Lecture 3
One-time Pad

One-Time Pad

- Basic Idea: Extend Vigenère cipher so that the key is as long as the plaintext
  - No repeat, cannot be broken by finding key length + frequency analysis
- Key is a *random string* that is at least as long as the plaintext
- Encryption is similar to Vigenère

One-Time Pad

- Key is chosen randomly
- Plaintext $X = (x_1\ x_2\ \ldots\ x_n)$
- Key $K = (k_1\ k_2\ \ldots\ k_n)$
- Ciphertext $Y = (y_1\ y_2\ \ldots\ y_n)$

- $e_k(X) = (x_1{+}k_1\ \ x_2{+}k_2\ \ldots\ \ x_n{+}k_n) \bmod m$
- $d_k(Y) = (y_1{-}\ k_1\ \ y_2{-}k_2\ \ldots\ \ y_n{-}k_n) \bmod m$

One-Time Pad

- Intuitively, it is secure …
- The key is random, so the ciphertext too will be completely random

Shannon (Information-Theoretic) Security

- Basic Idea: Ciphertext should provide no "information" about Plaintext
- We also say such a scheme has *perfect secrecy*.
- One-time pad has perfect secrecy
  - E.g., suppose that the ciphertext is "Hello", can we say any plaintext is more likely than another plaintext?

    (For example "Lucky", "Later", "Funny" … are all equally likely)
- Result due to Shannon, 1949.

Claude Elwood Shannon (1916 - 2001), an American electrical engineer and mathematician, has been called "the father of Information Theory"

Key Randomness in One-Time Pad

- One-Time Pad uses a *very* long key, what if the key is not chosen randomly, instead, texts from, e.g., a book is used.
  - this is not One-Time Pad anymore
  - this does not have perfect secrecy
  - this can be broken
- The key in One-Time Pad should never be reused.
  - If it is reused, it is Two-Time Pad, and is insecure!

Limitations of One-Time Pad

- Perfect secrecy $\Rightarrow$ key-length $\geq$ msg-length
- Difficult to use in practice

Limitations of One-Time Pad (2)

- *Example taken from «Security Engineering», Ross Anderson, 2nd edition (Wiley)*
- One-Time Pad was used in World War 2: one-time key material was printed on silk, which agents could conceal inside their clothing; whenever a key had been used, it was torn off and burnt
- Now suppose you intercepted a message from a wartime German agent which you know started with “Heil Hitler”, and the first 10 letters of cyphertext were DGTYI BWPJA
- This means that the first 10 letters of the one-time pad were *wclnb tdefj* since

- Plaintext: heilhitler
- Key: wclnbtdefj
- Ciphertext: DGTYIBWPJA

A spy’s message

Limitations of One-Time Pad (2)

- But once he has burnt the piece of silk with his key material, the spy can claim he's actually a member of the anti-Nazi underground resistance, and the message actually said «Hang Hitler». This is quite possible, as the key material could just as easily have been wggsb tdefj :

- Ciphertext: DGTYIBWPJA
- Key: wggsbtdefj
- Plaintext: hanghitler

What the spy *claimed* he said

Limitations of One-Time Pad (2)

- Now we rarely get anything for nothing in cryptology, and the price of the perfect secrecy of the one-time pad is that it fails completely to protect *message integrity.* Suppose for example that you wanted to get this spy into trouble, you could change the cyphertext to DCYTI BWPJA

• Ciphertext:	DCYTIBWPJA
• Key:	wclnbtdefj
• Plaintext:	hanghitler

Manipulating the message to entrap the spy

The Binary Version of One-Time Pad

- Plaintext space = Ciphtertext space =

  = Keyspace = $\{0,1\}^n$
- Key is chosen randomly
- For example:
  - Plaintext is 11011011
  - Key is 01101001
  - Then ciphertext is 10110010

