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Approximations of shape metrics and application to shape warping and empirical shape statistics

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Abstract: This article proposes a framework for dealing with several problems related to the analysis of shapes. Two related such problems are the definition of the relevant set of shapes and that of defining a metric on it. Following a recent research monograph by Delfour and Zolesio [11], we consider the characteristic functions of the subsets of $\mathbb{R}^2$ and their distance functions. The L^2 norm of the difference of characteristic functions, the L^∞ and the $W^{1,2}$ norms of the difference of distance functions define interesting topologies, in particular that induced by the well-known Hausdorff distance. Because of practical considerations arising from the fact that we deal with image shapes defined on finite grids of pixels we restrict our attention to subsets of $\mathbb{R}^2$ of positive reach in the sense of Federer [16], with smooth boundaries of bounded curvature. For this particular set of shapes we show that the three previous topologies are equivalent. The next problem we consider is that of warping a shape onto another by infinitesimal gradient descent, minimizing the corresponding distance. Because the distance function involves an *inf*, it is not differentiable with respect to the shape. We propose a family of smooth approximations of the distance function which are continuous with respect to the Hausdorff topology, and hence with respect to the other two topologies. We compute the corresponding Gâteaux derivatives. They define deformation flows that can be used to warp a shape onto another by solving an initial value problem. We show several examples of this warping and prove properties of our approximations that relate to the existence of local minima. We then use this tool to produce computational definitions of the empirical mean and covariance of a set of shape examples. They yield an analog of the notion of principal modes of variation. We illustrate them on a variety of examples.

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Key-words: Shape metrics, characteristic functions, distance functions, deformation flows, lower semi continuous envelope, shape warping, empirical mean shape, empirical covariance operator, principal modes of variation.

Approximations de métriques de formes et applications à la déformation et à l'estimation de statistiques empiriques

Résumé : Nous proposons dans cet article un cadre pour traiter un certain nombre de problèmes liés à l'analyse de formes. Nous commençons par ceux de la définition d'un espace de formes et d'une métrique sur celui-ci. À partir d'une monographie de recherche récenter due à Delfour et Zolésio [11], nous considérons les fonctions caractéristiques des sous-ensembles de $\mathbb{R}^2$ et leurs fonctions distance. La norme L^2 de la différence de deux fonctions caractéristiques, les normes L^∞ et $W^{1,2}$ de la différence de deux fonctions distance définissent des topologies intéressantes, en particulier celle induite par la célèbre distance de Hausdorff. Des considérations d'ordre pratique liées au fait que nous nous intéressons à des formes image définies sur des grilles finies de pixels, nous concentrons notre attention sur des sous-ensembles de $\mathbb{R}^2$ dits de Federer [16], avec une frontière lisse et de courbure bornée. Nous démontrons que pour cet espace de forme les trois topologies précédentes sont équivalentes. Le problème suivant est celui de la déformation d'une forme en une autre forme par une descente de gradient infinitésimale, en minimisant l'une des trois distances. La présence d'un *inf* dans la définition de la fonction distance rend celle-ci non-différentiable par rapport à la forme. Nous proposons donc une famille d'approximations régulières de la fonction distance, continues par rapport à la distance de Hausdorff, et donc par rapport aux deux autres distances considérées. Nous calculons les dérivées de Gâteaux correspondantes. Celle-ci définissent des champs de déformation avec lesquels on peut déformer une forme en une autre en résolvant un problème de Cauchy. Nous illustrons cette idée à l'aide de plusieurs exemples et prouvons des propriétés de notre approximation relatives à l'existence de minima locaux. Nous utilisons ensuite cet outil pour définir de manière computationnelle la moyenne empirique et la covariance d'un ensemble de formes. Ceci conduit à une notion analogue à celle des modes principaux de déformation que nous illustrons sur des exemples.

Mots-clés : Métriques de formes, fonctions caractéristiques d'ensembles, fonctions distance, flots de déformation, enveloppe semi continue inférieurement, déformation de formes, moyenne empirique de formes, operateur empirique de covariance, modes de déformations.

1 Introduction

Learning shape models from examples, using them to recognize new instances of the same class of shapes are fascinating problems that have attracted the attention of many scientists for many years. Central to this problem is the notion of a random shape which in itself has occupied people for decades. Frechet [19] is probably one of the first mathematicians to develop some interest for the analysis of random shapes, i.e. curves. He was followed by Matheron [33] who founded with Serra the french school of mathematical morphology and by David Kendall [24, 26, 27] and his colleagues, e.g. Small [42]. In addition, and independently, a rich body of theory and practice for the statistical analysis of shapes has been developed by Bookstein [4], Dryden and Mardia [13], Carne [5], Cootes, Taylor and colleagues [8]. Except for the mostly theoretical work of Frechet and Matheron, the tools developed by these authors are very much tied to the point-wise representation of the shapes they study: objects are represented by a finite number of salient points or landmarks. This is an important difference with our work which deals explicitely with curves as such, independently of their sampling or even parametrization.

In effect, our work bears more resemblance with that of several other authors. Like in Grenander's theory of patterns [21, 22], we consider shapes as points of an infinite dimensional manifold but we do not model the variations of the shapes by the action of Lie groups on this manifold, except in the case of such finite-dimensional Lie groups as rigid displacements (translation and rotation) or affine transformations (including scaling). For infinite dimensional groups such as diffeomorphisms [14, 49] which smoothly change the objects' shapes previous authors have been dependent upon the choice of parametrizations and origins of coordinates [50, 51, 48, 47, 34, 23]. For them, warping a shape onto another requires the construction of families of diffeomorphisms that use these parametrizations. Our approach, based upon the use of the distance functions, does not require the arbitrary choice of parametrizations and origins. From our viewpoint this is already very nice in two dimensions but becomes even nicer in three dimensions and higher where finding parametrizations and tracking origins of coordinates can be a real problem: this is not required in our case. Another piece of related work is that of Soatto and Yezzi [43] who tackle the problem of jointly extracting and characterizing the motion of a shape and its deformation. In order to do this they find inspiration in the above work on the use of diffeomorphisms and propose the use of a distance between shapes (based on the set-symmetric difference described in section 2.2). This distance poses a number of problems that we address in the same section where we propose two other distances which we believe to be more suitable.

Some of these authors have also tried to build a Riemannian structure on the set of shapes, i.e. to go from an infinitesimal metric structure to a global one. The infinitesimal structure is defined by an inner product in the tangent space (the set of normal deformation fields) and has to vary continuously from point to point, i.e. from shape to shape. The Riemannian metric is then used to compute geodesic curves between two shapes: these geodesics define a way of warping either shape onto the other. This is dealt with in the work of Trouvé and Younes [50, 51, 49, 48, 47, 52] and, more recently, in the work of Klassen and Srivastava [29], again at the cost of working with parametrizations. The problem with these approaches, beside that of having to deal with parametrizations of the shapes, is that there exist global metric structures on the set of shapes (see section 2.2) which are useful and relevant to the problem of the comparison of shapes but that do not derive from an infinitesimal structure. Our approach can be seen as taking the problem from exactly the opposite viewpoint from the previous one: we start with a global metric on the set of shapes and build smooth functions (in effect smooth approximations of these metrics) that are dissimilarity measures, or energy functions; we then minimize these functions using techniques of the calculus of variation by computing their gradient and performing infinitesimal gradient descent: this minimization defines another way of warping either shape onto the other. In this endeavour we build on the seminal work of Delfour and Zolesio who have introduced new families of sets, complete metric topologies, and compactness theorems. This work is now available in book form [11]. The book provides a fairly broad coverage and a synthetic treatment of the field along with many new important results, examples, and constructions which have not been published elsewhere. Its full impact on image processing and robotics has yet to be fully assessed.

In this article we also revisit the problem of computing empirical statistics on sets of 2D shapes and propose a new approach by combining several notions such as topologies on set of shapes, calculus of variations, and some measure theory. Section 2 sets the stage and introduces some notations and tools. In particular in section 2.2 we discuss three of the main topologies that can be defined on sets of shapes and motivate the choice of two of them. In section 3 we introduce the particular set of shapes we work with in this paper, show that it has nice compactness properties and that the three topologies defined in the previous section are in fact equivalent on this set of shapes. In section 4 we introduce one of the basic tools we use for computing shape statistics, i.e., given a measure of the dissimilarity between two shapes, the curve gradient flow that is used to deform a shape into another. Having motivated the introduction of the measures of dissimilarity, we proceed in section 5 with the construction of classes of such measures which are based on the idea of approximating some of the shape distances that have been presented in section 2.2; we also prove the continuity of our approximations with respect to these distances and compute the corresponding curve gradient flows. This being settled, we are in a position to warp any given shape onto another by solving the Partial Differential Equation (PDE) attached to the particular curve gradient flow. This problem is studied in section 6 where examples are also presented. In section 7.1 we use all these tools to define a mean-shape and to provide algorithms for computing it from sample shape examples. In section 7.2, we extend the notion of covariance matrix of a set of samples to that of a covariance *operator* of a set of sample shape examples from which the notion of principal modes of variation follows naturally. We discuss some details of our implementation of these algorithms in section 8 and conclude in section 9.

2 Shapes and shape topologies

To define fully the notion of a shape is beyond the scope of this article in which we use a limited, i.e purely *geometric*, definition. It could be argued that the perceptual shape of an object also depends upon the distributiion of illumination, the reflectance and texture of its surface; these aspects are not discussed in this paper. In our context we define a shape to be a measurable subset of $\mathbb{R}^2$. Since we are driven by image applications we also assume that all our shapes are contained in a hold-all open bounded subset of $\mathbb{R}^2$ which we denote by D. The reader can think of D as the "image".

In the next section we will restrict our interest to a more limited set of shapes but presently this is sufficient to allow us to introduce some methods for representing shapes.

2.1 Definitions

Since, as mentioned in the introduction, we want to be independent of any particular parametrisation of the shape, we use two main ingredients, the *characteristic function* of a shape Ω

$$\chi_\Omega(x) = 1 \quad \text{if} \quad x \in \Omega \quad \text{and} \quad 0 \quad \text{if} \quad x \notin \Omega,$$

and the *distance function* to a shape Ω

$$d_\Omega(x) = \inf_{y\in\Omega} |y-x| = \inf_{y\in\Omega} d(x,y) \quad \text{if} \quad \Omega \neq \emptyset \quad \text{and} \quad +\infty \quad \text{if} \quad \Omega = \emptyset.$$

Note the important property [11, chapter 4, theorem 2.1]:

$$d_{\Omega_1} = d_{\Omega_2} \Longleftrightarrow \overline{\Omega}_1 = \overline{\Omega}_2 \tag{1}$$

Also of interest is the distance function to the complement of the shape, $d_{\complement\Omega}$ and the distance function to its boundary, $d_{\partial\Omega}$. In the case where $\Omega = \partial\Omega$ and Ω is closed, we have

$$d_\Omega = d_{\partial\Omega} \quad d_{\complement\Omega} = 0$$

We note $C_d(D)$ the set of distance functions of nonempty sets of D. Similarly, we note $C_d^c(D)$ the set of distance functions to the complements of open subsets of D (for technical reasons which are irrelevant here, it is sufficient to consider open sets).

Another function of great interest is the *oriented distance function* b_Ω defined as

$$b_\Omega = d_\Omega - d_{\complement\Omega}$$

Note that for closed sets such that $\Omega = \partial\Omega$, one has $b_\Omega = d_\Omega$.

We briefly recall some well known results about these two functions. The integral of the characteristic function is equal to the measure (area) $m(\Omega)$ of Ω:

$$\int_\Omega \chi_\Omega(x)\, dx = m(\Omega)$$

Note that this integral does not change if we add to or subtract from Ω a measurable set of Lebesgue measure 0 (also called a negligible set).

Concerning the distance functions, they are continuous, in effect Lipschitz continuous with a Lipschitz constant equal to 1 [9, 11]:

$$|d_\Omega(x) - d_\Omega(y)| \leq |x-y| \quad \forall x,\, y,\, \in D.$$

Thanks to the Rademacher theorem [15], this implies that d_Ω is differentiable almost everywhere in D, i.e. outside of a negligible set, and that the magnitude of its gradient, when it exists, is less than or equal to 1

$$|\nabla d_\Omega(x)| \leq 1 \quad \text{a.e.}.$$

The same is true of $d_{\complement\Omega}$ and b_Ω (if $\partial\Omega \neq \emptyset$ for the second), [11, Chapter 5, theorem 8.1].

Closely related to the various distance functions (more precisely to their gradients) are the projections associated with $\overline{\Omega}$ and $\overline{\complement\Omega}$. These are also related to the notion of skeleton. We recall some definitions. The first one is adapted from [11, Chapter 4 definition 3.1]:

Definition 1 (Projections and skeletons).

- *Given $\Omega \subset D$, $\Omega \neq \emptyset$ (resp. $\complement\Omega \neq \emptyset$), the set of projections of $x \in D$ on Ω (resp. on $\complement\Omega$) is given by*

$$\Pi_\Omega(x) \stackrel{def}{=} \{p \in \overline{\Omega} : |p-x| = d_\Omega(x)\}$$
$$(resp.\, \Pi_{\complement\Omega}(x) \stackrel{def}{=} \{p \in \overline{\complement\Omega} : |p-x| = d_{\complement\Omega}(x)\})$$

  The elements of $\Pi_\Omega(x)$ (resp. $\Pi_{\complement\Omega}(x)$) are called projections onto $\overline{\Omega}$ (resp. $\overline{\complement\Omega}$).

- *Given $\Omega \subset D$, $\Omega \neq \emptyset$ (resp. $\complement\Omega \neq \emptyset$), the set of points where the projection on Ω (resp. $\complement\Omega$) is not unique is called the exterior (resp. interior) skeleton $Sk_{ext}(\Omega)$ (resp. $Sk_{int}(\Omega)$). We define $Sk(\Omega) = Sk_{ext}(\Omega) \cup Sk_{int}(\Omega)$.*

The following properties of the skeletons can be found e.g. in [11, Chapter 4, theorems 3.1 and 3.2]

Proposition 1. *The exterior (resp. interior) skeleton is exactly the subset of $\complement\overline{\Omega}$ (resp. of $int(\Omega)$) where the function d_Ω (resp. $d_{\complement\Omega}$) is not differentiable. Moreover the exterior and interior skeletons and the boundary $\partial\Omega$ is exactly the subset of D where $d_{\partial\Omega}$ is not differentiable.*

At each x of $\complement\Omega \backslash \mathrm{Sk}_{\mathrm{ext}}(\Omega)$, the gradient of the distance function $d_{\partial\Omega}$ is well-defined, of unit norm, and points away from the projection $y = \Pi_\Omega(x)$ of x onto Ω, see figure 1. Similar considerations apply to the case where $x \in \Omega$.

We introduce an additional definition that will be useful in the sequel.

Definition 2. *Given $\Omega \subset D$, $\Omega \neq \emptyset$, and a real number $h > 0$, the h-tubular neighbourhood of Ω is defined as*

$$U_h(\Omega) \stackrel{def}{=} \{y \in D : d_\Omega(y) < h\}$$

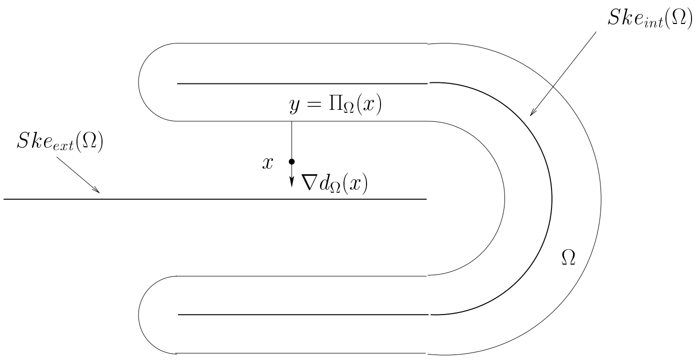

Figure 1: An example of skeletons.

2.2 Some shape topologies

The next question we want to address is that of the definition of the similarity between two shapes. This question of similarity is closely connected to that of metrics on sets of shapes which in turn touches that of what is known as shape topologies. We now briefly review three main similarity measures between shapes which turn out to define three distances.

2.2.1 Characteristic functions

The similarity measure we are about to define is based upon the characteristic functions of the two shapes we want to compare. We denote by $X(D)$ the set of characteristic functions of measurable subsets of D.

Given two such sets Ω_1 and Ω_2, we define their distance

$$\rho_2(\Omega_1, \Omega_2) = \|\chi_{\Omega_1} - \chi_{\Omega_2}\|_{L^2} = \left(\int_D \left(\chi_{\Omega_1}(x) - \chi_{\Omega_2}(x)\right)^2 \, dx\right)^{1/2}$$

This definition also shows that this measure does not "see" differences between two shapes that are of measure 0 (see figure 2 adapted from [11, Chapter 3, Figure 3.1]) since the integral does not change if we modify the values of χ_{Ω_1} or χ_{Ω_2} over negligible sets. In other words, this is not a distance between the two shapes Ω_1 and Ω_2 but between their equivalence classes $[\Omega_1]_m$ and $[\Omega_2]_m$ of measurable sets. Given a measurable subset Ω of D, we define its equivalence class $[\Omega]_m$ as $[\Omega]_m = \{\Omega' | \Omega' \text{ is measurable and } \Omega \,\Delta\, \Omega' \text{ is negligible}\}$, where $\Omega \,\Delta\, \Omega'$ is the symmetric difference

$$\Omega \Delta \Omega' = \complement_\Omega \Omega' \cup \complement_{\Omega'} \Omega$$

The proof that this defines a distance follows from the fact that the L^2 norm defines a distance over the set of equivalence classes of square integrable functions (see e.g. [39, 15]).

