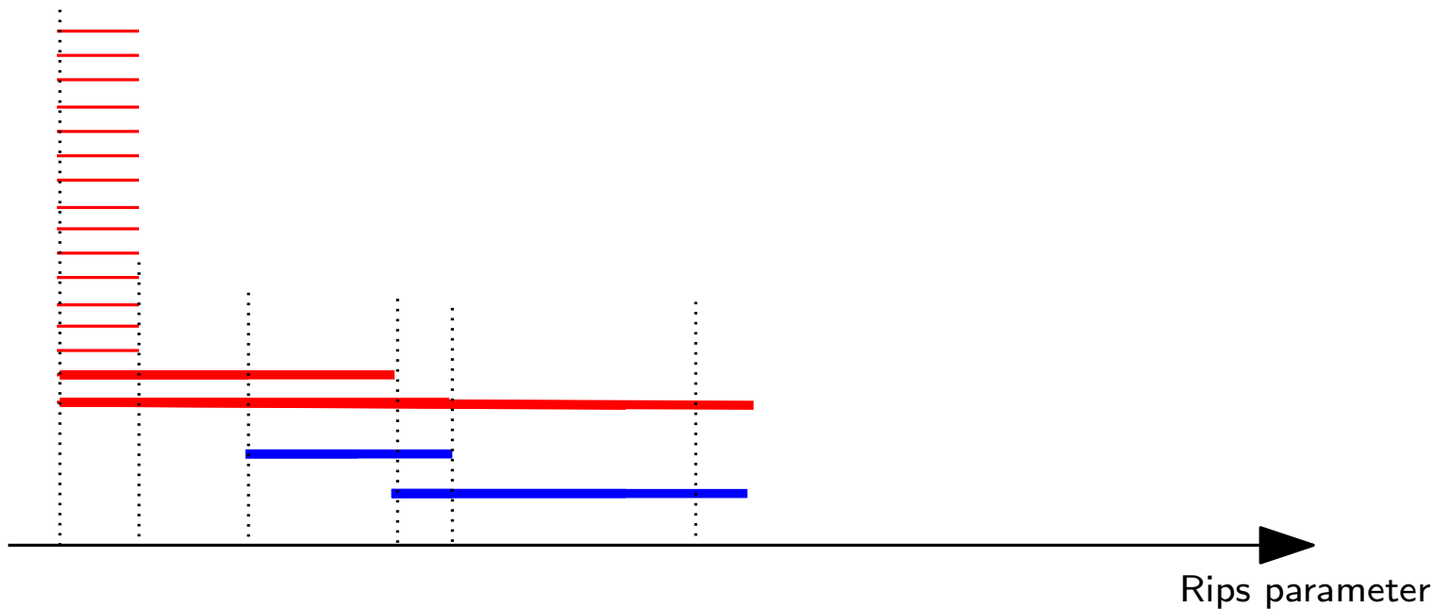


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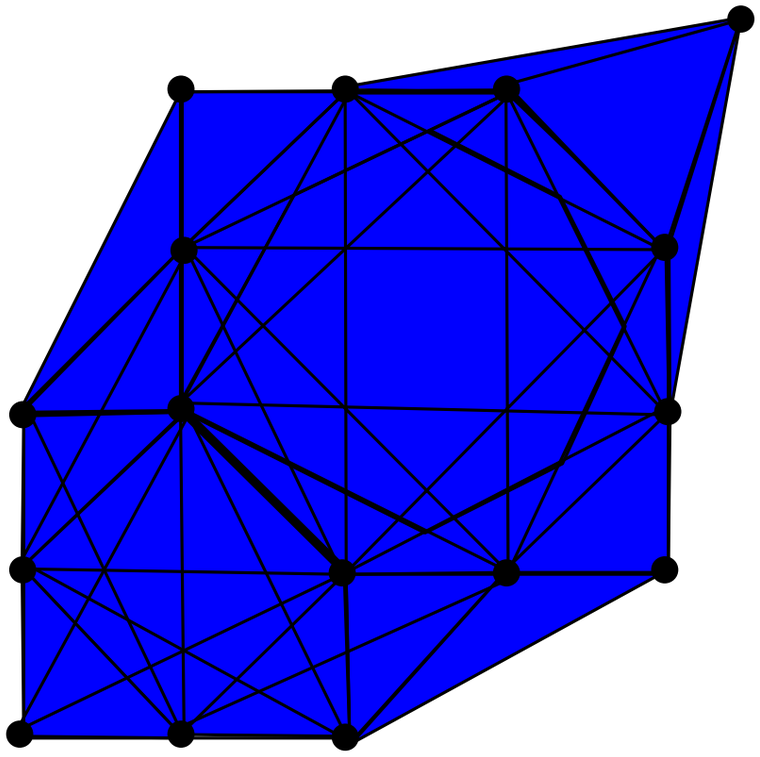