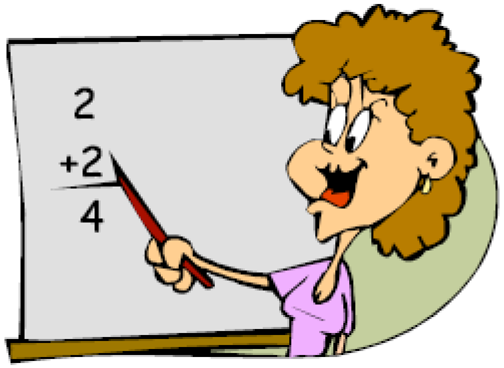
Bit Operators

- Bit AND
  - $0 \wedge 0 = 0 \quad 0 \wedge 1 = 0 \quad 1 \wedge 0 = 0 \quad 1 \wedge 1 = 1$
- Bit OR
  - $0 \vee 0 = 0 \quad 0 \vee 1 = 1 \quad 1 \vee 0 = 1 \quad 1 \vee 1 = 1$
- Addition mod 2 (also known as Bit XOR)
  - $0 \oplus 0 = 0$
  - $0 \oplus 1 = 1$
  - $1 \oplus 0 = 1$
  - $1 \oplus 1 = 0$

Unconditional Security

- The adversary has *unlimited* computational resources.
- Analysis is made by using probability theory.
- Perfect secrecy: observation of the ciphertext provides *no information* to an adversary.
- Result due to Shannon, 1949.
- *C. E. Shannon, "Communication Theory of Secrecy Systems", Bell System Technical Journal, vol.28-4, pp 656--715, 1949.*

Begin Math



Elements of Probability Theory

- A random experiment has an unpredictable outcome.
- **Definition**

  The sample space (S) of a random phenomenon is the set of all outcomes for a given experiment.
- **Definition**

  The event (E) is a subset of a sample space, an event is any collection of outcomes.

Basic Axioms of Probability

- If *E* is an event, *Pr(E) is the probability that event E* occurs, then
  - (a) $0 \le Pr(A) \le 1$ for any set ***A in S.***
  - (b) Pr(S) = 1 , where S is the sample space.
  - (c) If $E_1, E_2, \ldots E_n$ is a sequence of *mutually exclusive* events, that is $E_i \cap E_j = 0$, for all $\boldsymbol{i \neq j}$ then:

$$\Pr\left(E_1 \cup E_2 \cup \ldots \cup E_n\right) = \sum_{i=1}^{n} \Pr\left(E_i\right)$$

Probability: More Properties

- If *E* is an event and *Pr(E) is the* probability that the event E occurs then
  - *Pr(Ê) = 1 - Pr(E) where Ê is the* complimentary event of E
  - If outcomes in S are equally like, then
    Pr(E) = |E| / |S|
    (where |S| denotes the cardinality of the set S)

Random Variable

- **Definition**

  A **discrete random variable, X,** consists of a finite set X, and a probability distribution defined on X. The probability that the random variable **X** takes on the value x is denoted Pr[**X** =x]; sometimes, we will abbreviate this to Pr[x] if the random variable **X** is fixed. It must be that

$$0 \le \Pr[x] \qquad \forall x \in X$$

$$\sum_{x \in X} \Pr[x] = 1$$

Relationships between Two Random Variables

- **Definitions**

  Assume **X** and **Y** are two random variables, we define:

  - joint probability: Pr[x, y] is the probability that **X** takes value x <u>and</u> **Y** takes value y.
  - conditional probability: Pr[x|y] is the probability that **X** takes on the value x <u>given that</u> **Y** takes value y.
    - Note that joint probability can be related to conditional probability by the formula Pr[x, y] = Pr[x|y] Pr[y]
    - Interchanging x and y we have that Pr[x, y] = Pr[y|x] Pr[x]
    - This permits to obtain Bayes' Theorem
  - independent random variables: **X** and **Y** are said to be independent if Pr[x,y]=Pr[x]Pr[y], for all x ∈ **X** and all y ∈ **Y**

Elements of Probability Theory

- Find the conditional probability of event **X** given the conditional probability of event **Y** and the unconditional probabilities of events **X** and **Y**.
- **Bayes' Theorem**

  If Pr[y] > 0 then

$$\Pr[x \mid y] = \frac{\Pr[y \mid x]\Pr[x]}{\Pr[y]}$$

- **Corollary**

  X and **Y** are independent random variables if and only if Pr[x|y] = Pr[x], for all x ∈ **X** and all y ∈ **Y**.