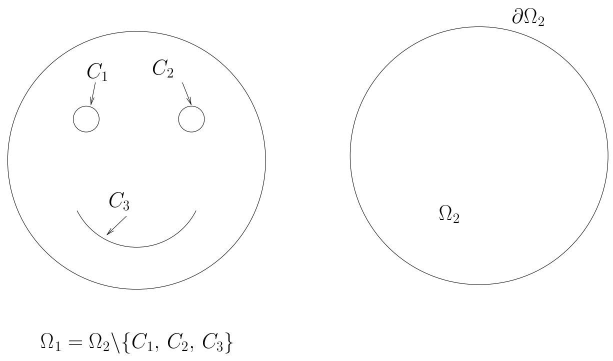

Figure 2: Two shapes whose distance ρ_2 is equal to 0; Ω_1 is obtained by removing from Ω_2 the three curves C_1, C_2, C_3: $\rho_2(\Omega_1, \Omega_2) = 0$

This is nice and one has even more ([11, Chapter 3, Theorem 2.1]: the set $X(D)$ is closed and bounded in $L^2(D)$ and $\rho_2(\cdot, \cdot)$ defines a complete metric structure on the set of equivalence classes of measurable subsets of D. Note that ρ_2 is closely related to the symmetric difference:

$$\rho_2(\Omega_1, \Omega_2) \;=\; m(\Omega_1 \, \Delta \, \Omega_2)^{\frac{1}{2}}$$

The completeness is important in applications: any Cauchy sequence of characteristic functions $\{\chi_{\Omega_n}\}$ converges for this distance to a characteristic function χ_Ω of a limit set Ω. Unfortunately in applications not all sequences are Cauchy sequences, for example the minimizing sequences of the energy functions defined in section 5, and one often requires more, i.e. that any sequence of characteristic functions contains a subsequence that converges to a characteristic function. This stronger property, called *compactness*, *is not* satisfied by $X(D)$ (see [11, Chapter 3]).

2.2.2 Distance functions

We therefore turn ourselves toward a different similarity measure which is based upon the distance function to a shape. As in the case of characteristic functions, we define equivalent sets and say that two subsets Ω_1 and Ω_2 of D are equivalent iff $\overline{\Omega}_1 = \overline{\Omega}_2$. We note $[\Omega]_d$ the corresponding equivalence class of Ω. Let $\mathcal{T}(D)$ be the set of these equivalence classes. The application

$$[\Omega]_d \to d_\Omega \quad \mathcal{T}(D) \to C_d(D) \subset C(\overline{D})$$

is injective according to (1). We can therefore identify the set $C_d(D)$ of distance functions to sets of D with the just defined set of equivalence classes of sets. Since $C_d(D)$ is a subset of the set $C(\overline{D})$ of continuous functions on $\overline{D}$, a Banach space[1] when endowed with the norm

$$\|f\|_{C(D)} = \sup_{x \in D} |f(x)|,$$

it can be shown (e.g. [11]), that the similarity measure

$$\rho([\Omega_1]_d, [\Omega_2]_d) = \|d_{\Omega_1} - d_{\Omega_2}\|_{C(D)} = \sup_{x \in D} |d_{\Omega_1}(x) - d_{\Omega_2}(x)|, \tag{2}$$

is a distance on the set of equivalence classes of sets which induces on this set a complete metric. Moreover, because we have assumed D bounded, the corresponding topology is identical to the one induced by the well-known Hausdorff metric (see [33, 40, 11])

$$\rho_H([\Omega_1]_d, [\Omega_2]_d) = \max\left\{ \sup_{x \in \Omega_2} d_{\Omega_1}(x), \sup_{x \in \Omega_1} d_{\Omega_2}(x) \right\} \tag{3}$$

In fact we have even more than the identity of the two topologies, see [11, Chapter 4, Theorem 2.2]:

Proposition 2. *If the hold-all set D is bounded $\rho = \rho_H$.*

An important improvement with respect to the situation in the previous section is the (see [11, Chapter 4, Theorem 2.2])

Theorem 3. *The set $C_d(D)$ is compact in the set $C(\overline{D})$ for the topology defined by the Hausdorff distance.*

In particular, from any sequence $\{d_{\Omega_n}\}$ of distance functions to sets Ω_n one can extract a sequence converging toward the distance function d_Ω to a subset Ω of D.

It would appear that we have reached an interesting stage and that the Hausdorff distance is what we want to measure shape similarities. Unfortunately this is not so because the convergence of areas and perimeters is lost in the Hausdorff metric, as shown in the following example taken from [11, Chapter 4, Example 4.1 and Figure 4.3]

Consider the sequence $\{\Omega_n\}$ of sets in the open square $]-1,2[^2$:

$$\Omega_n = \{(x,y) \in D : \frac{2k}{2n} \leq x \leq \frac{2k+1}{2n}, \, 0 \leq k < n\}$$

Figure 3 shows the sets Ω_4 and Ω_8. This defines n vertical stripes of equal width $1/2n$ each distant of $1/2n$. It is easy to verify that, for all $n \geq 1$, $m(\Omega_n) = 1/2$ and $|\partial\Omega_n| = 2n+1$. Moreover, if S is the unit square $[0,1]^2$, for all $x \in S$, $d_{\Omega_n}(x) \leq 1/4n$, hence $d_{\Omega_n} \to d_S$ in $C(\overline{D})$. The sequence $\{\Omega_n\}$ converges to S for the Hausdorff distance but since $m(\overline{\Omega}_n) = m(\Omega_n) = 1/2 \nrightarrow 1 = m(S)$, $\chi_{\Omega_n} \nrightarrow \chi_S$ in $L^2(D)$ and hence we do not have convergence for the ρ_2 topology. Note also that $|\partial\Omega_n| = 2n+1 \nrightarrow |\partial S| = 4$.

[1] A Banach space is a complete normed vector space.

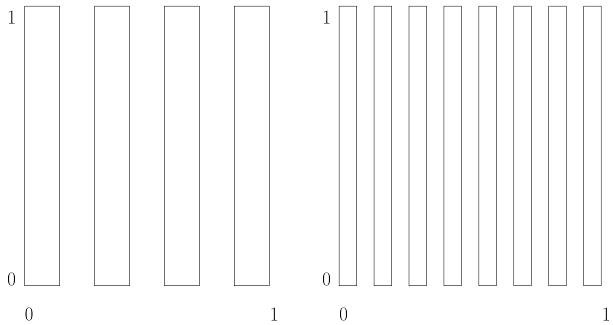

Figure 3: Two shapes in the sequence $\{\Omega_n\}$, see text: (left) Ω_4 and (right), Ω_8.

2.2.3 Distance functions and their gradients

In order to recover continuity of the area one can proceed as follows. If we recall that the gradient of a distance function is of magnitude equal to 1 except on a subset of measure 0 of D, one concludes that it is square integrable on D. Hence the distance functions and their gradients are square-integrable, they belong to the Sobolev space $W^{1,2}(D)$, a Banach space for the vector norm

$$\|f-g\|_{W^{1,2}(D)} = \|f-g\|_{L^2(D)} + \|\nabla f - \nabla g\|_{\mathbf{L}^2(D)},$$

where $\mathbf{L}^2(D) = L^2(D) \times L^2(D)$. This defines a similarity measure for two shapes

$$\rho_D([\Omega_1]_d, [\Omega_2]_d) \;=\; \|d_{\Omega_1} - d_{\Omega_2}\|_{W^{1,2}(D)}$$

which turns out to define a complete metric structure on $\mathcal{T}(D)$. The corresponding topology is called the $W^{1,2}$-topology. For this metric, the set $C_d(D)$ of distance functions is closed in $W^{1,2}(D)$, and the mapping

$$d_\Omega \to \chi_{\overline{\Omega}} = 1 - |\nabla d_\Omega| : C_d(D) \subset W^{1,2}(D) \to L^2(D)$$

is "Lipschitz continuous":

$$(4) \qquad \|\chi_{\overline{\Omega}_1} - \chi_{\overline{\Omega}_2}\|_{L^2(D)} \leq \|\nabla d_{\Omega_1} - \nabla d_{\Omega_2}\|_{\mathbf{L}^2(D)} \leq \|d_{\Omega_1} - d_{\Omega_2}\|_{W^{1,2}(D)},$$

which indeed shows that areas are continuous for the $W^{1,2}$-topology, see [11, Chapter 4, Theorem 4.1].

$C_d(D)$ is not compact for this topology but a subset of it of great practical interest is, see section 3.

3 The set $\mathcal{S}$ of all shapes and its properties

We now have all the necessary ingredients to be more precise in the definition of shapes.

3.1 The set of all shapes

We restrict ourselves to sets of D with compact boundary and consider three different sets of shapes. The first one is adapted from [11, Chapter 4, definition 5.1]:

Definition 4 (Set $\mathcal{DZ}$ of sets of bounded curvature). *The set $\mathcal{DZ}$ of sets of bounded curvature contains those subsets Ω of $\overline{D}$, Ω, $\complement\Omega \neq \emptyset$ such that ∇d_Ω and $\nabla d_{\complement\Omega}$ are in $BV(D)^2$, where $BV(D)$ is the set of functions of bounded variations.*

This is a large set (too large for our applications) which we use as a "frame of reference". $\mathcal{DZ}$ was introduced by Delfour and Zolésio [9, 10] and contains the sets $\mathcal{F}$ and $\mathcal{C}_2$ introduced below. For technical reasons related to compactness properties (see section 3.2) we consider the following subset of $\mathcal{DZ}$.

Definition 5 (Set $\mathcal{DZ}_0$). *The set $\mathcal{DZ}_0$ is the subset of $\mathcal{DZ}$ such that there exists $c_0 > 0$ such that for all $\Omega \in \mathcal{DZ}_0$,*

$$\|D^2 d_\Omega\|_{M^1(D)} \leq c_0 \text{ and } \|D^2 d_{\complement\Omega}\|_{M^1(D)} \leq c_0$$

$M^1(D)$ is the set of bounded measures on D and $\|D^2 d_\Omega\|_{M^1(D)}$ is defined as follows. Let $\mathbf{\Phi}$ be a 2×2 matrix of functions in $C^1(D)$, we have

$$\|D^2 d_\Omega\|_{M^1(D)} = \sup_{\mathbf{\Phi} \in C^1(D)^{2\times 2},\, \|\mathbf{\Phi}\|_C \leq 1} \left| \int_D \nabla d_\Omega \cdot \mathbf{div}\mathbf{\Phi}\, dx \right|,$$

where

$$\|\mathbf{\Phi}\|_C = \sup_{x \in D} |\mathbf{\Phi}(x)|_{\mathbb{R}^{2\times 2}},$$

and

$$\mathbf{div}\mathbf{\Phi} = [div\Phi_1,\, div\Phi_2],$$

where Φ_i, $i = 1, 2$ are the row vectors of the matrix $\mathbf{\Phi}$.

The set $\mathcal{DZ}_0$ has the following property (see [11, Chapter 4, Theorem 5.2])

Proposition 3. *Any $\Omega \in \mathcal{DZ}_0$ has a finite perimeter upper-bounded by $2c_0$.*

We next introduce three related sets of shapes.

Definition 6 (Sets of smooth shapes). *The set $\mathcal{C}_0$ (resp. $\mathcal{C}_1$, $\mathcal{C}_2$) of smooth shapes is the set of subsets of D whose boundary is non-empty and can be locally represented as the epigraph of a C^0 (resp. C^1, C^2) function. One further distinguishes the sets $\mathcal{C}_i^c$ and $\mathcal{C}_i^o$, $i = 0, 1, 2$ of subsets whose boundary is closed and open, respectively.*

Note that this implies that the boundary is a simple regular curve (hence compact) since otherwise it cannot be represented as the epigraph of a C^1 (resp. C^2) function in the vicinity of a multiple point. Another consequence of this definition is that the shape Ω_1 on

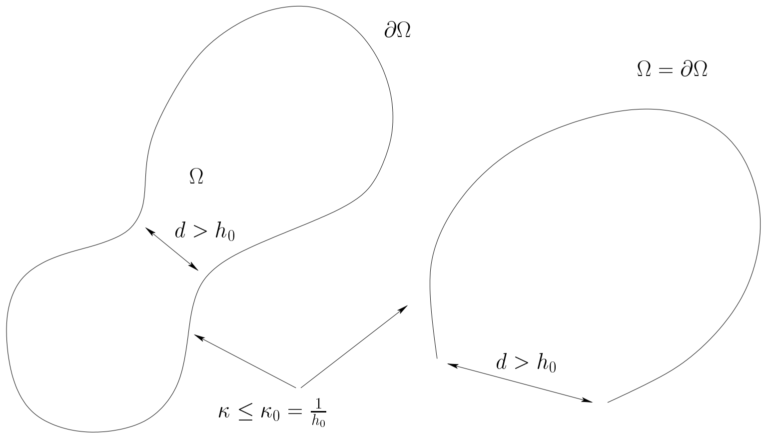

Figure 4: Examples of admissible shapes: a simple, closed, regular curve (left); a simple, open regular curve (right). In both cases the curvature is upperbounded by κ_0 and the pinch distance is larger than h_0.

the lefthand side of figure 2 is not in $\mathcal{C}_i$, $i = 0,\, 1,\, 2$ because C_1, C_2, C_3, parts of $\partial\Omega_1$, cannot be represented as epigraphs of a $C^i, i = 0,\, 1,\, 2$ function. Also note that $\mathcal{C}_2 \subset \mathcal{C}_1 \subset \mathcal{DZ}$ ([9, 10]).

The third set has been introduced by Federer [16].

Definition 7 (Set $\mathcal{F}$ of shapes of positive reach). *A nonempty subset Ω of D is said to have positive reach if there exists $h > 0$ such that $\Pi_\Omega(x)$ is a singleton for every $x \in U_h(\Omega)$. The maximum h for which the property holds is called the reach of Ω and is noted reach(Ω).*

We will also be interested in the subsets, called h_0-Federer's sets and noted $\mathcal{F}_{h_0}$, $h_0 > 0$, of $\mathcal{F}$ which contain all Federer's sets Ω such that reach$(\Omega) \geq h_0$. Note that $\mathcal{C}_i$, $i = 1, 2 \subset \mathcal{F}$ but $\mathcal{C}_i \not\subset \mathcal{F}_{h_0}$.

We are now ready to define the set of shapes of interest.

Definition 8 (Set of all shapes). *The set, noted $\mathcal{S}$, of all shapes (of interest) is the subset of $\mathcal{C}_2$ whose elements are also h_0-Federer's sets for a given and fixed $h_0 > 0$.*

$$\mathcal{S} \stackrel{def}{=} \mathcal{C}_2 \cap \mathcal{F}_{h_0}$$

This set contains the two subsets $\mathcal{S}^c$ and $\mathcal{S}^o$ obtained by considering $\mathcal{C}_2^c$ and $\mathcal{C}_2^o$, respectively.

Note that $\mathcal{S} \subset \mathcal{DZ}$. Note also that the curvature of $\partial\Omega$ is well defined and upperbounded by $1/h_0$, noted κ_0. Hence, c_0 in definition 5 can be chosen in such a way that $\mathcal{S} \subset \mathcal{DZ}_0$.

At this point, we can represent regular (i.e. C^2) simple curves with and without boundaries that do not curve or pinch too much (in the sense of κ_0 and h_0, see figure 4.

There are two reasons why we choose $\mathcal{S}$ as our frame of reference. The first one is because our implementations work with discrete objects defined on an underlying discrete square grid of pixels. As a result we are not able to describe details smaller than the distance between two pixels. This is our unit and h_0 is chosen to be smaller than or equal to it. The second reason is that $\mathcal{S}$ is included in $\mathcal{DZ}_0$ which, as shown in section 3.2, is compact. This will turn out to be important when minimizing shape functionals.

The question of the deformation of a shape by an element of a group of transformations could be raised at this point. What we have in mind here is the question of deciding whether a square and the same square rotated by 45 degrees are the same shape. There is no real answer to this question, more precisely the answer depends on the application. Note that the group in question can be finite dimensional, as in the case of the Euclidean and affine groups which are the most common in applications, or infinite dimensional. In this work we will, for the most part, not consider the action of groups of transformations on shapes.

3.2 Compactness properties

Interestingly enough, the definition of the set $\mathcal{DZ}_0$ (definition 5) implies that it is compact for all three topologies. This is the result of the following theorem whose proof can be found in [11, Chapter 4, Theorems 8.2, 8.3].

Theorem 9. *Let D be a nonempty bounded regular*[2] *open subset of $\mathbb{R}^2$ and $\mathcal{DZ}$ the set defined in definition 4. The embedding*

$$BC(D) = \{d_\Omega \in C_d(D) \cap C_d^c(D) : \nabla d_\Omega,\, \nabla d_{\complement\Omega} \in BV(D)^2\} \to W^{1,2}(D),$$

is compact.

This means that for any bounded sequence $\{\Omega_n\}$, $\emptyset \neq \Omega_n$ of elements of $\mathcal{DZ}$, i.e. for any sequence of $\mathcal{DZ}_0$, there exists a set $\Omega \neq \emptyset$ of $\mathcal{DZ}$ such that there exists a subsequence Ω_{n_k} such that

$$d_{\Omega_{n_k}} \to d_\Omega \quad \text{and} \quad d_{\complement\Omega_{n_k}} \to d_{\complement\Omega} \quad \text{in} \quad W^{1,2}(D).$$

Since $b_\Omega = d_\Omega - d_{\complement\Omega}$, we also have the convergence of $b_{\Omega_{n_k}}$ to b_Ω, and since the mapping $b_\Omega \to |b_\Omega| = d_{\partial\Omega}$ is continuous in $W^{1,2}(D)$ (see [11, Chapter 5, Theorem 5.1 (iv)]), we also have the convergence of $d_{\partial\Omega_{n_k}}$ to $d_{\partial\Omega}$. The convergence for the ρ_2 distance follows from equation (4):

$$\chi_{\Omega_{n_k}} \to \chi_\Omega \quad \text{in} \quad L^2(D),$$

and the convergence for the Hausdorff distance follows from theorem 3, taking subsequences if necessary.