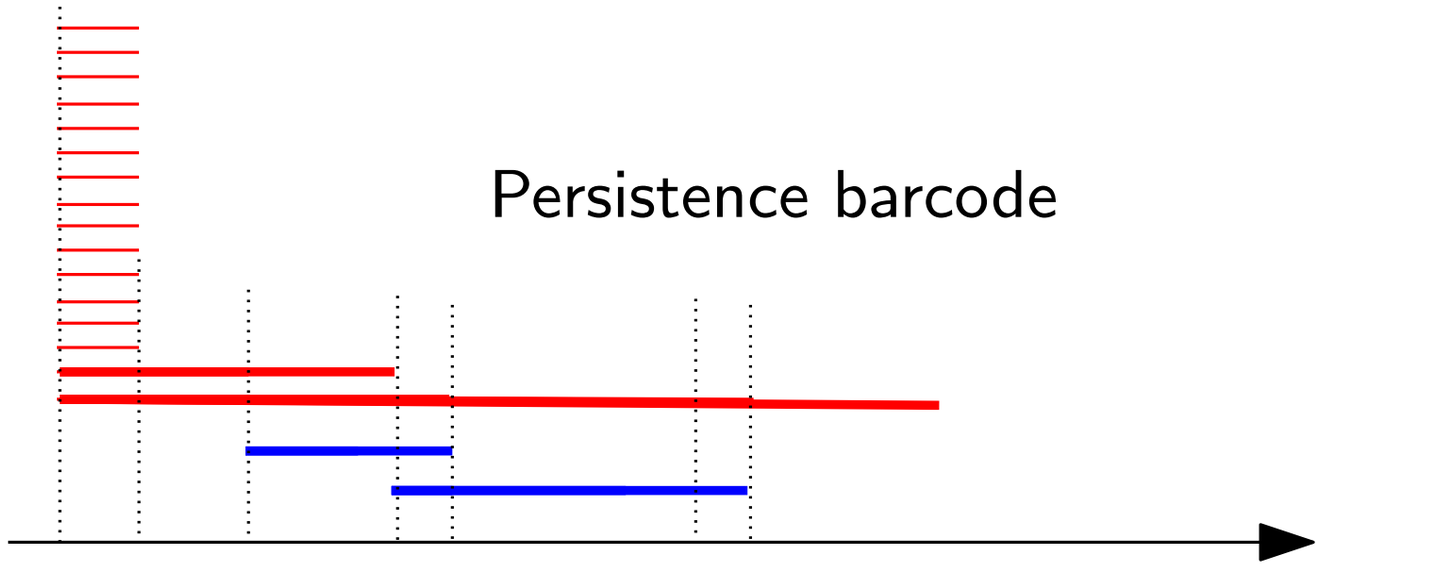


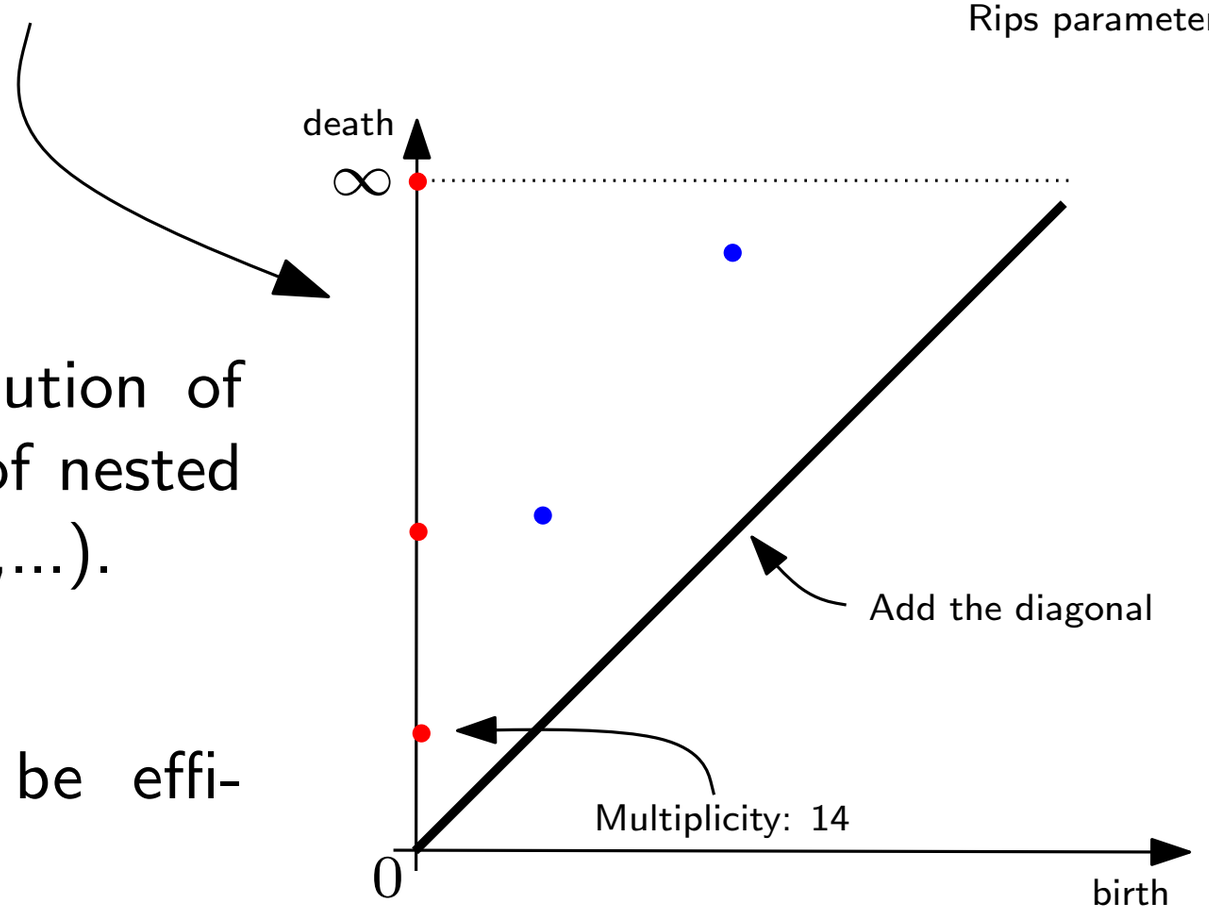


Persistence diagram

- An efficient way to encode the evolution of the topology (homology) of families of nested spaces (filtered complex, sublevel sets,...).
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# Stability properties

**"Stability theorem":** Close spaces/data sets have close persistence diagrams!

If $\mathbb{X}$ and $\mathbb{Y}$ are pre-compact metric spaces, then