End Math

Ciphers Modeled by Random Variables

- Consider a cipher (P, C, K, E, D). We assume that:
  1. there is an (a-priori) probability distribution on the plaintext (message) space
  2. the key space also has a probability distribution. We assume the key is chosen before one (Alice) knows what the plaintext will be, therefore **the key and the plaintext are independent random variables**
  3. The two probability distributions on P and K induce a probability distribution on C: the ciphertext is also a random variable

Example

- P = {a, b};
- Pr(a) = 1/4;   Pr(b) = 3/4

- K = {k1, k2, k3};
- Pr(k1) = 1/2;   Pr(k2) = Pr(k3) = 1/4

- C = {1, 2, 3, 4};
- $e_{k1}(a) = 1$; $e_{k1}(b) = 2$;
- $e_{k2}(a) = 2$; $e_{k2}(b) = 3$;
- $e_{k3}(a) = 3$; $e_{k3}(b) = 4$

P=Plaintext

C=Ciphertext

K=Key

Encryption Matrix

	a	b
k1	1	2
k2	2	3
k3	3	4

Example

- P = {a, b}; Pr(a) = 1/4; Pr(b) = 3/4
- K = {k1, k2, k3}; Pr(k1) = 1/2; Pr(k2) = Pr(k3) = 1/4
- C = {1, 2, 3, 4};
  - $e_{k1}(a) = 1$; $e_{k1}(b) = 2$;
  - $e_{k2}(a) = 2$; $e_{k2}(b) = 3$;
  - $e_{k3}(a) = 3$; $e_{k3}(b) = 4$;

Encryption Matrix

	a	b
k1	1	2
k2	2	3
k3	3	4

- We now compute the probability distribution of the **ciphertext:**
  - Pr(1) = Pr(k1) Pr(a) = 1/2 * 1/4 = **1/8**
  - Pr(2) = Pr(k1) P(b) + Pr(k2) Pr(a) = 1/2 * 3/4 + 1/4 *1/4 = **7/16**
  - Pr(3) = **1/4**
  - Pr(4) = **3/16**

Example

- P = {a, b}; Pr(a) = 1/4; Pr(b) = 3/4
- K = {k1, k2, k3}; Pr(k1) = 1/2; Pr(k2) = Pr(k3) = 1/4
- C = {1, 2, 3, 4};
- Distribution of the ciphertext:
  - Pr(1) = 1/8, Pr(2) = 7/16, Pr(3) = 1/4, Pr(4) = 3/16;

Encryption Matrix

	a	b
k1	1	2
k2	2	3
k3	3	4

- Now we can compute the *Conditional probability* distribution on the **Plaintext**, given that a certain ciphertext has been observed (we use Bayes)

$$\Pr[a \mid 1] = \frac{\Pr[1 \mid a]\Pr[a]}{\Pr[1]} = \frac{\frac{1}{2} \cdot \frac{1}{4}}{\frac{1}{8}} = 1$$

Pp(a|1)=1	Pp(b|1)=0
Pp(a|2)=1/7	Pp(b|2)=6/7
Pp(a|3)=1/4	**Pp(b|3)=3/4**
Pp(a|4)=0	Pp(b|4)=1

DOES THIS CRYPTOSYSTEM HAVE PERFECT SECRECY?

Perfect Secrecy

- **Definition**

  Informally, perfect secrecy means that an attacker can not obtain <u>any</u> information about the plaintext, by observing the ciphertext.

  What type of attack is this?

- **Definition**

  A cryptosystem has perfect secrecy if **$Pr[x|y] = Pr[x]$, for all $x \in P$ and $y \in C$,** where P is the set of plaintext and C is the set of ciphertext.

Perfect Secrecy

- What can I say about Pr[x|y] and Pr[x], for all x ∈ P and y ∈ C
- From Bayes' Theorem

Given

Don't know it, but can be computed

$$\Pr[x \mid y] = \frac{\Pr[x]\Pr[y \mid x]}{\Pr[y]}$$

Don't know it, but can be computed

Perfect Secrecy

- **KNOWN, Pr[x], Pr[k]**

  C(k): the set of all possible ciphertexts if key is k.

$$\Pr[y \mid x] = \sum_{k: x = d_k(y)} \Pr[k]$$

$$\Pr[y] = \sum_{k: y \in C(x)} \Pr[k]\Pr[x]$$

$$\Pr[x \mid y] = \frac{\Pr[x] \cdot \sum_{k: x = d_k(y)} \Pr[k]}{\sum_{k: y \in C(x)} \Pr[k]\Pr[x]}$$