In other words, the set $\mathcal{DZ}_0$ is compact for the topologies defined by the ρ_2, Hausdorff and $W^{1,2}$ distances.

Note that, even though $\mathcal{S} \subset \mathcal{DZ}_0$, this does not imply that it is compact for either one of these three topologies. But it does imply that its closure $\overline{\mathcal{S}}$ for each of these topologies is compact in the compact set $\mathcal{DZ}_0$.

[2] Regular means uniformly Lipschitzian in the sense of [11, Chapter 2, Definition 5.1].

3.3 Comparison between the three topologies on $\mathcal{S}$

The three topologies we have considered turn out to be closely related on $\mathcal{S}$. This is summarized in the following

Theorem 10. *The three topologies defined by the three distances ρ_2, ρ_D and ρ_H are equivalent on $\mathcal{S}^c$. The two topologies defined by ρ_D and ρ_H are equivalent on $\mathcal{S}^o$.*

This means that, for example, given a set Ω of $\mathcal{S}^c$, a sequence $\{\Omega_n\}$ of elements of $\mathcal{S}^c$ converging toward $\Omega \in \mathcal{S}^c$ for any of the three distances ρ_2, ρ (ρ_H) and ρ_D also converges toward the *same* Ω for the other two distances.

We now proceed with the proof of theorem 10. Being a bit lengthy, we have split it in a series of lemmas and propositions.

We start with a lemma.

Lemma 11. *Let $\{f_n\}$ be a sequence of uniformly Lipschitz functions $K \to \mathbb{R}^m$, K a compact of $\mathbb{R}^2$, converging for the L^2 norm toward a Lipschitz continuous function f. Then, the convergence is uniform.*

Proof. The L^2 convergence of continuous functions implies the convergence a.e.. Let us show that this implies the convergence everywhere. We note L the Lipschitz constant. Let x_0 be a point of K such that $f_n(x_0)$ does not converge toward $f(x_0)$. There exists $\varepsilon_0 > 0$ such that for all $n_0 \geq 0$, $\exists n > n_0$, $|f_n(x_0) - f(x_0)| > \varepsilon_0$.

f being continuous at x_0, there exists $\eta > 0$ such that for all y in K such that $d(x_0, y) < \eta$, $|f(y) - f(x_0)| < \varepsilon_0/3$.

Consider now the ys of K such that $d(x_0, y) < \inf(\varepsilon_0/3L, \eta)$. There exists at least one of them, noted y_0, such that $f_n(y_0)$ converges to $f(y_0)$ because the convergence is a.e..

We write

$$|f_n(x_0) - f(x_0)| \leq |f_n(x_0) - f_n(y_0)| + |f_n(y_0) - f(y_0)| + |f(y_0) - f(x_0)|$$

The first term in the right handside of this inequality is less than or equal to $\varepsilon_0/3$ because of the uniform Lipschitz hypothesis. Because $f_n(y_0)$ converges to $f(y_0)$ there exists $N_0(\varepsilon_0, y)$ such that for all $n \geq N_0$, $|f_n(y_0) - f(y_0)| \leq \varepsilon_0/3$. The third term is also less than or equal to $\varepsilon_0/3$ because of the hypothesis on f. Hence

$$|f_n(x_0) - f(x_0)| \leq \varepsilon_0 \, \forall n \geq N_0,$$

a contradiction. The sequence $\{f_n\}$ converges toward f everywhere in K and since the f_ns are uniformly Lipschitz, the convergence is uniform (see e.g. [12]). □

This lemma is useful for proving the following

Proposition 4. *In $\mathcal{S}$, the $W^{1,2}$ convergence of sequences of distance functions implies their Hausdorff convergence.*

Proof. The $W^{1,2}$ convergence implies the L^2 convergence of the distance functions. According to lemma 11 this implies the uniform convergence of the distance functions and hence the Hausdorff convergence. □

We also have the converse

Proposition 5. *In $\mathcal{S}$, the Hausdorff convergence of sequences of distance functions implies their $W^{1,2}$ convergence.*

Proof. We consider the boundary Γ of a shape Ω of $\mathcal{S}$. The inequality

$$\|d_{\Gamma_1} - d_{\Gamma_2}\|_{L^2} \leq \rho(\Gamma_1, \Gamma_2) m(D)^{1/2}$$

shows that the Hausdorff convergence implies the L^2 convergence of the distance functions. For the $W^{1,2}$ topology we also need the convergence of the L^2 norm of the gradient.

Consider a sequence $\{\Omega_n\}$ of elements of $\mathcal{S}$ converging for the Hausdorff distance toward $\Omega \in \mathcal{S}$. If we prove the convergence a.e. of $\nabla(d_{\Omega_n} - d_\Omega)$ to 0, the Lebesgue dominated convergence theorem will give us the L^2 convergence toward 0 since

$$|\nabla(d_{\Omega_n} - d_\Omega)| \leq 2 \quad \text{a.e.}$$

Because we are in $\mathcal{S}$, all skeletons are negligible (zero Lebesgue measure), [6]. Consider the union $Sk = \partial\Omega \cup Sk(\Omega) \cup_n Sk(\Omega_n) \cup_n \partial\Omega_n$; as a denombrable union of negligible sets it is negligible. Let x be a point of $D \backslash Sk$, y_n its projection on Ω_n, y its projection on Ω. According to definition 1 and proposition 1, all distance functions of interest are differentiable at x. We prove that the angle between the vectors $\vec{xy_n}$ and $\vec{xy}$ goes to 0 by proving that $y_n \to y$. By compactness of $\overline{D}$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging toward $z \in \Omega$. If $z = y$ we are done. If $z \neq y$ we prove a contradiction. Indeed, since the distance is continuous

$$lim_{k\to\infty} d(x, y_{n_k}) = d(x, y).$$

But we also have, by definition, $d(x, y_{n_k}) = d_{\Omega_{n_k}}(x)$; since $\Omega_{n_k} \to \Omega$ for the Hausdorff distance, $d_{\Omega_{n_k}} \to d_\Omega$ everywhere in D and therefore $\lim_{k\to\infty} d_{\Omega_{n_k}}(x) = d_\Omega(x)$. Hence $d(x, y) = d(x, z)$ and $x \in Sk(\Omega)$, a contradiction.

We have shown that all converging subsequences of $\{y_n\}$ converged to $z = \Pi_{\partial\Omega}(x)$. In order to conclude, we must show that the sequence $\{y_n\}$ converges to z. Indeed, let us assume that there exists a subsequence $\{y_{n_k}\}$ not converging. There exists an $\varepsilon_0 > 0$ such that there is an infinity of values of k for which y_{n_k} is outside the open disc $B(z, \varepsilon_0)$. Let us note $\{y_{n_l}\}$ the corresponding subsequence. Because of compactness again there exists a converging subsequence of $\{y_{n_l}\}$ which has to converge toward z but this is impossible since all y_{n_l} are ouside of $B(z, \varepsilon_0)$. Hence the sequence $\{y_n\}$ converges toward z and we have proved that $\nabla(d_{\Omega_n} - d_\Omega) \to 0$ a.e.. □

We now compare the topologies induced by the ρ_2 and the Hausdorff distances. This makes only sense in $\mathcal{S}^c$. The first result is in the following

Proposition 6. *In $\mathcal{S}^c$, the $W^{1,2}$ convergence of sequences of distance functions implies the L^2 convergence of the corresponding characteristic functions.*

Proof. The proof can be found in [11] for more general shapes than those in $\mathcal{S}^c$. It goes as follows. Given a subset Ω of $\mathbb{R}^2$ such that $\Omega \neq \emptyset$, for almost all x

$$\chi_{\overline{\Omega}}(x) = 1 - |\nabla d_\Omega(x)|$$

Given two nonempty subets Ω_1 and Ω_2 of D

$$|\nabla d_{\Omega_2}(x)| \le |\nabla d_{\Omega_1}(x)| + |\nabla d_{\Omega_2}(x) - \nabla d_{\Omega_1}(x)|,$$

from which it follows that

$$\chi_{\overline{\Omega}_1}(x) \le \chi_{\overline{\Omega}_2}(x) + |\nabla d_{\Omega_2}(x) - \nabla d_{\Omega_1}(x)|,$$

and therefore

$$\int_D |\chi_{\overline{\Omega}_1}(x) - \chi_{\overline{\Omega}_2}(x)|^2\, dx \le \|d_{\Omega_1} - d_{\Omega_2}\|^2_{W^{1,2}(D)}.$$

□

We also prove the converse for the Hausdorff distance in the

Proposition 7. *In $\mathcal{S}^c$, the ρ_2 convergence of sequences of characteristic functions implies the Hausdorff convergence of the distance functions of the boundaries of the corresponding sets.*

In the proof we will need the following two lemmas and proposition.

Lemma 12. *Let Γ be a C^2 curve whose curvature is upperbounded by κ_0. Let C_1 and C_2 be two points of Γ, δ the length of the curve between C_1 and C_2:*

$$0 \le \delta - d(C_1, C_2) \le \frac{\delta^2 \kappa_0}{2} \tag{5}$$

Proof. The first inequality in (5) follows from the fact that the straight line is the shortest path between two points in the plane.

We parameterize Γ with its arc length s. We recall the Frenet formulae

$$\frac{d\Gamma}{ds} = \mathbf{t} \quad \frac{d\mathbf{t}}{ds} = \kappa\mathbf{n} \quad \frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t},$$

where $\mathbf{t}$ and $\mathbf{n}$ are the unit tangent and normal vectors to Γ, respectively. We then write the second order Taylor expansion without remainder of $\Gamma(s_2) = C_2$ at $\Gamma(s_1) = C_1$

(6)

$$C_2 = C_1 + (s_2 - s_1)\mathbf{t}(s_1) + (s_2 - s_1)^2 \int_0^1 (1-\zeta)\,\kappa(s_1 + \zeta(s_2 - s_1))\,\mathbf{n}(s_1 + \zeta(s_2 - s_1))\, d\zeta.$$

The second inequality in (5) follows from the fact that $|\kappa| \le \kappa_0$ and $\delta = |s_2 - s_1|$. □

An easy consequence of this lemma is the

Proposition 8. *The length of a closed curve in $\mathcal{S}$ is greater than or equal to $2h_0$.*

Proof. We use the second inequality in (5) with $d(C_1, C_2) = 0$ from which the conclusion follows. □

The second lemma tells us that in a disc of small enough radius we cannot have too large a piece of a boundary of an element of $\mathcal{S}$.

Lemma 13. *Let $\varepsilon > 0$ be such that $2\varepsilon\kappa_0 << 1$. Then any disc of radius ε does not contain a connected piece of boundary of an element of $\mathcal{S}$ of length greater than h_0.*

Proof. The proof follows from the previous lemma. Let us first assume that the piece in question has a boundary, hence two different endpoints C_1 and C_2. By definition, $d(C_1, C_2) \leq 2\varepsilon$. Using (5) we conclude that the length δ of the curve between C_1 and C_2 must satisfy

$$\frac{\delta^2\kappa_0}{2} - \delta + 2\varepsilon \geq 0.$$

The lefthand side is a second degree polynomial in the variable δ, noted $P(\delta)$, which has two positive roots $\delta_1 \leq \delta_2$:

$$\delta_1 = h_0(1 - \sqrt{1 - 2\varepsilon\kappa_0})$$

Since $P(0) > 0$, δ can continuously vary from 0 to its maximal value, and $\delta_1 < \delta_2$, we must have $\delta \leq \delta_1$. Moreover, since $2\varepsilon\kappa_0 << 1$, $\delta_1 < h_0/2$.

Let us now assume that the connected piece does not have a boundary, hence is a closed simple curve. We choose two distinct arbitrary points C_1 and C_2 on the curve, apply the previous analysis to the each of the two connected components, and conclude that the length of the curve is less than h_0. Because of proposition 8, this is impossible. □

We now prove proposition 7.

Proof. Let Ω_1 and Ω_2 two shapes of $\mathcal{S}^c$ with boundaries Γ_1 and Γ_2, $\varepsilon > 0$ such that $\rho_2(\Omega_1, \Omega_2) \leq \varepsilon$ and $2\kappa_0\varepsilon << 1$. We assume that there exists a point A of Γ_1 such that $d_{\Gamma_2}(A) > \varepsilon$ and prove a contradiction.

Consider the open disc $B(A, \varepsilon)$ of center A and radius ε. This disc does not contain any point of Γ_2 by hypothesis, since otherwise we would have $d_{\Gamma_2}(A) \leq \varepsilon$. Moreover, the curve Γ_1 is not included in $B(A, \varepsilon)$ because of the hypothesis $2\kappa_0\varepsilon << 1$ and lemma 13, therefore there must be a strictly positive even number of points of intersection between Γ_1 and the border of $B(A, \varepsilon)$. If there are more than two, the same reasoning as in the proof of proposition 10 below shows that there is a piece of skeleton of Γ_1 within $B(A, \varepsilon)$ and hence $\Omega_1 \notin \mathcal{F}_{h_0}$.

Let A_1 and A_2 be the endpoints of the arc of Γ_1 going through A. This arc divides $B(A, \varepsilon)$ in two parts, one of them belongs to $\Omega_1 \Delta \Omega_2$. The idea is that since $2\varepsilon\kappa_0 << 1$, the arc A_1AA_2 is equivalent to a line segment and each area is approximately equal to $\pi\varepsilon^2/2$, hence $\|\chi_{\Omega_1} - \chi_{\Omega_2}\|_{L^2} \geq \varepsilon\sqrt{\pi/2} > \varepsilon$, a contradiction.

In order to prove this, we parameterize Γ_1 between A_1 and A_2 by its arc-length s and compute an upperbound on the distance of $A(s)$ to the tangent line to $\partial\Omega_1$ at A. We choose A as the origin of arclength on Γ_1 and use equation (6):

$$A(s) = A + s\mathbf{t}(s) + s^2 \int_0^1 (1-\zeta)\,\kappa(\zeta s)\,\mathbf{n}(\zeta s)\,d\zeta.$$

The distance of $A(s)$ to the line $(A,\,\mathbf{t}(0))$ is given by

$$(A(s) - A)\cdot\mathbf{n}(0) = s^2 \int_0^1 (1-\zeta)\,\kappa(\zeta s)\,\mathbf{n}(\zeta s)\cdot\mathbf{n}(0)\,d\zeta.$$

We obtain an upper bound on its magnitude by

$$|(A(s) - A)\cdot\mathbf{n}(0)| \leq \frac{s^2}{2}\kappa_0 \tag{7}$$

The upper bound is maximal for $s = s_1$ ($A_1 = A(s_1)$ and $|s_1| \stackrel{\text{def}}{=} \delta_1$) or $s = s_2$ ($A_2 = A(s_2)$ and $s_2 \stackrel{\text{def}}{=} \delta_2$). We obtain upper bounds from (5); δ_1 and δ_2 must satisfy

$$\delta^2\frac{\kappa_0}{2} - \delta + \varepsilon \geq 0.$$

In order for this to be true we must have

$$0 \leq \delta \leq \frac{1}{\kappa_0}(1 - \sqrt{1 - 2\varepsilon\kappa_0}) \stackrel{\text{def}}{=} \delta_m \quad \text{or} \quad \delta \geq \frac{1}{\kappa_0}(1 + \sqrt{1 - 2\varepsilon\kappa_0})$$

The second alternative is impossible since $B(A,\,\varepsilon)$ cannot contain an arc whose length is larger than $1/\kappa_0$ (lemma 13). There remains only the first alternative. Returning to (7), we find that $\delta_m^2\kappa_0/2$ is an upper bound on the distance of $A(s)$ to the tangent. Referring to figure 5 we conclude that the area of interest is bounded below by

$$\frac{\pi\varepsilon^2}{2} - \varepsilon\kappa_0\delta_m^2$$

Since $2\varepsilon\kappa_0 << 1$, we have $\delta_m = \frac{1}{\kappa_0}(\varepsilon\kappa_0 + o(\varepsilon\kappa_0)$ and therefore

$$\varepsilon\kappa_0\delta_m^2 = \frac{\varepsilon}{\kappa_0}((\varepsilon\kappa_0)^2 + o((\varepsilon\kappa_0)^2) = \varepsilon^2(\varepsilon\kappa_0 + o(\varepsilon\kappa_0)).$$

The area of interest is lower bounded by

$$\varepsilon^2(\frac{\pi}{2} - \varepsilon\kappa_0 + o(\varepsilon\kappa_0)),$$

and therefore, for $\varepsilon\kappa_0$ sufficiently small, its square root is strictly larger than ε. $\square$

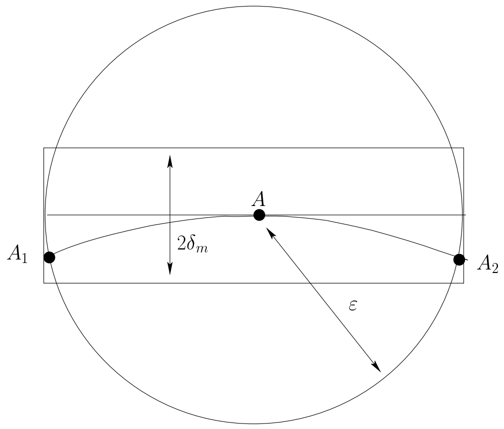

Figure 5: A lower bound on the area of $\Omega_1 \Delta \Omega_2$ (see text).