$$\mathrm{d_b}(\mathsf{dgm}(\mathrm{Rips}(\mathbb{X})), \mathsf{dgm}(\mathrm{Rips}(\mathbb{Y}))) \leq d_{GH}(\mathbb{X}, \mathbb{Y}).$$

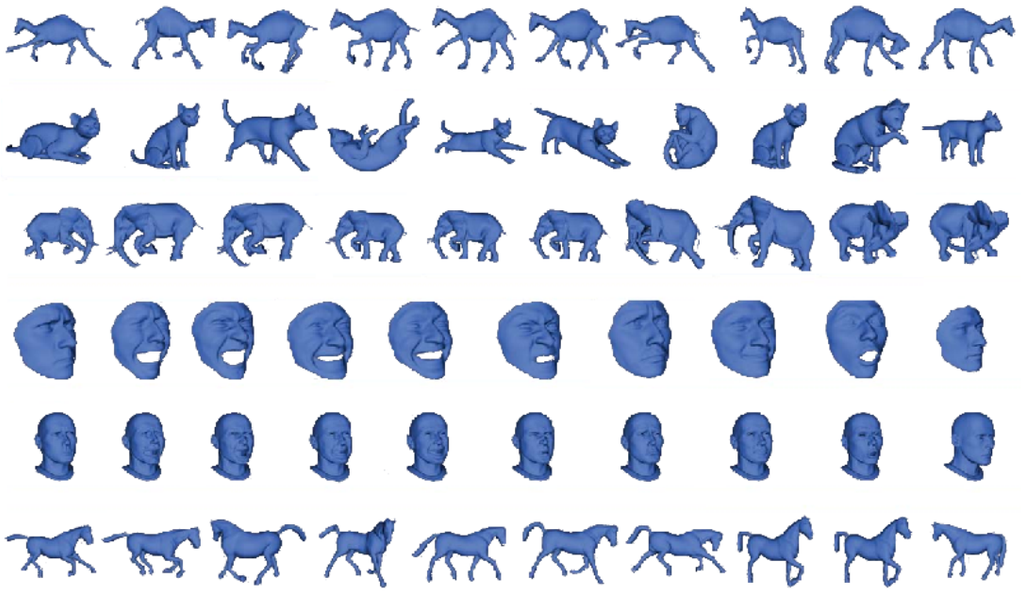
Bottleneck distance: a matching distance between diagrams

Gromov-Hausdorff distance

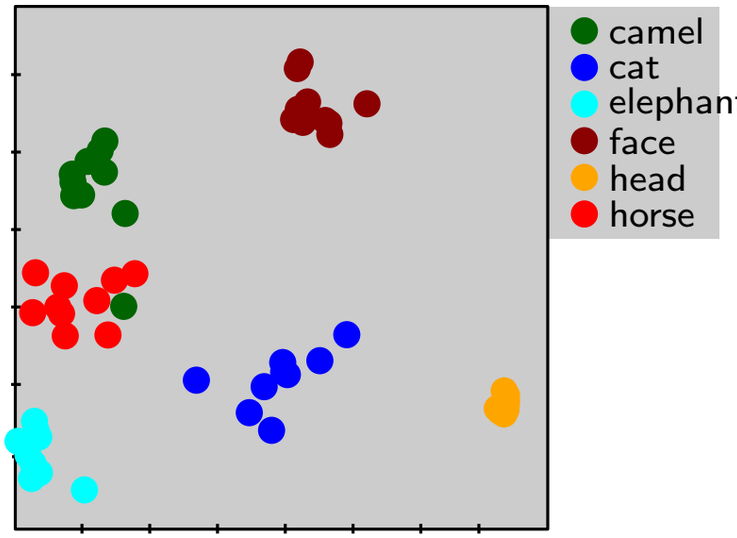
**Rem:** this a particular case of a more general theorem [C.-de Silva-Oudot 2013].