Example

- P = {a, b}; Pr(a) = 1/4; Pr(b) = 3/4
- K = {k1, k2, k3}; Pr(k1) = 1/2; Pr(k2) = Pr(k3) = 1/4
- C = {1, 2, 3, 4};
  - $e_{k1}(a) = 1$; $e_{k1}(b) = 2$;
  - $e_{k2}(a) = 2$; $e_{k2}(b) = 3$;
  - $e_{k3}(a) = 3$; $e_{k3}(b) = 4$;

- Distribution of the ciphertext:
  - Pr(1) = Pr(k1) Pr(a) = 1/2 * 1/4 = 1/8
  - Pr(2) = Pr(k1) P(b) + Pr(k2) Pr(a) = 1/2 * 3/4 + 1/4 *1/4 = 7/16
  - Similarly: Pr(3) = 1/4; Pr(4) = 3/16;
- Conditional probability distribution of the ciphertext (we use Bayes)
  - Pr(a|1) = Pr(1|a)Pr(a)/Pr(1) = 1/2*1/4/(1/8) = 1
  - Similarly: Pr(a|2) = 1/7; Pr(a|3) = 1/4; Pr(a|4) = 0;
  - Pr(b|1) = 0; Pr(b|2) = 6/7; Pr(b|3) = 3/4; Pr(b|4) = 1

DOES THIS CRYPTOSYSTEM HAVE PERFECT SECRECY?

Names connected with OTP

- Co-inventors of One-time-pad
  - **Joseph Mauborgne** (1881-1971) became a Major General in the United States Army
  - **Gilbert Sandford Vernam** (1890 - 1960) was AT&T Bell Labs engineer
- Security of OTP
  - **Claude Elwood Shannon** (1916 - 2001), American electronic engineer and mathematician, was "the father of information theory.

Perfect secrecy of One-Time Pad

One-Time Pad has Perfect Secrecy

- P = C = K = $\{0,1\}^n$, the key is chosen randomly, the key used only once per message
- Proof: We need to show that for <u>any</u> probability of the plaintext, $\forall x\ \forall y$, Pr [x|y] = Pr[x]

$$\Pr[x \mid y] = \frac{\Pr[x]\Pr[y \mid x]}{\Pr[y]} =$$

$$= \frac{\Pr[x]\Pr[k]}{\sum_{x \in X} \Pr[x]\Pr[k]} = \frac{\Pr[x]\, {}^{1}\!/_{2^n}}{\sum_{x \in X} \Pr[x]\, {}^{1}\!/_{2^n}} = \frac{\Pr[x]}{\sum_{x \in X} \Pr[x]} = \Pr[x]$$

Modern Cryptography

- One-time pad requires the length of the key to be the length of the plaintext and the key to be used only once. Difficult to manage.
- Alternative: design cryptosystems where a key is used more than once.
- What about the attacker? Resource constrained, make it infeasible for adversary to break the cipher.

Stream Ciphers

- In OTP, a key is described by a random bit string of length n
- Stream ciphers:
- Idea: replace "rand" by "pseudo rand"
- Use Pseudo Random Number Generator (PRNG)
- PRNG: $\{0, 1\}^s \rightarrow \{0, 1\}^n$
  - expand a short (e.g., 128-bit) random seed into a long (e.g., 10^6 bit) string that "looks random"
  - Secret key is the seed
  - $E_{seed}[M] = M \oplus PRNG(seed)$

Properties of Stream Ciphers

- Does not have perfect secrecy
  - security depends on PRNG
- PRNG must be "unpredictable"
  - given consecutive sequence of bits output (but not seed), next bit must be hard to predict
- Typical stream ciphers are very fast
- Used in many places, often incorrectly
  - SSL(Rivest Cipher 4, or RC4), DVD (LFSR), WEP (RC4), etc.

Fundamental Weaknesses of Stream Ciphers

- If the same key-stream is used twice ever, then easy to break.
- Highly malleable
  - easy to change ciphertext so that plaintext changes in predictable, e.g., flip bits
- Weaknesses exist even if the PRNG is strong