This completes the proof of theorem 10.

An interesting and practically important consequence of this theorem is the following. Consider the set $\mathcal{S}$, included in $\mathcal{DZ}_0$, and its closure $\overline{\mathcal{S}}$ for any one of the three topologies of interest. $\overline{\mathcal{S}}$ is a closed subset of the compact metric space $\mathcal{DZ}_0$ and is therefore compact as well. Given a continuous function $f : \mathcal{S} \to \mathbb{R}$ we consider its lower semi-continuous (l.s.c.) envelope $\underline{f}$ defined on $\overline{\mathcal{S}}$ as follows

$$\underline{f}(x) = \begin{cases} f(x) \quad \text{if} \quad x \in \mathcal{S} \\ \liminf_{y \to x,\, y \in \mathcal{S}} f(y) \end{cases}$$

The useful result for us is summarized in the

Proposition 9. *$\underline{f}$ is l.s.c. in $\overline{\mathcal{S}}$ and therefore has at least a minimum in $\overline{\mathcal{S}}$.*

Proof. In a metric space E, a real function f is said to be l.s.c. if and only if

$$f(x) \leq \liminf_{y \to x} f(y) \quad \forall x \in E.$$

Therefore $\underline{f}$ is l.s.c. by construction. The existence of minimum of an l.s.c. function defined on a compact metric space is well-known (see e.g. [7, 15]) and will be needed later to prove that some of our minimization problems are well-posed. □

4 Deforming shapes

The problem of continuously deforming a shape so that it turns into another is central to this paper. The reasons for this will become more clear in the sequel. Let us just mention here that it can be seen as an instance of the warping problem: given two shapes Ω_1 and Ω_2, how do I deform Ω_1 onto Ω_2? The applications in the field of medical image processing and analysis are immense (see for example [46, 45]). It can also be seen as an instance of the famous (in computer vision) correspondence problem: given two shapes Ω_1 and Ω_2, how do I find the corresponding point P_2 in Ω_2 of a given point P_1 in Ω_1? Note that a solution of the warping problem provides a solution of the correspondence problem if we can track the evolution of any given point during the smooth deformation of the first shape onto the second.

In order to make things more quantitative, we assume that we are given a function $E : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathbb{R}^+$, called the Energy, which is continuous on $\mathcal{S} \times \mathcal{S}$ for one of the shape topologies of interest. This Energy can also be thought of as a measure of the dissimilarity between the two shapes. By smooth, we mean that it is continuous with respect to this topology and that its derivatives are well-defined in a sense we now make more precise.

We first need the notion of a normal deformation flow of a curve Γ in $\mathcal{S}$. This is a smooth (i.e. C^0) function $\beta : [0,1] \to \mathbb{R}$ (when $\Gamma \in \mathcal{S}^o$, one further requires that $\beta(0) = \beta(1)$). Let $\Gamma : [0,1] \to \mathbb{R}^2$ be a parameterization of Γ, $\mathbf{n}(p)$ the unit normal at the point $\Gamma(p)$ of Γ; the normal deformation flow β associates the point $\Gamma(p) + \beta(p)\mathbf{n}(p)$ to $\Gamma(p)$. The resulting shape is noted $\Gamma + \boldsymbol{\beta}$, where $\boldsymbol{\beta} = \beta\mathbf{n}$. There is no guarantee that $\Gamma + \boldsymbol{\beta}$ is still a shape in $\mathcal{S}$ in general but if β is C^0 and ε is small enough, $\Gamma + \boldsymbol{\beta}$ is in $\mathcal{C}_0$. Given two shapes Γ and Γ_0, the corresponding Energy $E(\Gamma, \Gamma_0)$, and a normal deformation flow β of Γ, the Energy $E(\Gamma + \varepsilon\boldsymbol{\beta}, \Gamma_0)$is now well-defined for ε sufficiently small. The derivative of $E(\Gamma, \Gamma_0)$ with respect to Γ in the direction of the flow $\boldsymbol{\beta}$ is then defined, when it exists, as

$$\mathcal{G}_\Gamma(E(\Gamma, \Gamma_0), \boldsymbol{\beta}) = \lim_{\varepsilon \to 0} \frac{E(\Gamma + \varepsilon\boldsymbol{\beta}, \Gamma_0) - E(\Gamma, \Gamma_0)}{\varepsilon} \tag{8}$$

This kind of derivative is also known as a Gâteaux semi-derivative. In our case the function $\beta \to \mathcal{G}_\Gamma(E(\Gamma, \Gamma_0), \boldsymbol{\beta})$ is linear and continuous (it is then called a Gâteaux derivative) and defines a continuous linear form on the vector space of normal deformation flows of Γ. This is a vector subspace of the Hilbert space $L^2(\Gamma)$ with the usual Hilbert product $\langle \beta_1, \beta_2 \rangle = \frac{1}{|\Gamma|} \int_\Gamma \beta_1 \, \beta_2 = \frac{1}{|\Gamma|} \int_\Gamma \beta_1(x)\beta_2(x) \, d\Gamma(x)$, where $|\Gamma|$ is the length of Γ. Given such an inner product, we can apply Riesz's representation theorem [39] to the Gâteaux derivative $\mathcal{G}_\Gamma(E(\Gamma, \Gamma_0), \boldsymbol{\beta})$: There exists a deformation flow, noted $\nabla E(\Gamma, \Gamma_0)$, such that

$$\mathcal{G}_\Gamma(E(\Gamma, \Gamma_0), \boldsymbol{\beta}) = \langle \nabla E(\Gamma, \Gamma_0), \beta \rangle.$$

This flow is called the gradient of $E(\Gamma, \Gamma_0)$.

We now return to the initial problem of smoothly deforming a curve Γ_1 onto a curve Γ_2. We can state it as that of defining a family $\Gamma(t)$, $t \geq 0$ of shapes such that $\Gamma(0) = \Gamma_1$, $\Gamma(T) = \Gamma_2$ for some $T > 0$ and for each value of $t \geq 0$ the deformation flow of the current shape $\Gamma(t)$ is equal to minus the gradient $\nabla E(\Gamma, \Gamma_2)$ defined previously. This is equivalent to solving the following PDE

$$\begin{aligned} \Gamma_t &= -\nabla E(\Gamma, \Gamma_2)\mathbf{n} \\ \Gamma(0) &= \Gamma_1 \end{aligned} \tag{9}$$

In this paper we do not address the question of the existence of solutions to (9).

Natural candidates for the Energy function E are the distances defined in section 2.2. The problem we are faced with is that none of these distances are Gâteaux differentiable. This is why the next section is devoted to the definition of smooth approximations of some of them.

5 How to approximate shape distances

The goal of this section is to provide smooth approximations of some of these distances, i.e. that admit Gâteaux derivatives. We start with some notations.

5.1 Averages

Let Γ be a given curve in $\mathcal{C}^1$ and consider an integrable function $f : \Gamma \to \mathbb{R}^n$. We denote by $\langle f \rangle_\Gamma$ the average of f along the curve Γ:

$$\langle f \rangle_\Gamma = \frac{1}{|\Gamma|} \int_\Gamma f = \frac{1}{|\Gamma|} \int_\Gamma f(x)\, d\Gamma(x) \tag{10}$$

For real positive integrable functions f, and for any continuous strictly monotonous (hence one to one) function φ from $\mathbb{R}^+$ or $\mathbb{R}^{+*}$ to $\mathbb{R}^+$ we will also need the φ-average of f along Γ which we define as

$$\langle f \rangle_\Gamma^\varphi = \varphi^{-1}\left(\frac{1}{|\Gamma|} \int_\Gamma \varphi \circ f\right) = \varphi^{-1}\left(\frac{1}{|\Gamma|} \int_\Gamma \varphi(f(x))\, d\Gamma(x)\right) \tag{11}$$

Note that φ^{-1} is also strictly monotonous and continous from $\mathbb{R}^+$ to $\mathbb{R}^+$ or $\mathbb{R}^{+*}$. Also note that the unit of the φ-average of f is the same as that of f, thanks to the normalization by $|\Gamma|$.

The discrete version of the φ-average is also useful: let $a_i, i = 1, \dots, n$ be n positive numbers, we note

$$\langle a_1, \cdots, a_n \rangle^\varphi = \varphi^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \varphi(a_i)\right), \tag{12}$$

their φ-average.

5.2 Approximations of the Hausdorff distance

We now build a series of smooth approximations of the Hausdorff distance $\rho_H(\Gamma, \Gamma')$ of two shapes Γ and Γ'. According to (3) we have to consider the functions $d_{\Gamma'} : \Gamma \to \mathbb{R}^+$ and $d_\Gamma : \Gamma' \to \mathbb{R}^+$. Let us focus on the second one. Since d_Γ is Lipschitz continuous on the bounded hold-all set D it is certainly integrable on the compact set Γ' and we have [39, Chapter 3, problem 4]

$$\lim_{\beta\to+\infty}\left(\frac{1}{|\Gamma'|}\int_{\Gamma'} d_\Gamma^\beta(x')\,d\Gamma'(x')\right)^{\frac{1}{\beta}} = \sup_{x'\in\Gamma'} d_\Gamma(x'). \tag{13}$$

Moreover, the function $\mathbb{R}^+ \to \mathbb{R}^+$ defined by $\beta \to \left(\frac{1}{|\Gamma'|}\int_{\Gamma'} d_\Gamma^\beta(x')\,d\Gamma'(x')\right)^{\frac{1}{\beta}}$ is monotonously increasing [39, Chapter 3, problem 5].

Similar properties hold for $d_{\Gamma'}$.

If we note p_β the function $\mathbb{R}^+ \to \mathbb{R}^+$ defined by $p_\beta(x) = x^\beta$ we can rewrite (13)

$$\lim_{\beta\to+\infty} \langle d_\Gamma\rangle_{\Gamma'}^{p_\beta} = \sup_{x'\in\Gamma'} d_\Gamma(x').$$

$\langle d_\Gamma\rangle_{\Gamma'}^{p_\beta}$ is therefore a monotonically increasing approximation of $\sup_{x'\in\Gamma'} d_\Gamma(x')$. We go one step further and approximate $d_{\Gamma'}(x)$.

Consider a continuous strictly monotonously decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^{+*}$. Because φ is strictly monotonously decreasing

$$\sup_{x'\in\Gamma'} \varphi(d(x,x')) = \varphi(\inf_{x'\in\Gamma'} d(x,x')) = \varphi(d_{\Gamma'}(x)),$$

and moreover

$$\lim_{\alpha\to+\infty}\left(\frac{1}{|\Gamma'|}\int_{\Gamma'} \varphi^\alpha(d(x,x'))\,d\Gamma'(x')\right)^{\frac{1}{\alpha}} = \sup_{x'\in\Gamma'} \varphi(d(x,x')).$$

Because φ is continuous and strictly monotonously decreasing, it is one to one and φ^{-1} is strictly monotonously decreasing and continuous. Therefore

$$d_{\Gamma'}(x) = \lim_{\alpha\to+\infty} \varphi^{-1}\left(\left(\frac{1}{|\Gamma'|}\int_{\Gamma'} \varphi^\alpha(d(x,x'))\,d\Gamma'(x')\right)^{\frac{1}{\alpha}}\right)$$

We can simplify this equation by introducing the function $\varphi_\alpha = p_\alpha \circ \varphi$:

$$d_{\Gamma'}(x) = \lim_{\alpha\to+\infty} \langle d(x,\cdot)\rangle_{\Gamma'}^{\varphi_\alpha} \tag{14}$$

Any $\alpha > 0$ provides us with an approximation, noted $\tilde{d}_{\Gamma'}$, of $d_{\Gamma'}$:

$$\tilde{d}_{\Gamma'}(x) = \langle d(x,\cdot)\rangle_{\Gamma'}^{\varphi_\alpha} \tag{15}$$

We have a similar expression for $\tilde{d}_\Gamma$.

Note that because $\left(\frac{1}{|\Gamma'|}\int_{\Gamma'} \varphi^\alpha(d(x,x'))\,d\Gamma'(x')\right)^{\frac{1}{\alpha}}$ increases with α toward its limit $\sup_{x'} \varphi(d(x,x')) = \varphi(d_{\Gamma'}(x))$, $\varphi^{-1}\left(\left(\frac{1}{|\Gamma'|}\int_{\Gamma'} \varphi^\alpha(d(x,x'))\,d\Gamma'(x')\right)^{\frac{1}{\alpha}}\right)$ decreases with α toward its limit $d_{\Gamma'}(x)$.

Examples of functions φ are

$$\begin{aligned}
\varphi_1(z) &= \frac{1}{z+\varepsilon} \quad \varepsilon > 0,\, z \geq 0 \\
\varphi_2(z) &= \mu \exp(-\lambda z) \quad \mu,\, \lambda > 0,\, z \geq 0 \\
\varphi_3(z) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{z^2}{2\sigma^2}) \quad \sigma > 0,\, z \geq 0
\end{aligned}$$

Putting all this together we have the following result

$$\begin{aligned}
\sup_{x\in\Gamma} d_{\Gamma'}(x) &= \lim_{\alpha,\,\beta\to+\infty} \langle\langle d(\cdot,\,\cdot)\rangle_{\Gamma'}^{\varphi_\alpha}\rangle_{\Gamma}^{p_\beta} \\
\sup_{x\in\Gamma'} d_{\Gamma}(x) &= \lim_{\alpha,\,\beta\to+\infty} \langle\langle d(\cdot,\,\cdot)\rangle_{\Gamma}^{\varphi_\alpha}\rangle_{\Gamma'}^{p_\beta}
\end{aligned}$$

Any positive values of α and β yield approximations of $\sup_{x\in\Gamma} d_{\Gamma'}(x)$ and $\sup_{x\in\Gamma'} d_{\Gamma}(x)$.

The last point to address is the max that appears in the definition of the Hausdorff distance. We use (12), choose $\varphi = p_\gamma$ and note that, for a_1 and a_2 positive,

$$\lim_{\gamma\to+\infty} \langle a_1,\, a_2\rangle^{p_\gamma} = \max(a_1,\, a_2).$$

This yields the following expression for the Hausdorff distance between two shapes Γ and Γ'

$$\rho_H(\Gamma,\,\Gamma') = \lim_{\alpha,\,\beta,\,\gamma\to+\infty} \left\langle \langle\langle d(\cdot,\,\cdot)\rangle_{\Gamma'}^{\varphi_\alpha}\rangle_{\Gamma}^{p_\beta},\, \langle\langle d(\cdot,\,\cdot)\rangle_{\Gamma}^{\varphi_\alpha}\rangle_{\Gamma'}^{p_\beta} \right\rangle^{p_\gamma}$$

This equation is symmetric and yields approximations $\tilde{\rho}_H$ of the Hausdorff distance for all positive values of α, β and γ:

$$\tilde{\rho}_H(\Gamma,\,\Gamma') = \left\langle \langle\langle d(\cdot,\,\cdot)\rangle_{\Gamma'}^{\varphi_\alpha}\rangle_{\Gamma}^{p_\beta},\, \langle\langle d(\cdot,\,\cdot)\rangle_{\Gamma}^{\varphi_\alpha}\rangle_{\Gamma'}^{p_\beta} \right\rangle^{p_\gamma} \tag{16}$$

This approximation is "nice" in several ways, the first one being the obvious one, stated in the following

Proposition 10. *For each triplet $(\alpha,\,\beta,\,\gamma)$ in $(\mathbb{R}^{+*})^3$ the function $\tilde{\rho}_H : \mathcal{S}\times\mathcal{S} \to \mathbb{R}^+$ defined by equation (16) is continuous for the Hausdorff topology.*

We first recall the following properties of the squared distance function $\eta_{\partial\Omega}$ of the boundary of an element Ω of $\mathcal{S}$ (see [2]):

Proposition 11. *$\eta_{\partial\Omega}$ is smooth, i.e. C^2, in $U_{h_0}(\partial\Omega)$ and for all $x \in \partial\Omega$, the Hessian matrix $\nabla^2\eta_{\partial\Omega}(x)$ is the (matrix of) orthogonal projection onto the normal to $\partial\Omega$ at x.*

We now prove proposition 10.

Proof. For each shape Ω of $\mathcal{S}$, we consider the square of the distance function of $\partial\Omega$, noted $\eta_{\partial\Omega}$. We next prove that the length is continuous for the Hausdorff topology on $\mathcal{S}$. Consider two shapes Ω_1 and Ω_2 of $\mathcal{S}$ and assume that $\rho_H(\Omega_1, \Omega_2) < \varepsilon$. Let Γ_1 and Γ_2 be the boundaries of Ω_1 and Ω_2. Let $p \in [0,1] \to \Gamma_1(p)$ be a C^2 parametrization of Γ_1, we prove that the mapping

$$p \in [0,1] \to \Gamma_2(p) = \Gamma_1(p) - \frac{1}{2}\nabla\eta_{\Gamma_2}(\Gamma_1(p)), \tag{17}$$

is a one to one parametrization of Γ_2. If we choose $\varepsilon < h_0$, $\nabla\eta_{\Gamma_2}(\Gamma_1(p))$ is well-defined and continuous for all ps (proposition 11), hence $p \to \Gamma_1(p) - \frac{1}{2}\nabla\eta_{\Gamma_2}(\Gamma_1(p))$ is continuous.