# Stability properties

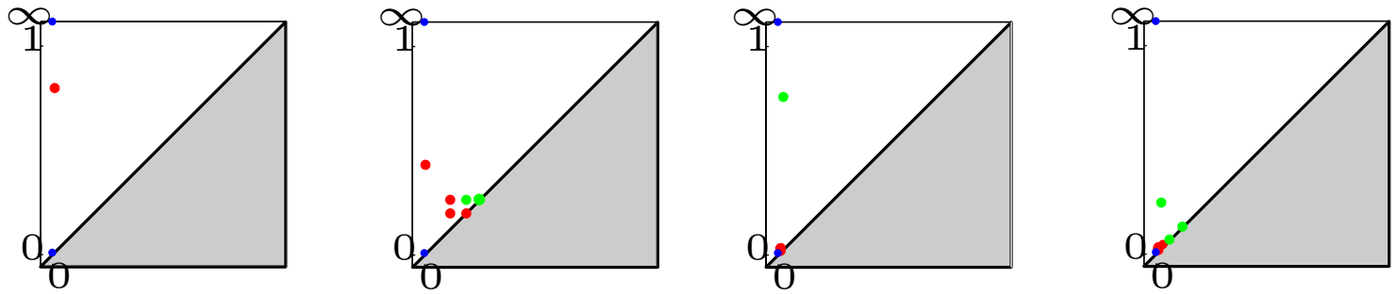
**Example:** Application to non rigid shape classification.



[C., Cohen-Steiner, Guibas, Mémoli, Oudot '09]



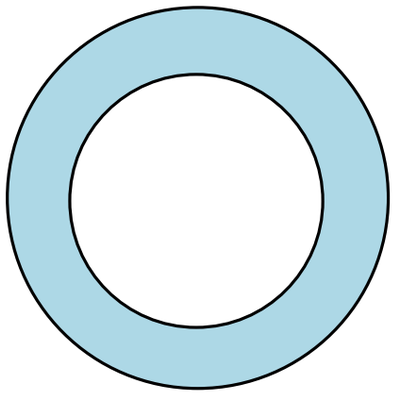

Use the metric on the space of persistence diagrams.




- Non rigid shapes in a same class are almost isometric, but computing Gromov-Hausdorff distance between shapes is extremely expensive.
- Compare diagrams of sampled shapes instead of shapes themselves.

$\rightarrow$ Other applications in image classifications, object recognition, clustering,...

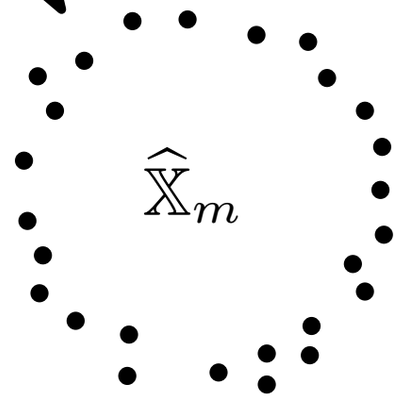
# Convergence rates of persistence diagrams



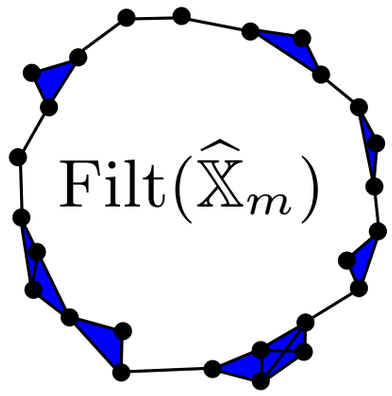
$(\mathbb{M}, \rho)$ metric space

$\mu$ a probability measure with compact support $\mathbb{X}_\mu$.

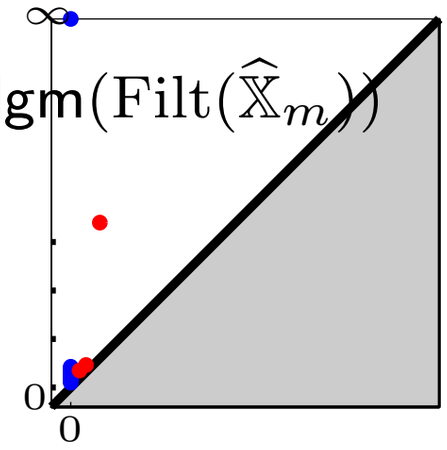
Sample $m$ points according to $\mu$.