It is injective: assume that there exist p_1 and p_2 in $[0, 1]$, $p_1 \neq p_2$ such that $\Gamma_2(p_1) = \Gamma_2(p_2)$, see figure 6 (if Γ_1 and Γ_2 are closed, $p_1,\ p_2 \notin \{0, 1\}$). Since the curvature of Γ_1 and Γ_2 is bounded by $1/h_0$, we choose $\varepsilon << h_0$. The two points $\Gamma_1(p_1)$ and $\Gamma_1(p_2)$ are in the disc of center $\Gamma_2(p_1)$ and radius ε since their distances to Γ_2 are by construction equal to $d(\Gamma_1(p_1), \Gamma_2(p_1))$ and $d(\Gamma_1(p_2), \Gamma_2(p_2))$, respectively, and are less than ε. Because of our choice of ε, the curvatures of the two curves within the disc are negligible and we can consider they are straight lines, as shown in figure 6. Therefore there must be a piece of the skeleton of Ω_1 within the disc and this contradicts the hypothesis that $\Omega_1 \in \mathcal{F}_{h_0}$.

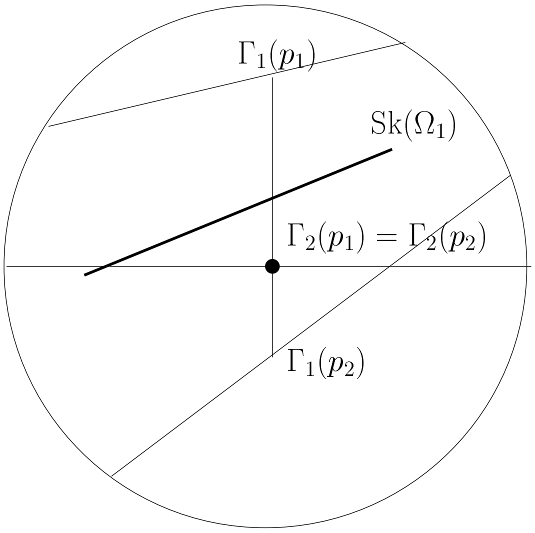

Figure 6: The mapping is injective.

It is surjective: we proceed by contradiction. Let us assume it is not surjective. Since the mapping (17) is continuous its image is connected and compact. Its complement $\hat{\Gamma}_2$ (assumed here to be non empty) is thus an open interval of Γ_2 (possibly two, if Γ_2 is an open curve). Let Γ_2^0 be one of the endpoints of this interval. There exists a value p_0 of p such that

$$\Gamma_2^0 = \Gamma_1(p_0) - \frac{1}{2}\nabla\eta_{\Gamma_2}(\Gamma_1(p_0))$$

Two cases can occur. Either $\Gamma_2^0 = \Gamma_1(p_0)$ and this implies that Γ_2 is not simple (see figure 7, left), or $\Gamma_2^0 \neq \Gamma_1(p_0)$ and this implies that $\Gamma_1(p_0)$ is on the skeleton of Γ_2, a contradiction if ε is small with respect to h_0 (see figure 7, right).

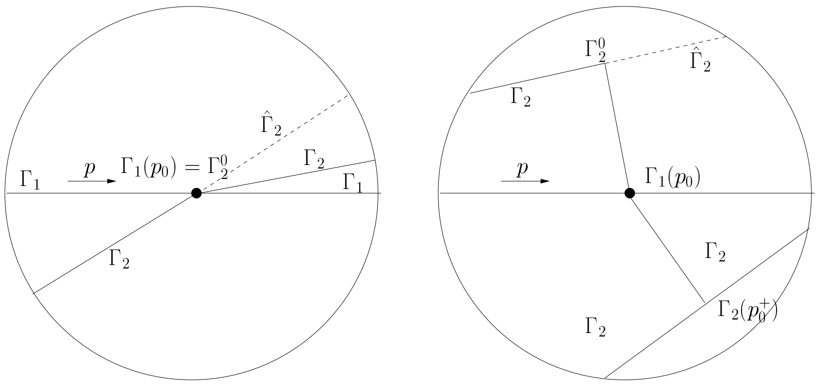

Figure 7: The mapping is surjective: The dotted line represents a piece of $\hat{\Gamma}_2$, see text.

Using this parameterization, we now prove that the length is continuous for the Hausdorff metric. Given a shape Ω and a sequence $\{\Omega_n\}$ of shapes of $\mathcal{S}$ converging to Ω for the Hausdorff topology, we show that $\lim_{n\to\infty} ||\partial\Omega_n| - |\partial\Omega|| = 0$. If n is large enough, we use the first part of the proof to parametrize Γ_n:

$$\Gamma_n(p) = \Gamma(p) - \frac{1}{2}\nabla\eta_{\Gamma_n}(\Gamma(p)), \tag{18}$$

and proceed from there:

$$| \, |\partial\Omega_n| - |\partial\Omega| \, | = \left| \int_0^1 |\Gamma_n'(p)| \, dp - \int_0^1 |\Gamma'(p)| \, dp \right| \leq \int_0^1 | \, |\Gamma_n'(p)| - |\Gamma'(p)| \, | \, dp \leq \int_0^1 |\Gamma_n'(p) - \Gamma'(p)| \, dp.$$

We take the derivative of (18) with respect to p:

$$\Gamma_n'(p) = \Gamma'(p) - \frac{1}{2}\nabla^2\eta_{\Gamma_n}(\Gamma(p))\Gamma'(p),$$

where $\nabla^2\eta_{\Gamma_n}$ is the second order derivative of η_{Γ_n}.

We are only interested in comparing the lengths of Γ and Γ_n where they differ. We can therefore exclude from the integral $\int |\Gamma_n'(p) - \Gamma'(p)| \, dp$ the values of p for which $\Gamma_n(p) = \Gamma(p)$ and assume that $\Gamma_n(p) \neq \Gamma(p)$. At these points, the first and second order derivatives of the distance function d_{Γ_n} are well-defined and (because $\Gamma_n \in \mathcal{S}$ and $\varepsilon << h_0$) there exists $M > 0$, independent of n, such that

$$|\ \nabla^2 d_{\Gamma_n}(x)\ | \leq M\ \forall x \notin \Gamma_n$$

Using the chain rule we obtain

$$\frac{1}{2}\nabla\eta_{\Gamma_n} = d_{\Gamma_n}\nabla d_{\Gamma_n} \quad \frac{1}{2}\nabla^2\eta_{\Gamma_n} = d_{\Gamma_n}\nabla^2 d_{\Gamma_n} + \nabla d_{\Gamma_n}\nabla^T d_{\Gamma_n},$$

and therefore

$$(19) \quad |\Gamma_n'(p) - \Gamma'(p)| \leq |\nabla d_{\Gamma_n}(\Gamma(p))\cdot\Gamma'(p)|\,|\nabla d_{\Gamma_n}(\Gamma(p))| + \\ d_{\Gamma_n}(\Gamma(p))\,\|\nabla^2 d_{\Gamma_n}(\Gamma(p))\Gamma'(p)\| \leq |\nabla d_{\Gamma_n}(\Gamma(p))\cdot\Gamma'(p)| + M d_{\Gamma_n}(\Gamma(p))\,|\Gamma'(p)|$$

Consider the term $\nabla d_{\Gamma_n}(\Gamma(p))\cdot\Gamma'(p)$. We write the following first order Taylor expansion without remainder

$$0 = d_{\Gamma_n}(\Gamma_n(p)) = d_{\Gamma_n}(\Gamma(p)) + \\ \left(\int_0^1 (1-\zeta)\nabla d_{\Gamma_n}(\Gamma(p) + \zeta(\Gamma_n(p) - \Gamma(p)))\,d\zeta\right)\cdot(\Gamma_n(p) - \Gamma(p))$$

We take the derivative with respect to p:

$$\nabla d_{\Gamma_n}(\Gamma(p))\cdot\Gamma'(p) + \left(\int_0^1 (1-\zeta)\nabla d_{\Gamma_n}(\Gamma(p) + \zeta(\Gamma_n(p) - \Gamma(p)))\,d\zeta\right)\cdot(\Gamma_n'(p) - \Gamma'(p)) + \\ \left(\int_0^1 (1-\zeta)\nabla^2 d_{\Gamma_n}(\Gamma(p) + \zeta(\Gamma_n(p) - \Gamma(p)))(\Gamma'(p) + \zeta(\Gamma_n'(p) - \Gamma'(p)))\,d\zeta\right)\cdot(\Gamma_n(p) - \Gamma(p)),$$

and obtain the upper bound

$$|\nabla d_{\Gamma_n}(\Gamma(p))\cdot\Gamma'(p)| \leq \frac{1}{2}|\Gamma_n'(p) - \Gamma'(p)| + Ad(\Gamma_n(p), \Gamma(p)).$$

We use it in (19) to obtain

$$\frac{1}{2}|\Gamma_n'(p) - \Gamma'(p)| \leq Ad(\Gamma_n(p), \Gamma(p)) + M d_{\Gamma_n}(\Gamma(p))\,|\Gamma'(p)|,$$

from which follows

$$(20) \qquad |\,|\partial\Omega_n| - |\partial\Omega|\,| \leq 2(A + M|\partial\Omega|)\varepsilon.$$

We next prove that for all Lipschitz continuous functions f on D, the integral $\int_\Gamma f(x)\,d\Gamma(x)$ is continuous for the Hausdorff topology. Consider a shape Ω and a sequence $\{\Omega_n\}$ of shapes of $\mathcal{S}$ converging to Ω for the Hausdorff topology, we show that $\lim_{n\to\infty}\left|\int_{\Gamma_n} f(x)\,d\Gamma_n(x) - \int_\Gamma f(y)\,d\Gamma(y)\right| = 0$. We use once more the parametrization (18) and write

$$\begin{aligned}\left|\int_{\Gamma_n} f(x)\,d\Gamma_n(x) - \int_\Gamma f(y)\,d\Gamma(y)\right| = &\\ \left|\int_0^1 \left(f(\Gamma_n(p))|\Gamma_n'(p)| - f(\Gamma(p))|\Gamma'(p)|\right)\,dp\right| \le &\\ \int_0^1 \left|f(\Gamma_n(p))|\Gamma_n'(p)| - f(\Gamma(p))|\Gamma'(p)|\right|\,dp \le &\\ \int_0^1 |f(\Gamma_n(p))|\,\left||\Gamma_n'(p)| - |\Gamma'(p)|\right|\,dp + \int_0^1 |f(\Gamma_n(p)) - f(\Gamma(p))|\,|\Gamma'(p)|\,dp &\end{aligned}$$

f is continuous on the compact set $\overline{D}$ and is therefore upperbounded, $|f(x)| \le K$, $\forall x \in \overline{D}$. It is also Lipschitz continuous, hence $|f(\Gamma_n(p)) - f(\Gamma(p))| \le Ld(\Gamma_n(p), \Gamma(p)) \le L\varepsilon$. We combine this with (20) and obtain

$$\left|\int_{\Gamma_n} f(x)\,d\Gamma_n(x) - \int_\Gamma f(y)\,d\Gamma(y)\right| \le \varepsilon((L + 2KM)|\partial\Omega| + 2KA).$$

We have used the Lipschitz hypothesis in the proof. It easy to verify that this hypothesis is satisfied since we are integrating along curves functions of the type $\varphi \circ d(\cdot, x)$. The functions φ are defined and at least C^1, hence Lipschitz continuous on $[0, diam(D)]$, where $diam(D)$ is the diameter of D. Hence $|\varphi\circ d(x_1, x) - \varphi\circ d(x_2, x)| \le L_\varphi |d(x_1, x) - d(x_2, x)| \le L_\varphi d(x_1, x_2)$. □

5.3 Computing the gradient of the approximation to the Hausdorff distance

We now proceed with showing that the approximation $\tilde{\rho}_H(\Gamma, \Gamma_0)$ of the Hausdorff distance $\rho_H(\Gamma, \Gamma_0)$ is differentiable with respect to Γ and compute its gradient $\nabla\tilde{\rho}_H(\Gamma, \Gamma_0)$, in the sense of section 4. To simplify notations we rewrite (16) as

$$\tilde{\rho}_H(\Gamma, \Gamma_0) = \left\langle \left\langle \langle d(\cdot,\cdot)\rangle^{\varphi}_{\Gamma_0}\right\rangle^{\psi}_{\Gamma},\, \left\langle \langle d(\cdot,\cdot)\rangle^{\varphi}_{\Gamma}\right\rangle^{\psi}_{\Gamma_0} \right\rangle^{\theta}, \tag{21}$$

and state the result, the reader interested in the proof being referred to the appendix A.

Proposition 12. *The gradient of $\tilde{\rho}_H(\Gamma, \Gamma_0)$ at any point y of Γ is given by*

$$\nabla\tilde{\rho}_H(\Gamma, \Gamma_0)(y) = \frac{1}{\theta'(\tilde{\rho}_H(\Gamma, \Gamma_0))}\left(\alpha(y)\kappa(y) + \beta(y)\right), \tag{22}$$

where $\kappa(y)$ is the curvature of Γ at y, the functions $\alpha(y)$ and $\beta(y)$ are given by

$$\begin{aligned}(23)\quad \alpha(y) = \nu \int_{\Gamma_0} \frac{\psi'}{\varphi'}\left(\langle d(x,\cdot)\rangle_\Gamma^\varphi\right) \; [\, \varphi \circ \langle d(x,\cdot)\rangle_\Gamma^\varphi - \varphi \circ d(x,y)\,]\, d\Gamma_0(x) \\ + |\Gamma_0|\eta\, \left[\psi\left(\left\langle\langle d(\cdot,\cdot)\rangle_{\Gamma_0}^\varphi\right\rangle_\Gamma^\psi\right) - \psi\left(\langle d(\cdot,y)\rangle_{\Gamma_0}^\varphi\right)\,\right],\end{aligned}$$

$$(24)\quad \beta(y) = \int_{\Gamma_0} \varphi' \circ d(x,y)\, \left[\nu \frac{\psi'}{\varphi'}\left(\langle d(x,\cdot)\rangle_\Gamma^\varphi\right) + \eta \frac{\psi'}{\varphi'}\left(\langle d(\cdot,y)\rangle_{\Gamma_0}^\varphi\right)\right]\, \frac{y-x}{d(x,y)} \cdot n(y)\; d\Gamma_0(x),$$

where $\nu = \dfrac{1}{|\Gamma|\,|\Gamma_0|} \dfrac{\theta'}{\psi'}\left(\left\langle\langle d(\cdot,\cdot)\rangle_\Gamma^\varphi\right\rangle_{\Gamma_0}^\psi\right)$ *and* $\eta = \dfrac{1}{|\Gamma|\,|\Gamma_0|}\dfrac{\theta'}{\psi'}\left(\left\langle\langle d(\cdot,\cdot)\rangle_{\Gamma_0}^\varphi\right\rangle_\Gamma^\psi\right)$.

Note that the function $\beta(y)$ is well-defined even if y belongs to Γ_0 since the term $\frac{y-x}{d(x,y)}$ is of unit norm.

The first two terms of the gradient show explicitely that minimizing the energy implies homogenizing the distance to Γ_0 along the curve Γ, that is to say the algorithm will take care in priority of the points of Γ which are the furthest from Γ_0.

We should also note that this result holds independently of the fact that the curves are open or closed (see appendix A).

5.4 Other alternatives related to the Hausdorff distance

There exist several alternatives to the method presented in the previous sections if we use ρ (equation (2)) rather than ρ_H (equation (3)) to define the Hausdorff distance. A first alternative is to use the following approximation

$$\tilde{\rho}(\Gamma,\Gamma') = \langle |d_\Gamma - d_{\Gamma'}|\rangle_D^{p_\alpha},$$

where the bracket $\langle\, f(.)\,\rangle_D^\varphi$ is defined the obvious way for any integrable function $f : D \to \mathbb{R}^+$

$$\langle\, f\,\rangle_D^\varphi = \varphi^{-1}\left(\frac{1}{m(D)}\int_D \varphi(f(x))\, dx\right),$$

and which can be minimized, as in section 5.6, with respect to d_Γ. A second alternative is to approximate ρ using:

$$(25)\qquad \tilde{\rho}(\Gamma,\Gamma') = \langle |\langle d(\cdot,\,\cdot)\rangle_{\Gamma'}^{\varphi_\beta} - \langle d(\cdot,\,\cdot)\rangle_\Gamma^{\varphi_\beta}|\rangle_D^{p_\alpha},$$

and to compute is derivative with respect to Γ as we did in the previous section for $\tilde{\rho}_H$.

5.5 Approximations to the $W^{1,2}$ norm and computation of their gradient

The previous results can be used to construct approximations $\tilde{\rho}_D$ to the distance ρ_D defined in section 2.2.3:

$$\tilde{\rho}_D(\Gamma_1, \Gamma_2) = \|\tilde{d}_{\Gamma_1} - \tilde{d}_{\Gamma_2}\|_{W^{1,2}(D)}, \tag{26}$$

where $\tilde{d}_{\Gamma_i}$, $i = 1, 2$ is obtained from (15).

This approximation is also "nice" in the usual way and we have the

Proposition 13. *For each α in $\mathbb{R}^{+*}$ the function $\tilde{\rho}_D : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$ is continuous for the $W^{1,2}$ topology.*

Its proof is left to the reader.

The gradient $\nabla\tilde{\rho}_D(\Gamma, \Gamma_0)$, of our approximation $\tilde{\rho}_D(\Gamma, \Gamma_0)$ of the distance $\rho_D(\Gamma, \Gamma_0)$ given by (26) in the sense of section 4 can be computed. The interested reader is referred to appendix B. We simply state the result in the

Proposition 14. *The gradient of $\tilde{\rho}_D(\Gamma, \Gamma_0)$ at any point y of Γ is given by*

$$\nabla\tilde{\rho}_D(\Gamma, \Gamma_0)(y) = \int_D \left[B(x,y) \left(C_1(x) - \frac{\varphi''}{\varphi'}(\tilde{d}_\Gamma(x)) \left(\mathbf{C}_2(x) \cdot \nabla\tilde{d}_\Gamma(x) \right) \right) + \mathbf{C}_2(x) \cdot \nabla B(x,y) \right] dx, \tag{27}$$

where

$$B(x,y) = \kappa(y) \left(\langle \varphi \circ d(x,\cdot) \rangle_\Gamma - \varphi \circ d(x,y) \right) + \varphi'(d(x,y)) \frac{y-x}{d(x,y)} \cdot \mathbf{n}(y),$$

$\kappa(y)$ is the curvature of Γ at y,

$$C_1(x) = \frac{1}{|\Gamma|\ \varphi'(\tilde{d}_\Gamma(x))} \|\tilde{d}_\Gamma - \tilde{d}_{\Gamma_0}\|^{-1}_{L^2(D)} \ \left(\tilde{d}_\Gamma(x) - \tilde{d}_{\Gamma_0})(x) \right),$$

and

$$\mathbf{C}_2(x) = \frac{1}{|\Gamma|\ \varphi'(\tilde{d}_\Gamma(x))} \|\nabla(\tilde{d}_\Gamma - \tilde{d}_{\Gamma_0})\|^{-1}_{\mathbf{L}^2(D)} \ \nabla(\tilde{d}_\Gamma - \tilde{d}_{\Gamma_0})(x),$$

5.6 Direct minimization of the $W^{1,2}$ norm

An alternative to the method presented in the previous section is to evolve not the curve Γ but its distance function d_Γ. Minimizing $\rho_D(\Gamma, \Gamma_0)$ with respect to d_Γ implies computing the corresponding Euler-Lagrange equation EL. The reader will verify that the result is

$$EL = \frac{d_\Gamma - d_{\Gamma_0}}{\|d_\Gamma - d_{\Gamma_0}\|_{L^2(D)}} - div \left(\frac{\nabla\left(d_\Gamma - d_{\Gamma_0}\right)}{\|\nabla(d_\Gamma - d_{\Gamma_0})\|_{\mathbf{L}^2(D)}} \right) \tag{28}$$

To simplify notations we now use d instead of d_Γ. The problem of warping Γ_1 onto Γ_0 is then transformed into the problem of solving the following PDE

$$d_t = -EL$$
$$d(0,\cdot) = d_{\Gamma_1}(\cdot).$$

The problem that this PDE does not preserve the fact that d is a distance function is alleviated by "reprojecting" at each iteration the current function d onto the set of distance functions by running a few iterations of the "standard" restoration PDE [44]

$$d_t = (1 - |\nabla d|)sign(d_0)$$
$$d(0,\cdot) = d_0$$

6 Application to curve evolutions: Hausdorff warping

In this section we show a number of examples of solving equation (9) with the gradient given by equation (22). Our hope is that, starting from Γ_1, we will follow the gradient (22) and smoothly converge to the curve Γ_2 where the minimum of $\tilde{\rho}_H$ is attained. Let us examine more closely these assumptions. First, it is clear from the expression (16) of $\tilde{\rho}_H$ that in general $\tilde{\rho}_H(\Gamma,\Gamma) \neq 0$, which implies in particular that $\tilde{\rho}_H$, unlike ρ_H, is not a distance. But worse things can happen: there may exist a shape Γ' such that $\tilde{\rho}_H(\Gamma,\Gamma')$ is strictly less than $\tilde{\rho}_H(\Gamma,\Gamma)$ or there may not exist any minima for the function $\Gamma \to \tilde{\rho}_H(\Gamma,\Gamma')$! This sounds like the end of our attempt to warp a shape onto another using an approximation of the Hausdorff distance. But things turn out not to be so bad. First, the existence of a minimum is guaranteed by proposition 10 which says that $\tilde{\rho}_H$ is continuous on $\mathcal{S}$ for the Hausdorff topology, theorem 9 which says that $\mathcal{DZ}_0$ is compact for this topology, and proposition 9 which tells us that the l.s.c. extension of $\tilde{\rho}_H(\cdot,\Gamma)$ has a minimum in the closure $\overline{\mathcal{S}}$ of $\mathcal{S}$ in $\mathcal{DZ}_0$.

We show in the next section that phenomena like the one described above are for all practical matters "invisible" since confined to an arbitrarily small Hausdorff ball centered at Γ.

6.1 Quality of the approximation $\tilde{\rho}_H$ of ρ_H

In this section we make more precise the idea that $\tilde{\rho}_H$ can be made arbitrarily close to ρ_H. Because of the form of (21) we seek upper and lower bounds of such quantities as $\langle f\rangle_\Gamma^\psi$, where f is a continuous real function defined on Γ. We note $f_{\max}$ and $f_{\min}$ the maximum and minimum values of f on Γ.

The expression

$$\langle f\rangle_\Gamma^\psi = \psi^{-1}\left(\frac{1}{|\Gamma|}\int_\Gamma \psi \circ f\right),$$

yields, if ψ is strictly increasing:

$$\langle f\rangle_\Gamma^\psi \leqslant \psi^{-1}\left(\frac{1}{|\Gamma|}\int_\Gamma \psi\circ f_{\max}\right) = f_{\max}$$

and, similarly:

$$\langle f\rangle_\Gamma^\psi \geqslant f_{\min}$$

If $f \geqslant f_{\text{moy}}$ on a set F of the curve Γ, of length $|F|$ $(\leqslant |\Gamma|)$:

$$\begin{aligned}
\langle f\rangle_\Gamma^\psi &= \psi^{-1}\left(\frac{1}{|\Gamma|}\int_F \psi\circ f + \frac{1}{|\Gamma|}\int_{\Gamma\setminus F}\psi\circ f\right)\\
&\geqslant \psi^{-1}\left(\frac{|F|}{|\Gamma|}\psi\circ f_{\text{moy}} + \frac{|\Gamma|-|F|}{|\Gamma|}\psi\circ f_{\min}\right)\\
&\geqslant \psi^{-1}\left(\frac{|F|}{|\Gamma|}\psi\circ f_{\text{moy}}\right)
\end{aligned}$$

To analyse this lower bound, we introduce the following notation. Given Δ, $\alpha \geqslant 0$, we note $\mathcal{P}(\Delta,\alpha)$ the following property:

$$\mathcal{P}(\Delta,\alpha): \quad \forall x\in\mathbb{R}^+,\ \Delta\psi(x)\geqslant\psi(\alpha x)$$

This property is satisfied for $\psi(x) = x^\beta$, $\beta \geq 0$. The best pairs (Δ,α) verifying $\mathcal{P}$ are such that $\Delta = \alpha^\beta$. In the sequel, we say that a function ψ is admissible for $\mathcal{P}$ if

$$\forall\Delta\in]0;1[,\ \exists\alpha\in]0;1[, \mathcal{P}(\Delta,\alpha),$$

and, conversely,

$$\forall\alpha\in]0;1[,\ \exists\Delta\in]0;1[, \mathcal{P}(\Delta,\alpha)$$

Let us assume that ψ is admissible and note that we can rewrite $\mathcal{P}(\Delta,\alpha)$

$$\forall x\in\mathbb{R}^+,\ \psi^{-1}\left(\Delta\psi(x)\right)\geqslant\alpha x.$$

For $\Delta = \frac{|F|}{|\Gamma|}$ and $x = f_{\text{moy}}$ we obtain for the largest $\alpha(\Delta)$ the following lowerbound

$$\langle f\rangle_\Gamma^\psi \geqslant \psi^{-1}\left(\Delta\psi(f_{\text{moy}})\right)\geqslant \alpha f_{\text{moy}}.$$

In words, for each arbitrary percentage Δ there exists an α such that if $|\{f \geqslant f_{\text{moy}}\}| \geqslant \Delta|\Gamma|$, then $\langle f\rangle_\Gamma^\psi \geqslant \alpha f_{\text{moy}}$. Conversely, for a given value of α, there exists a Δ such that it is sufficient that $|\{f \geqslant f_{\text{moy}}\}| \geqslant \Delta|\Gamma|$ to have $\langle f\rangle_\Gamma^\psi \geqslant \alpha f_{\text{moy}}$.

For each choice of (Δ,α), the bracket $\langle f\rangle_\Gamma^\psi$ acts as a filter which only "looks" at the values of f along Γ such that the subset F of Γ where they are reached is of relative length

$\frac{|F|}{|\Gamma|} \geqslant \Delta$, meaning that one neglects the "details of relative importance $\leqslant \Delta$", and that the accuracy of the filter is relative, since it depends upon the product of α ($\leqslant 1$) with f_{moy}.

One has even more: the above admissible family of functions ψ allows one to select an arbitrary accuracy, i.e. to choose both Δ as close as possible to 0, and α as close as possible to 1, the best pairs (α, Δ) for $\psi(x) = x^\beta$ satisfying $\Delta = \alpha^\beta$, it is sufficient to choose β large enough.

Similar properties hold for such brackets as $\langle f \rangle_\Gamma^\varphi$ where φ is strictly decreasing. We have, as in the previous case:

$$f_{\min} \leqslant \langle f \rangle_\Gamma^\varphi \leqslant f_{\max}$$

Proceeding as before,if $|\{f \leqslant f_{\text{moy}}\}| \geqslant \Delta|\Gamma|$ and the pair (Δ, α) satisfies $\mathcal{P}$ for the function φ, we obtain:

$$\begin{aligned} \frac{1}{|\Gamma|}\int_\Gamma \varphi \circ f &\geqslant \Delta\varphi(f_{\text{moy}}) \\ \langle f \rangle_\Gamma^\varphi &\leqslant \varphi^{-1}\left(\Delta\varphi(f_{\text{moy}})\right) \\ \langle f \rangle_\Gamma^\varphi &\leqslant \alpha f_{\text{moy}} \end{aligned}$$

Admissible functions are $\varphi(x) = x^{-\beta}$, $\beta > 0$; the accuracy increases when α tends to 1^- and Δ to 0^+; this is always possible by choosing large values of β, and $\Delta = \alpha^{-\beta}$.

We now have all the ingredients for comparing $\tilde{\rho}_H$ and ρ_H. We start with two definitions.

Definition 14. *Let Γ be a shape. For each point P of D we note (see figure 8):*

$$d_\Delta(P, \Gamma) = \inf\left\{x \in \mathbb{R}^+ ; |\{Q \in \Gamma; d(P,Q) \leqslant x\}| \geqslant \Delta|\Gamma|\right\}$$

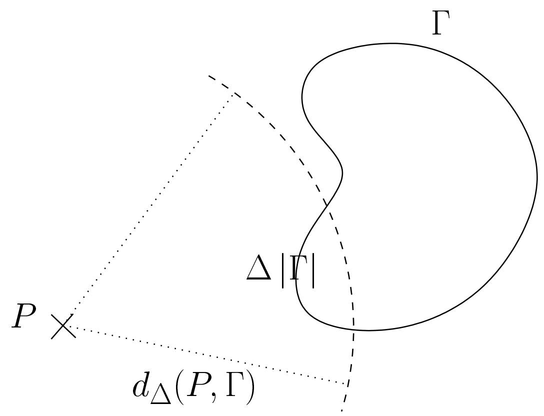

Figure 8: Geometric interpretation of $d_\Delta(P,\Gamma)$: Δ is the "percentage" of points of Γ whose distance to P is less than $d_\Delta(P,\Gamma)$.

Definition 15. *Let Γ and Γ' be two shapes, we define (see figure 9)*

$$d^{\Delta}(\Gamma',\Gamma) = \sup\left\{x \in \mathbb{R}^+ ; |\{Q \in \Gamma ; d(Q,\Gamma') \geqslant x\}| \geqslant \Delta|\Gamma|\right\}$$

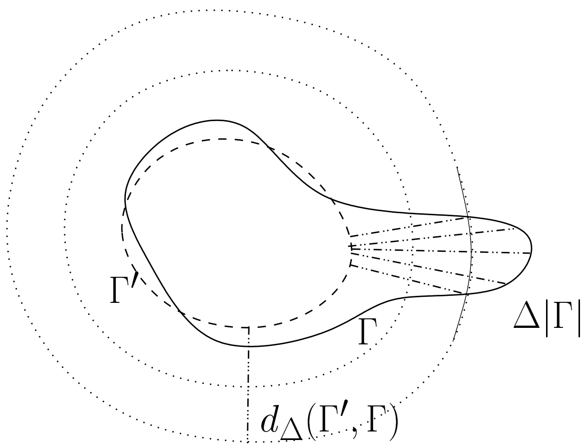

Figure 9: Geometric interpretation of $d^{\Delta}(\Gamma',\Gamma)$: Δ is the "percentage" of points of Γ whose distance to Γ' is greater than $d^{\Delta}(\Gamma',\Gamma)$.

If φ (respectively ψ)is an admissible function, we note $(\Delta_\varphi,\, \alpha_\varphi)$ (respectively $(\Delta_\psi,\, \alpha_\psi)$) a pair (Δ,α) for the bracket $\langle\cdot\rangle_\Gamma^\varphi$ (respectively, $\langle\cdot\rangle_\Gamma^\psi$).

The following proposition relates $\tilde\rho_H$ to d_Δ and d^Δ.

Proposition 15. *The following relation is satisfied by $\tilde\rho_H$, d_Δ and d^Δ:*

$$\alpha_\psi\alpha_\psi \max\left(d^{\Delta_\psi}(\Gamma,\Gamma'), d^{\Delta_\psi}(\Gamma',\Gamma)\right) \leqslant \tilde\rho_H(\Gamma,\Gamma') \leqslant \alpha_\varphi \max\left(\sup_{P\in\Gamma'} d_{\Delta_\varphi}(P,\Gamma), \sup_{P\in\Gamma} d_{\Delta_\varphi}(P,\Gamma')\right)$$

Proof. We notice that

$$\forall P \in \mathbb{R}^2, \quad d(P,\Gamma) \leqslant \langle d(P,\cdot)\rangle_\Gamma^\varphi \leqslant \alpha_\varphi d_{\Delta_\varphi}(P,\Gamma),$$

and therefore

$$\alpha_\psi \;\; d^{\Delta_\psi}(\Gamma,\Gamma') \;\; \leqslant \;\; \langle\langle d(\cdot,\cdot)\rangle_\Gamma^\varphi\rangle_{\Gamma'}^\psi \;\; \leqslant \;\; \alpha_\varphi \;\; \sup_{P\in\Gamma'} d_{\Delta_\varphi}(P,\Gamma).$$

$\tilde\rho_H$ is a discrete bracket $\langle\cdot,\cdot\rangle_\theta$ of two such terms, θ an increasing function. We note α_θ an α associated to $\Delta = \frac{1}{2}$ through $\mathcal{P}$ for θ. For all positive a and b we have

$$\theta^{-1}\left(\frac{1}{2}\theta(\max(a,b))\right) \leqslant \langle a,b\rangle_\theta \leqslant \max(a,b)$$

$$\alpha_\theta \max(a,b) \leqslant \langle a,b \rangle_\theta \leqslant \max(a,b),$$

from where the conclusion follows. ☐

We now relate d_Δ and d^Δ to the Hausdorff distance ρ_H.

Proposition 16. *For all $P \in D$ and for all shapes Γ and Γ' we have*

$$d(P,\Gamma) \leqslant d_\Delta(P,\Gamma) \leqslant d(P,\Gamma) + \frac{\Delta}{2}|\Gamma|,$$

and

$$d_H(\Gamma,\Gamma') - \Delta\frac{|\Gamma|+|\Gamma'|}{2} \leqslant d^\Delta(\Gamma',\Gamma) \leqslant d_H(\Gamma,\Gamma') + \Delta\frac{|\Gamma|+|\Gamma'|}{2}$$

Proof. The lowerbound on $d_\Delta(P,\Gamma)$ is easy to obtain, the upperbound can be obtained by contradiction as follows: let us assume that there exists a point P and a curve Γ such that the upperbound is not satisfied. Hence

$$d_\Delta(P,\Gamma) > d(P,\Gamma) + \frac{\Delta}{2}|\Gamma|$$

Γ being compact, there exists a point Q de Γ such that $d(P,Q) = d(P,\Gamma)$. Let us now consider Γ as a C^2 function from $[0,1]$ to $\mathbb{R}^2$ such that $|\Gamma'(p)| = \mathrm{c}^{\mathrm{ste}} = |\Gamma|$ for all ps in $[0,1]$. Let $q \in [0,1]$ such that $\Gamma(q) = Q$, and consider the image by Γ of $I = \{p|\ |p-q| \leqslant \Delta/2\}$ (assuming $q \in]\Delta/2, 1-\Delta/2[$, otherwise the proof can be easily modified). By construction

$$|\Gamma(I)| = |I||\Gamma| = \Delta|\Gamma|,$$

and for all point R of $\Gamma(I)$ of parameter r

$$PR \leqslant PQ + QR \leqslant d(P,A) + |r-q||\Gamma| \leqslant d(P,A) + \frac{1}{2}\Delta|A|.$$

We have found a measurable subset of the curve Γ of length larger than or equal to $\Delta|\Gamma|$ such that all its points are at a distance of P less than $d(P,\Gamma) + \frac{1}{2}\Delta|\Gamma|$, a contradiction.