Build topol. structure




Persistent homology

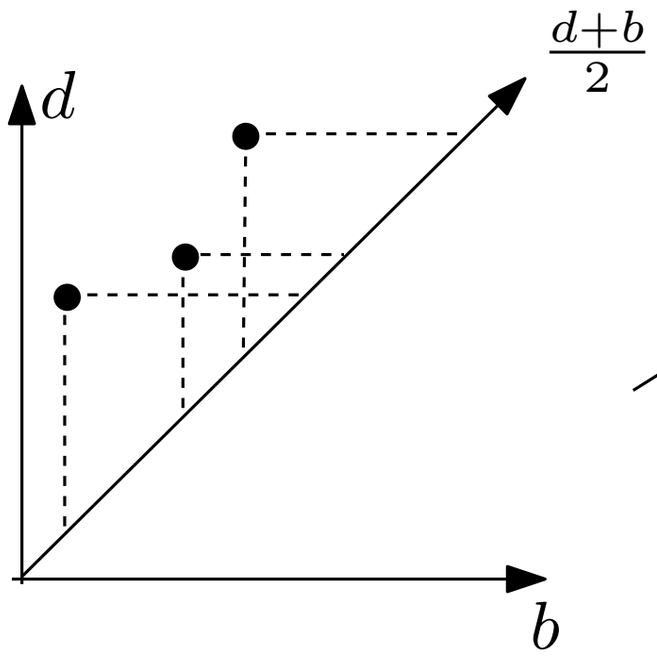


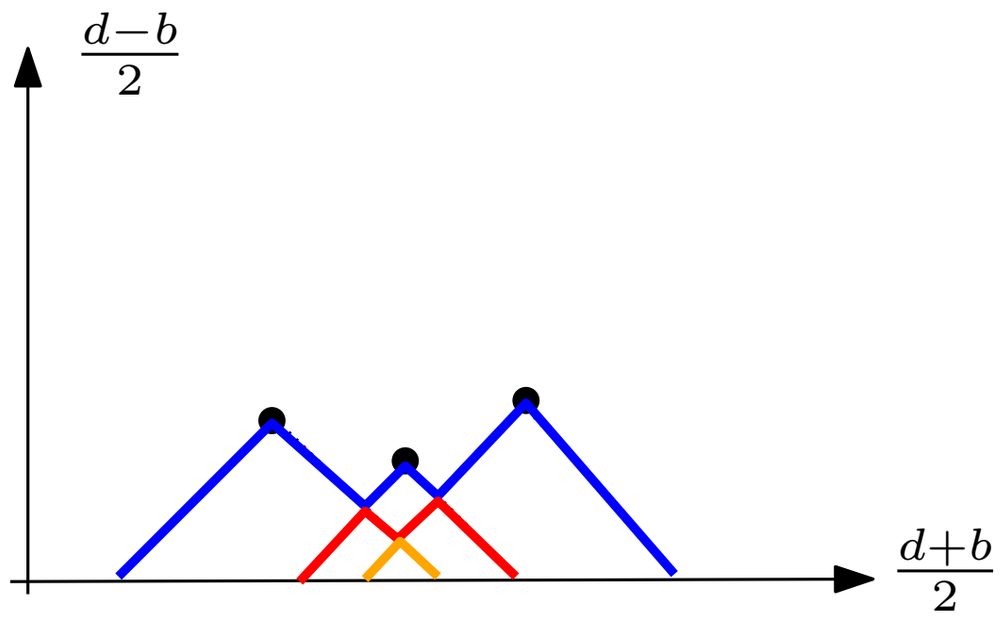

Assume that $\mu$ is $(a, b)$-standard: $\forall x \in \mathbb{X}_\mu,\ \forall r > 0,\ \mu(B(x, r)) \geq \min(ar^b, 1)$.

$$\mathbb{E}\left[\mathrm{d_b}(\mathrm{dgm}(\mathrm{Rips}(\mathbb{X}_\mu)), \mathrm{dgm}(\mathrm{Rips}(\widehat{\mathbb{X}}_m)))\right] \leq C\left(\frac{\ln m}{m}\right)^{1/b}$$

The convergence rate is optimal among $(a, b)$ standard measures, whatever the choice of the estimator of $\mathrm{dgm}(\mathrm{Rips}(\mathbb{X}_\mu))$ (minimax convergence rate).