The proof of the second set of inequalities proceeds in a similar fashion by considering subsets of the curves Γ and Γ' centered at points P of Γ and Q of Γ' such that $\rho_H(\Gamma,\Gamma') = d(P,Q)$; this is always possible since Γ and Γ' are compact. ☐

By combining propositions 15 and 16 we obtain the

Proposition 17. *$\tilde{\rho}_H(\Gamma,\Gamma')$ has the following upper and lower bounds*

$$\alpha_\theta\alpha_\psi(\rho_H(\Gamma,\Gamma') - \Delta_\psi\frac{|\Gamma|+|\Gamma'|}{2}) \leq \tilde{\rho}_H(\Gamma,\Gamma') \leq \alpha_\varphi(\rho_H(\Gamma,\Gamma') + \Delta_\varphi\frac{|\Gamma|+|\Gamma'|}{2}). \tag{29}$$

We can now characterize the shapes Γ and Γ' such that

$$\tilde{\rho}_H(\Gamma,\,\Gamma') < \tilde{\rho}_H(\Gamma,\,\Gamma). \tag{30}$$

Theorem 16. *The condition (30) is equivalent to*

$$\rho_H(\Gamma, \Gamma') < 4c_0\Delta,$$

where the constant c_0 is defined in definition 5 and theorem 3, and Δ in the proof.

Proof. We use the upper and lower bounds (29) derived in proposition 17 and write

$$\alpha_\theta\alpha_\psi(\rho_H(\Gamma, \Gamma') - \Delta_\psi \frac{|\Gamma| + |\Gamma'|}{2}) < \alpha_\varphi\Delta_\varphi|\Gamma|$$

To simplify the analysis, let us assume that $\alpha_\theta\alpha_\psi = \alpha_\varphi$ and $\Delta_\psi = \Delta_\varphi = \Delta$, we obtain

$$\rho_H(\Gamma, \Gamma') < (\frac{3}{2}|\Gamma| + \frac{1}{2}|\Gamma'|)\Delta,$$

and hence (proposition 3)

$$\rho_H(\Gamma, \Gamma') < 4c_0\Delta,$$

Conversely, if Γ' is not in the Hausdorff ball with center Γ and radius $4c_0\Delta$, we necessarily have $\tilde{\rho}_H(\Gamma, \Gamma') > \tilde{\rho}_H(\Gamma, \Gamma)$. □

From this we conclude that, since Δ can be made arbitrarily close to 0, and the length of shapes is bounded, strange phenomena such as a shape Γ' closer to a shape Γ than Γ itself (in the sense of $\tilde{\rho}_H$) cannot occur or rather will be "invisible" to our algorithms.

6.2 Applying the theory

In practice, the Energy that we minimize is not $\tilde{\rho}_H$ but in fact a "regularized" version obtained by combining $\tilde{\rho}_H$ with a term E_L which depends upon the lengths of the two curves. A natural candidate for E_L is $\max(|\Gamma|, |\Gamma'|)$ since it acts only if $|\Gamma|$ becomes larger than $|\Gamma'|$, thereby avoiding undesirable oscillations. To obtain smoothness, we approximate the max with a Ψ-average:

$$E_L(|\Gamma|, |\Gamma'|) = \langle |\Gamma|, |\Gamma'| \rangle^\Psi \tag{31}$$

We know that the function $\Gamma \to |\Gamma|$ is in general l.s.c.. It is in fact continuous on $\mathcal{S}$ (see the proof of proposition 10) and takes its values in the interval $[0, 2c_0]$, hence

Proposition 18. *The function $\mathcal{S} \to \mathbb{R}^+$ given by $\Gamma \to E_L(\Gamma, \Gamma')$ is continuous for the Hausdorff topology.*

Proof. It is clear since E_L is a combination of continuous functions. □

We combine E_L with $\tilde{\rho}_H$ the expected way, i.e. by computing their $\tilde{\Psi}$ average so that the final energy is

$$E(\Gamma, \Gamma') = \langle \tilde{\rho}_H(\Gamma, \Gamma'), E_L(|\Gamma|, |\Gamma'|) \rangle^{\tilde{\Psi}} \tag{32}$$

The function $E : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$ is continuous for the Hausdorff metric because of propositions 10 and 18 and therefore

Proposition 19. *The function* $\Gamma \to E(\Gamma, \Gamma')$ *defined on the set of shapes* $\mathcal{S}$ *has at least a minimum in the closure* $\overline{\mathcal{S}}$ *of* $\mathcal{S}$ *in* $\mathcal{L}_0$.

Proof. This is a direct application of proposition 9 applied to the function E. □

We call the resulting warping technique the *Hausdorff* warping. A first example, the Hausdorff warping of two circles, is shown in figure 10. A second example, the Hausdorff warping of two hand silhouettes, is shown in figure 11

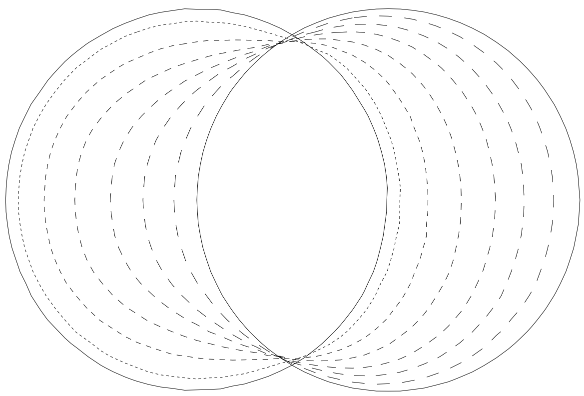

Figure 10: The result of the Hausdorff warping of two circles. The two circles are represented in continuous line while the intermediate shapes are represented in dotted lines.

We have borrowed the example in figure 12 from the database (`www.ee.surrey.ac.uk/Research/VSSP/imagedb/demo.html`) of fish silhouettes collected by the researchers of the University of Surrey at the center for Vision, Speech and Signal Processing (`www.ee.surrey.ac.uk/Research/VSSP`). This database contains 1100 silhouettes. A few steps of the result of Hausdorff warping one of these silhouettes onto another are shown in figure 12. Another similar example is shown in figure 13. Note that, prior to warpint, the two shapes have been normalized in such a way as to align their centers of gravity and their principle axes.

Figures 14 and 15 give a better understanding of the behavior of Hausdorff warping. A slightly displaced detail "warps back" to its original place (figure 14). Displaced further, the same detail is considered as another one and disappears during the warping process while the original one reappears (figure 15).

Finally, figure 16 shows the Hausdorff warping of an open curve to another.

Note also that other warpings are given by the minimization of other approximations of the Hausdorff distance. Figure 17 shows the warping of a rough curve to the silhouette of a fish and bubbles given by the minimization of the $W^{1,2}$ norm as explained in section

Figure 11: The result of the Hausdorff warping of two hand silhouettes. The two hands are represented in continuous line while the intermediate shapes are represented in dotted lines.

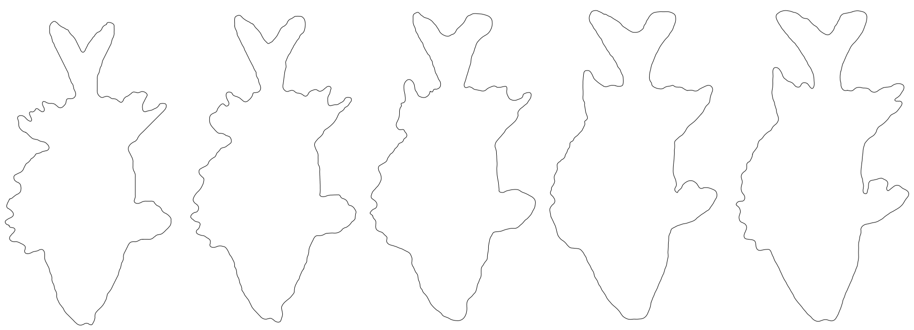

Figure 12: Hausdorff warping a fish onto another.

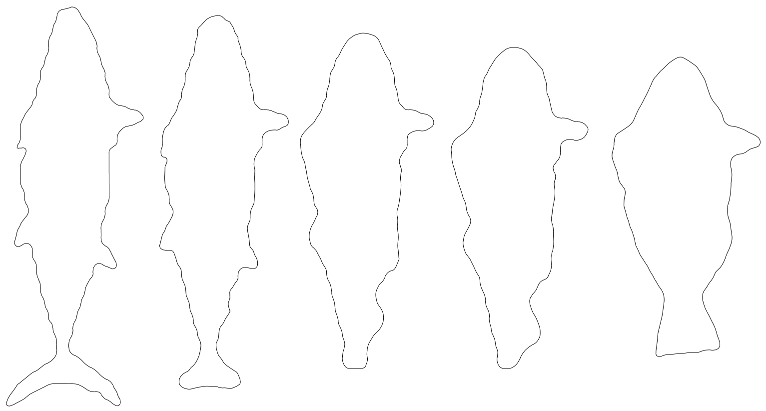

Figure 13: Another example of fish Hausdorff warping.

5.6. Our "level sets" implementation (see section 8) can deal with the splitting of the source curve while warping onto the target one.

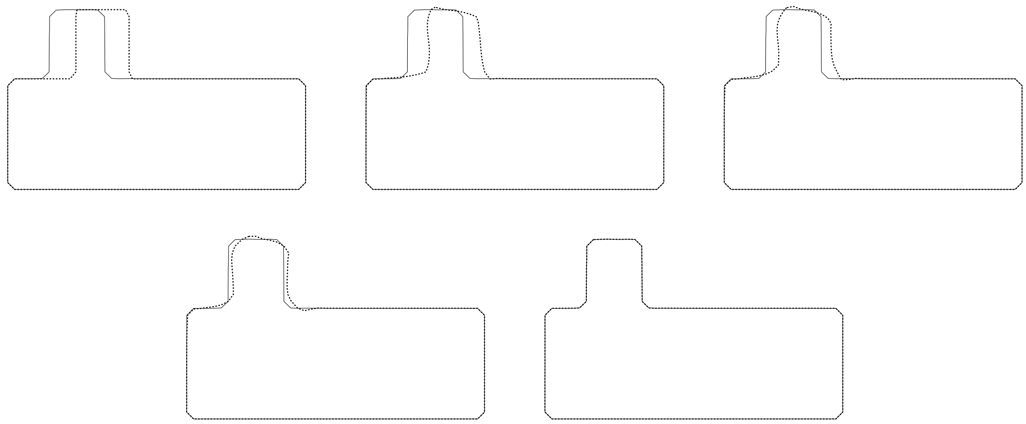

Figure 14: Hausdorff warping boxes (i). A translation-like behaviour.

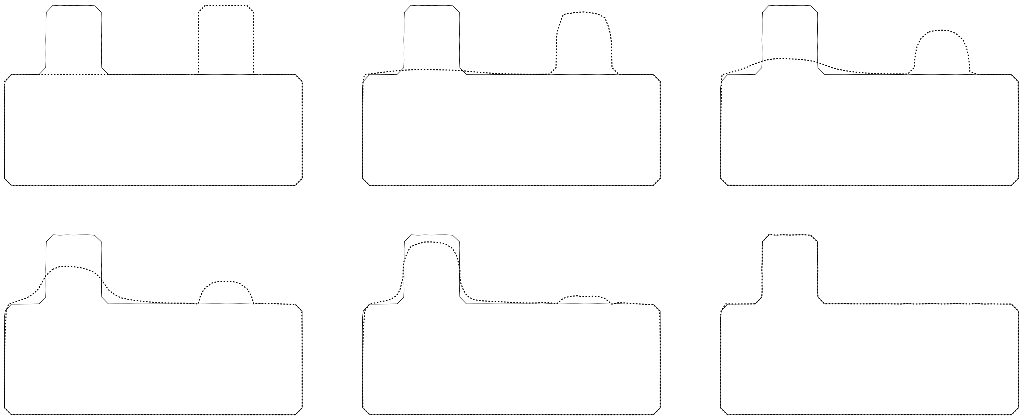

Figure 15: Hausdorff warping boxes (ii). A different behaviour: a detail disappears while another one appears.

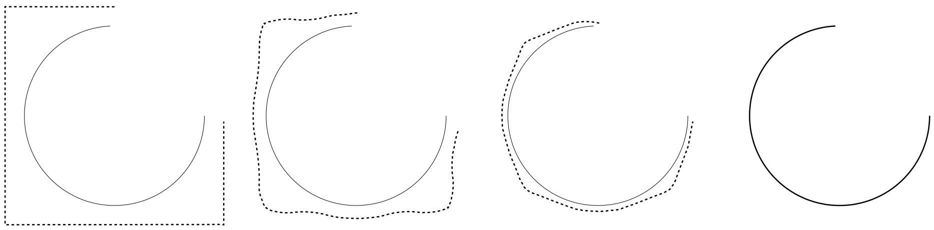

Figure 16: Hausdorff warping an open curve to another one.

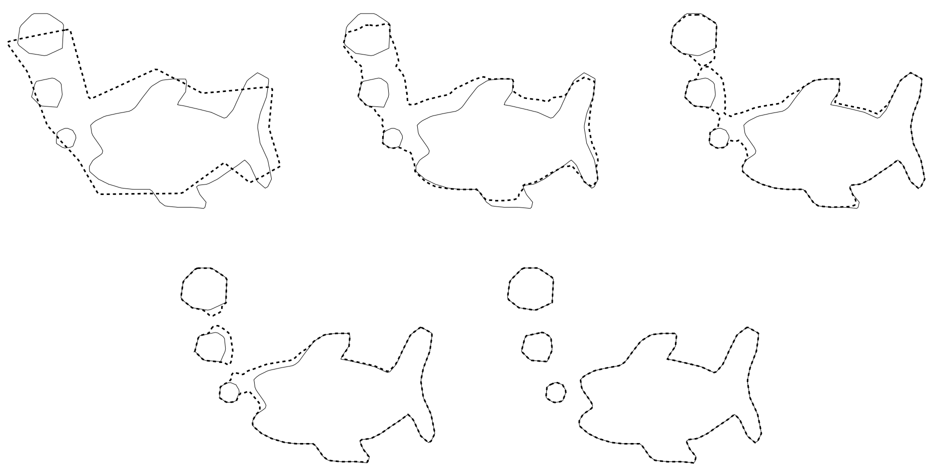

Figure 17: Splitting while warping.

7 Application to the computation of the empirical mean and covariance of a set of shape examples

We have now developed the tools for defining several concepts relevant to a theory of stochastic shapes as well as providing the means for their effective computation. They are based on the use of the function E defined by (32).

7.1 Empirical mean

The first one is that of the mean of a set of shapes. Inspired by the work of Fréchet [17, 18], Karcher [25], Kendall [28], and Pennec [38], we provide the following (classical)

Definition 17. *Given* $\Gamma_1, \cdots, \Gamma_N$*, N shapes, we define their empirical mean as any shape* $\hat{\Gamma}$ *that achieves a local minimimum of the function* $\mu : \mathcal{S} \to \mathbb{R}^+$ *defined by*

$$\Gamma \to \mu(\Gamma, \Gamma_1, \cdots, \Gamma_N) = \frac{1}{N} \sum_{i=1,\cdots,N} E^2(\Gamma, \Gamma_i)$$

Note that there may exist several means. We know from proposition 19 that there exists at least one. An algorithm for computing approximations to an empirical mean of N shapes readily follows from the previous section: start from an initial shape Γ_0 and solve the PDE

$$\begin{aligned} \Gamma_t &= -\nabla\mu(\Gamma, \Gamma_1, \cdots, \Gamma_N)\,\mathbf{n} \\ \Gamma(0, .) &= \Gamma_0(.) \end{aligned} \tag{33}$$

We show some examples of means computed by this algorithm in figure 18.

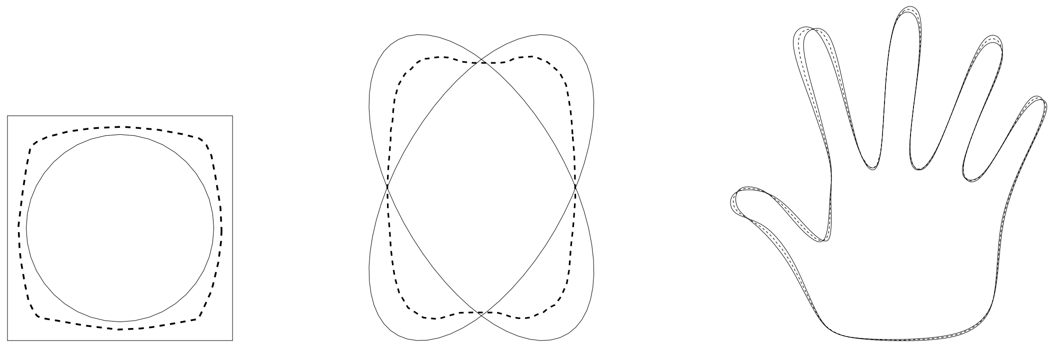

Figure 18: Examples of means of several curves: a square and a circle (left), two ellipses (middle), and two hands (right).

When the number of shapes grows larger, the question of the local minima of μ becomes a problem (see figure 19) and the choice of Γ_0 in (33) an important issue. We have not

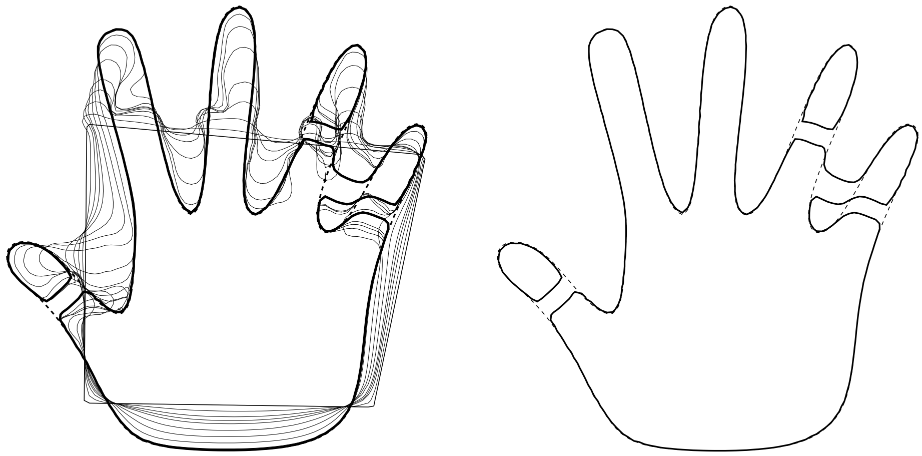

Figure 19: Example of an evolution which leads to a local minimum : mean of the same nine hands obtained by initializing with another curve (the bold quadrilateral, the expected mean being the dashed curve).

explored these questions in great detail but observed that the following heuristics led to "visually satisfying" results.