# To do more statistics: persistence landscapes




$D = \{(\frac{d_i+b_i}{2}, \frac{d_i+b_i}{2})\} i \in I$

For $p = (\frac{b+d}{2}, \frac{d-b}{2}) \in D$,

Persistence landscape [Bubenik 2012]:

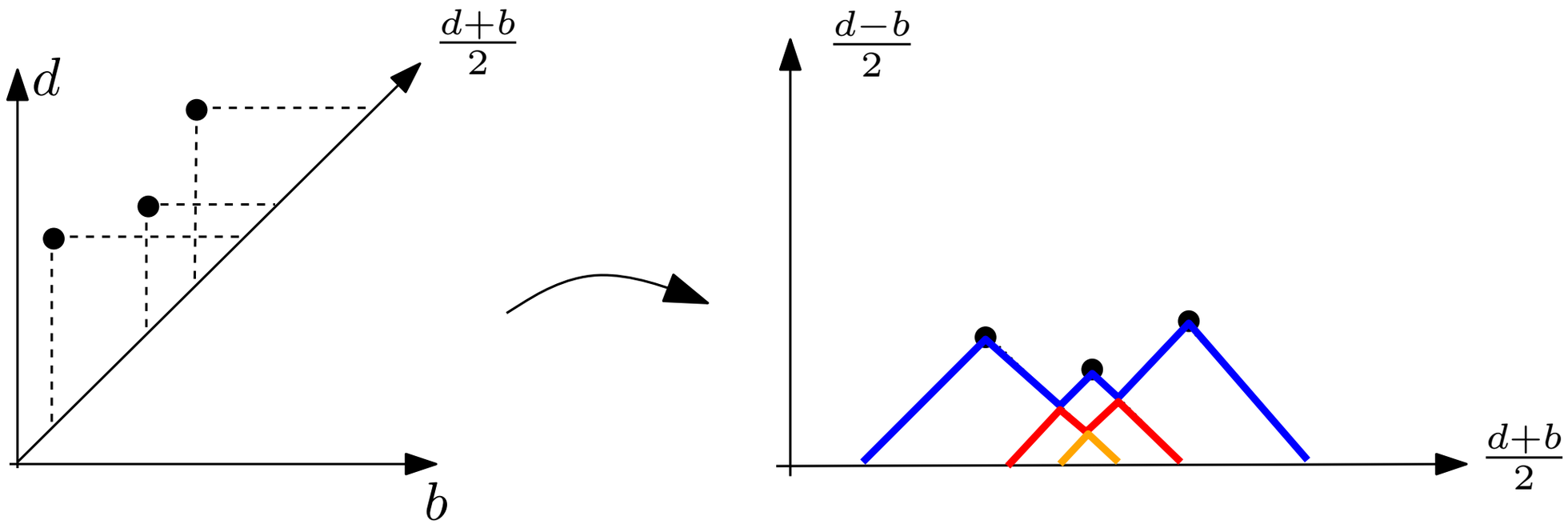
$$\Lambda_p(t) = \begin{cases} t-b & t \in [b, \frac{b+d}{2}] \\ d-t & t \in (\frac{b+d}{2}, d] \\ 0 & \text{otherwise.} \end{cases}$$

$$\lambda_D(k,t) = \underset{p \in D}{\text{kmax}}\ \Lambda_p(t), \quad t \in \mathbb{R}, k \in \mathbb{N},$$

where kmax is the $k$th largest value in the set.

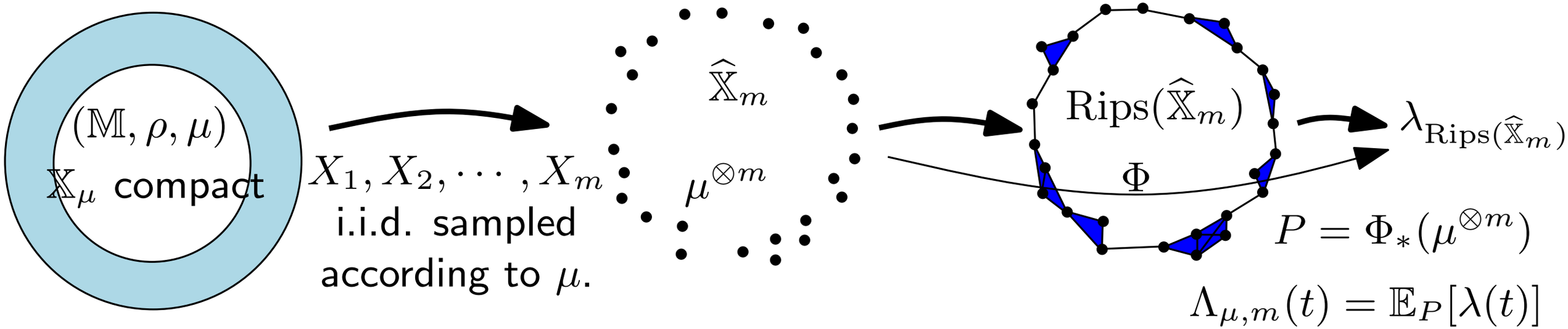
**Stability:** For any $t \in \mathbb{R}$ and any $k \in \mathbb{N}$, $|\lambda_D(k,t) - \lambda_{D'}(k,t)| \leq d_B(D, D')$.

# To do more statistics: persistence landscapes




- Persistence encoded as an element of a functional space (vector space!).
- Expectation of distribution of landscapes is well-defined and can be approximated from average of sampled landscapes.
- process point of view: convergence results and convergence rates $\rightarrow$ confidence intervals can be computed using bootstrap.

# (Sub)sampling and stability of expected landscapes




**Theorem:** Let $(\mathbb{M}, \rho)$ be a metric space and let $\mu$, $\nu$ be proba measures on $\mathbb{M}$ with compact supports. We have

$$\|\Lambda_{\mu,m} - \Lambda_{\nu,m}\|_\infty \leq m^{\frac{1}{p}} W_p(\mu, \nu)$$

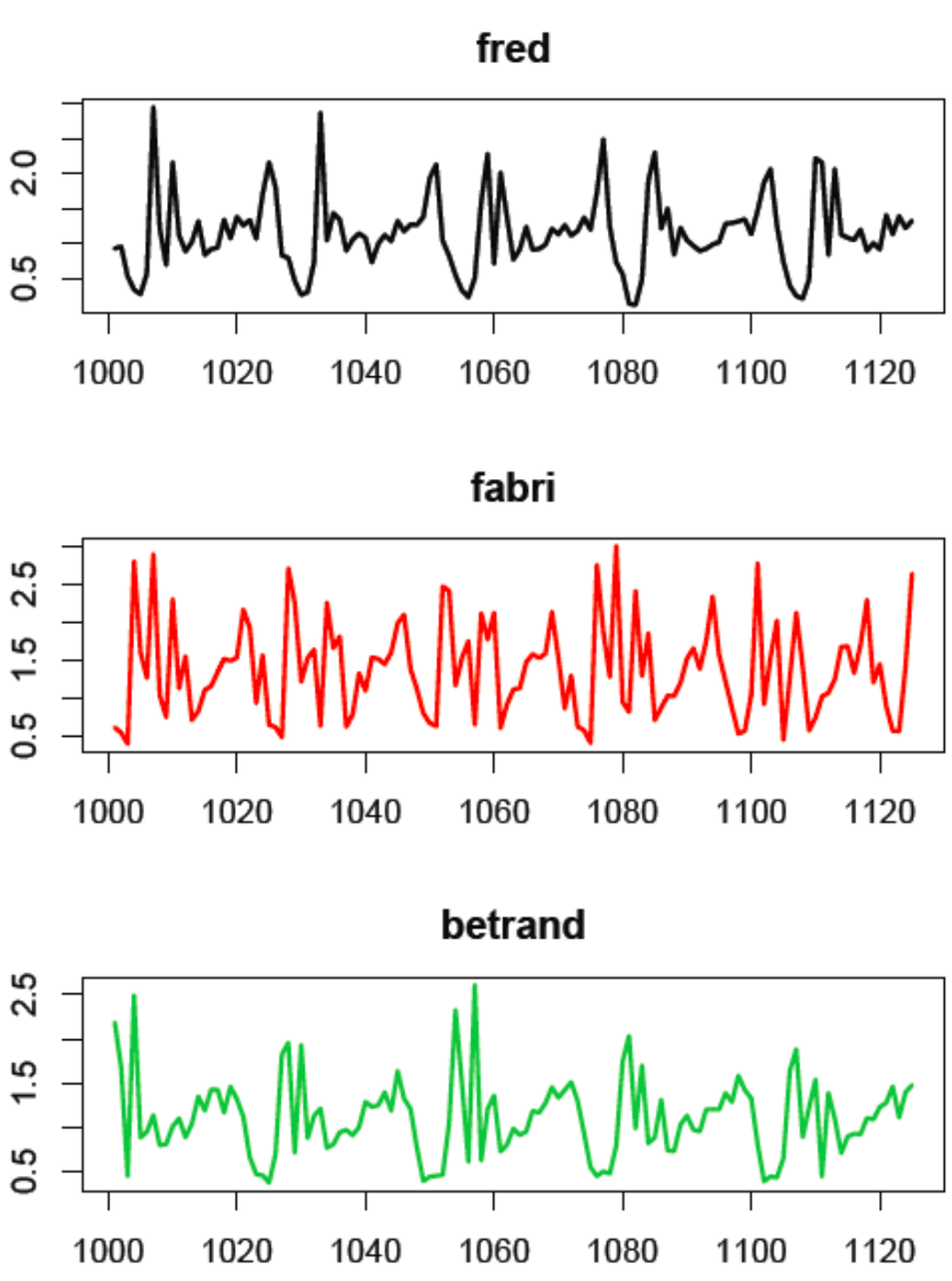
where $W_p$ denotes the Wasserstein distance with cost function $\rho(x, y)^p$.

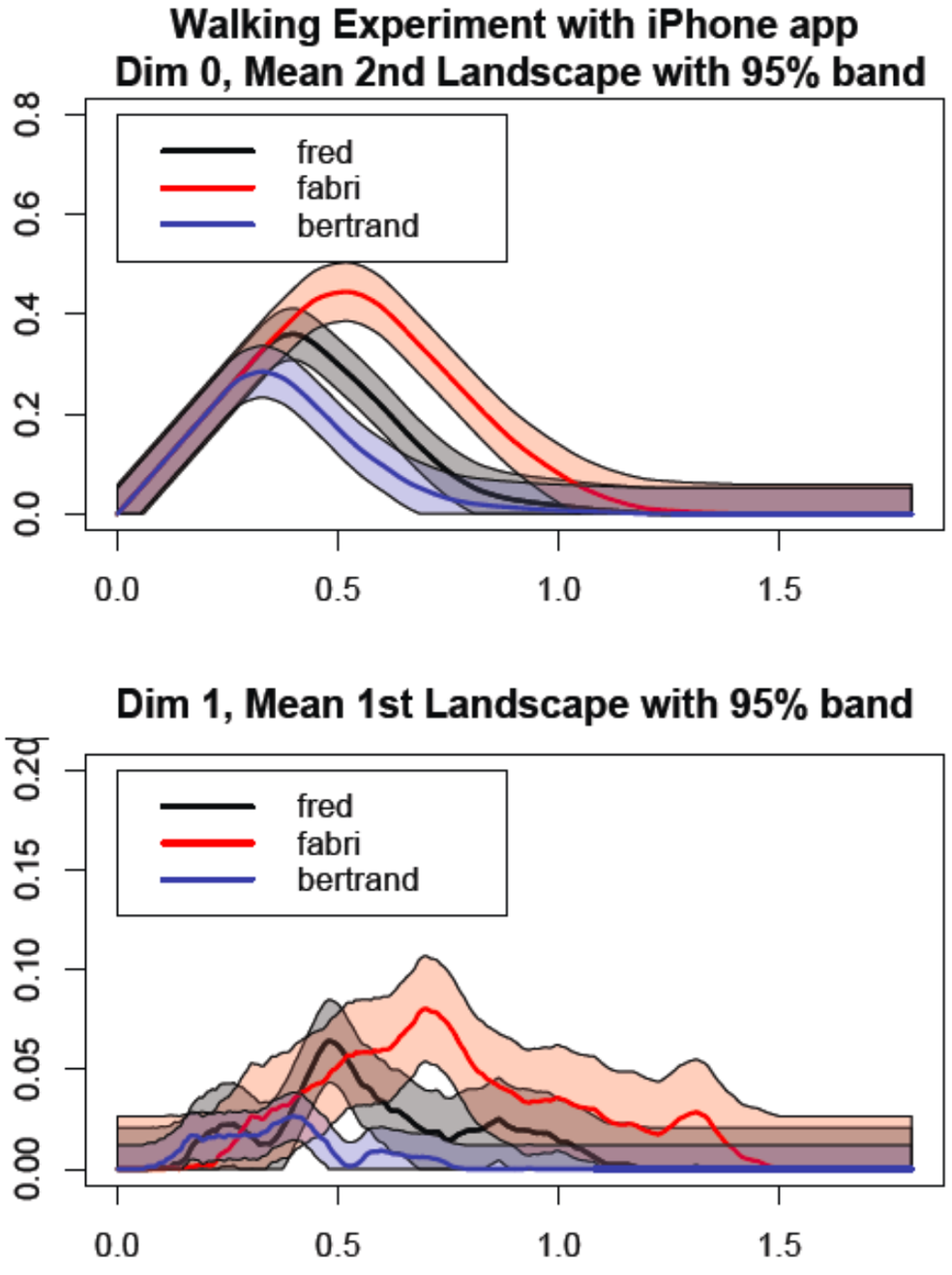
**Consequences:**

- Subsampling: efficient and easy to parallelize algorithm to infer topol. information from huge data sets.
- Robustness to outliers.
- R package (released soon) +Gudhi library: https://project.inria.fr/gudhi/software/

# (Sub)sampling and stability of expected landscapes

**(Toy) Example:** Accelerometer data from smartphone.




- spatial time series (accelerometer data from the smarphone of users).
- no registration/calibration preprocessing step needed to compare!

# Conclusion

- Persistent diagrams of geometric complexes built on top of data provide a very general, flexible way to infer relevant multiscale topological information.
- Although they live in a "non-friendly" metric space, persistence diagrams have good statistical properties.
- The convergence results are indeed more general:
  - extend to other families of filtered simplicial complexes (Čech-complexes, witness complexes,...)
  - extend to non-metric spaces endowed with a similarity measure.
- Subsampling, averaging:
  $\rightarrow$ robust topological inference under perturbations of the measure $\mu$.
  $\rightarrow$ very fast and easy to parallelize computation of topological feature for huge data.

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Thank you for your attention!

**References:**


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