Suppose that the example shapes are given in some order, according to the way they are indexed from 1 to N. Initialize $\hat{\Gamma}^{(1)}$ to Γ_1, solve

$$\Gamma_t^{(i+1)} = -\nabla\mu(\Gamma, i\hat{\Gamma}^{(i)}, \Gamma_i)\,\mathbf{n}$$
$$\Gamma^{(i+1)}(0,.) = \hat{\Gamma}^i(.),$$

and choose $\hat{\Gamma}^{(i+1)} = \Gamma^{(i+1)}$ at convergence, for $i = 1, \cdots, N-1$. Of course, there is not guarantee that either the result will be independent of the order of presentation (this may or may not be important, depending on the application) or that it will indeed be a local minimum of $\mu(\Gamma, \Gamma_1, \cdots, \Gamma_N)$. Another alternative is to solve (33) by choosing Γ_0 to be one of the given shapes.

We show the result of computing the mean of nine hands with this method in figure 20

Another example of mean is obtained from the previous fish silhouettes database: we have used eight silhouettes, normalized them so that their centers of gravity and principle axes were aligned, and computed their mean, as shown in figure 21. The initial curve, Γ_0 was chosen to be an enclosing circle.

7.2 Empirical covariance

We can go beyond the definition of the mean and in effect define something similar to the covariance matrix of a set of N shapes.

The function $\mathcal{S} \to \mathbb{R}^+$ defined by $\Gamma \to E^2(\Gamma, \Gamma_i)$ has a gradient which defines a normal velocity field, noted β_i, defined on Γ, such that if we consider the infinitesimal deformation $\Gamma - \beta_i \mathbf{n} d\tau$ of Γ, it decreases the value of $E^2(\Gamma, \Gamma_i)$. Each such β_i belongs to $L^2(\Gamma)$, the set of square integrable real functions defined on Γ. Each Γ_i defines such a normal velocity field β_i. We consider the mean velocity $\hat{\beta} = \frac{1}{N}\sum_{i=1}^N \beta_i$ and define the linear operator $\Lambda : L^2(\Gamma) \to L^2(\Gamma)$ such that $\beta \to \sum_{i=1,N} < \beta, \beta_i - \hat{\beta} > (\beta_i - \hat{\beta})$. We have the following

Definition 18. *Given N shapes of $\mathcal{S}$, the covariance operator of these N shapes relative to any shape Γ of $\mathcal{S}$ is the linear operator of $L^2(\Gamma)$ defined by*

$$\Lambda(\beta) = \sum_{i=1,N} < \beta, \beta_i - \hat{\beta} > (\beta_i - \hat{\beta}),$$

where the β_i are defined as above, relatively to the shape Γ.

This operator has some interesting properties which we study next.

Proposition 20. *The operator Λ is a continuous mapping of $L^2(\Gamma)$ into $L^2(\Gamma)$.*

Proof. We have $\| \sum_{i=1,N} < \beta, \beta_i - \hat{\beta} > (\beta_i - \hat{\beta})\|_2 \leq \sum_{i=1,N} | < \beta, \beta_i - \hat{\beta} > | \|\beta_i - \hat{\beta}\|_2$ and, because of Schwarz inequality, $| < \beta, \beta_i - \hat{\beta} > | \leq \|\beta\|_2 \|\beta_i - \hat{\beta}\|_2$. This implies that $\| \sum_{i=1,N} < \beta, \beta_i - \hat{\beta} > (\beta_i - \hat{\beta})\|_2 \leq K\|\beta\|_2$ with $K = \sum_{i=1,N} \|\beta_i - \hat{\beta}\|_2^2$. □

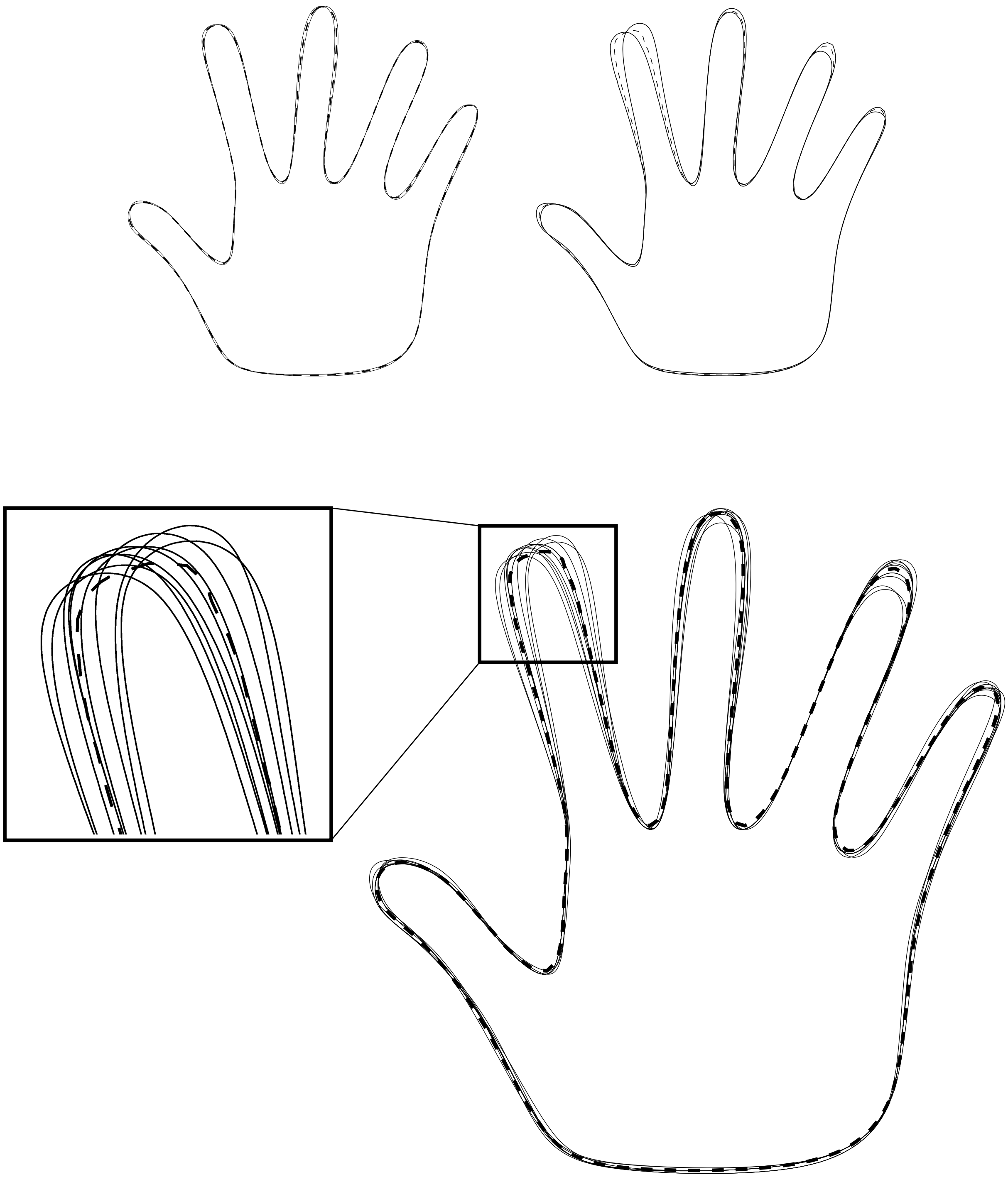

Figure 20: The mean (the dashed curve) of nine hand silhouettes (the continuous curves) obtained by the sequential suboptimal method described in the text: first step (mean of the two first curves), fifth step (weighted mean of the sixth curve and of the mean of the five first curves from the previous step), and final result.

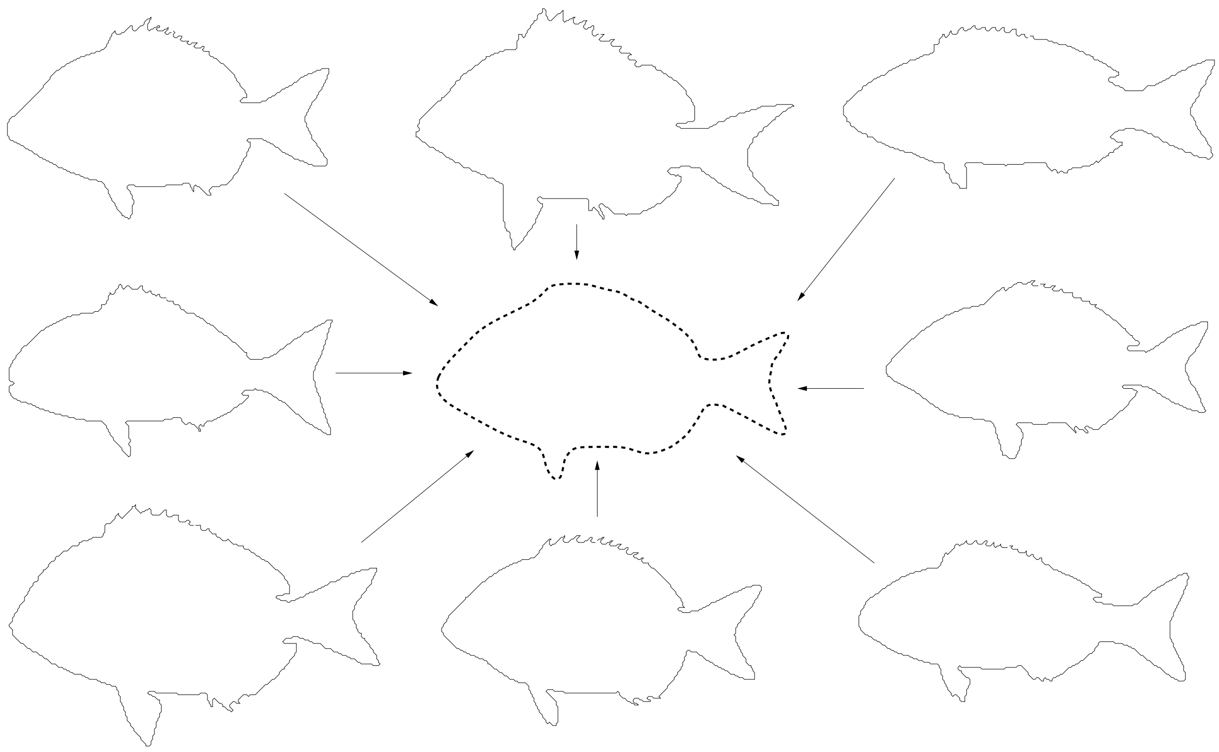

Figure 21: The mean of eight fishes.

Λ is in effect a mapping from $L^2(\Gamma)$ into its Hilbert subspace $A(\Gamma)$ generated by the N functions $\beta_i - \hat{\beta}$. Note that if Γ is one of the empirical means of the shapes Γ_i, by definition we have $\hat{\beta} = 0$.

This operator acts on what can be thought of as the tangent space to the manifold of all shapes at the point Γ. We then have the

Proposition 21. *The covariance operator is symmetric positive definite.*

Proof. This follows from the fact that $< \Lambda(\beta), \beta > = < \beta, \Lambda(\beta) > = \sum_{i=1,N} < \beta, \beta_i - \hat{\beta} >^2$. □

It is also instructive to look at the eigenvalues and eigenvectors of Λ. For this purpose we introduce the $N \times N$ matrix $\hat{\Lambda}$ defined by $\hat{\Lambda}_{ij} = < \beta_i - \hat{\beta}, \beta_j - \hat{\beta} >$. We have the

Proposition 22. *The $N \times N$ matrix $\hat{\Lambda}$ is symmetric semi positive definite. Let $p \leq N$ be its rank, $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_p^2 > 0$ its positive eigenvalues, $\mathbf{u}_1, \cdots, \mathbf{u}_N$ the corresponding eigenvectors. They satisfy*

$$\begin{aligned}\mathbf{u}_i \cdot \mathbf{u}_j &= \delta_{ij} \quad i, j = 1, \cdots, N \\ \hat{\Lambda}\mathbf{u}_i &= \sigma_i^2 \mathbf{u}_i \quad i = 1, \cdots, p \\ \hat{\Lambda}\mathbf{u}_i &= 0 \quad p+1 \leq i \leq N\end{aligned}$$

Proof. The matrix $\hat{\Lambda}$ is clearly symmetric. Let now $\boldsymbol{\alpha} = [\alpha_1, \cdots, \alpha_N]^T$ be a vector of $\mathbb{R}^N$, $\boldsymbol{\alpha}^T \hat{\Lambda} \boldsymbol{\alpha} = \|\beta\|_2^2$, where $\beta = \sum_{i=1}^N \alpha_i(\beta_i - \hat{\beta})$. The remaining of the proposition is simply a statement of the existence of an orthonormal basis of eigenvectors for a symmetric matrix of $\mathbb{R}^N$. □

The N-dimensional vectors $\mathbf{u}_j$, $j = 1, \cdots, p$ and the p eigenvalues σ_k^2, $k = 1, \cdots, p$ define p modes of variation of the shape Γ. These modes of variation are normal deformation flows which are defined as follows. We note u_{ij}, $i, j = 1, \cdots, N$ the ith coordinate of the vector $\mathbf{u}_j$ and v_j the element of $A(\Gamma)$ defined by

$$v_j = \frac{1}{\sigma_j} \sum_{i=1}^{N} u_{ij}(\beta_i - \hat{\beta}) \tag{34}$$

In the case $\Gamma = \hat{\Gamma}$, $\hat{\beta} = 0$. We have the proposition

Proposition 23. *The functions v_j, $j = 1, \cdots, p$ are an orthonormal set of eigenvectors of the operator Λ and form a basis of $A(\Gamma)$.*

Proof. Let us form the product $< v_j, v_k >$:

$$< v_j, v_k >= \frac{1}{\sigma_j \sigma_k} < \sum_{l=1}^N u_{lj}(\beta_l - \hat{\beta}),\ \sum_{m=1}^N u_{mk}(\beta_m - \hat{\beta}) >=$$

$$\frac{1}{\sigma_j \sigma_k} \sum_{l=1}^N u_{lj} \sum_{m=1}^N < \beta_l - \hat{\beta}, \beta_m - \hat{\beta} > u_{mk} = \frac{1}{\sigma_j \sigma_k} \sum_{l=1}^N u_{lj} (\hat{\Lambda} \mathbf{u}_k)_l = \frac{1}{\sigma_j \sigma_k} \mathbf{u}_j \cdot (\hat{\Lambda} \mathbf{u}_k)$$

According to proposition 22, $\hat{\Lambda} \mathbf{u}_k = \sigma_k^2 \mathbf{u}_k$ and $\mathbf{u}_j \cdot \mathbf{u}_k = \delta_{jk}$, which proves the orthonormality and therefore the linear independence. There remains to show that they generate the whole of $A(\Gamma)$. In order to see this, we consider the element $\beta = \sum_{i=1}^N \alpha_i (\beta_i - \hat{\beta})$ of $A(\Gamma)$ and look for the coefficients μ_k, $k = 1, \cdots, p$ such that

$$\sum_{i=1}^N \alpha_i (\beta_i - \hat{\beta}) = \sum_{k=1}^p \mu_k v_k \tag{35}$$

We take the Hilbert product of both sides of this equation with $\beta_j - \hat{\beta}$ to obtain

$$(\hat{\Lambda} \boldsymbol{\alpha})_j = \sum_{k=1}^p \mu_k < v_k, \beta_j - \hat{\beta} > . \tag{36}$$

We then use (34), replace v_k with

$$v_k = \frac{1}{\sigma_k} \sum_{i=1}^N u_{ik} (\beta_i - \hat{\beta}),$$

and obtain

$$< v_k, \beta_j - \hat{\beta} >= \frac{1}{\sigma_k} \sum_{i=1}^N u_{ik} < \beta_i - \hat{\beta},\ \beta_j - \hat{\beta} >= \frac{1}{\sigma_k} (\hat{\Lambda} \mathbf{u}_k)_j = \sigma_k u_{kj}.$$

Replacing this value in (36) yields

$$(\hat{\Lambda} \boldsymbol{\alpha})_j = \sum_{k=1}^p \mu_k \sigma_k u_{kj},$$

or, in matrix form

$$\hat{\Lambda} \boldsymbol{\alpha} = \mathbf{U}\, diag(\sigma_1, \cdots, \sigma_p) \boldsymbol{\mu}$$

where the $N \times p$ matrix $\mathbf{U}$ is equal to $[\mathbf{u}_1, \cdots, \mathbf{u}_p]$. Because the matrix $\mathbf{U}$ satisfies $\mathbf{U}^T \mathbf{U} = I_p$, the $p \times p$ identity matrix, and $\hat{\Lambda} \mathbf{U} = \mathbf{U} diag(\sigma_1^2, \cdots, \sigma_p^2)$, we obtain the values of the μ_k:

$$diag(\sigma_1, \cdots, \sigma_p) \boldsymbol{\mu} = \mathbf{U}^T \hat{\Lambda} \boldsymbol{\alpha} = (\hat{\Lambda} \mathbf{U})^T \boldsymbol{\alpha} = (\mathbf{U} diag(\sigma_1^2, \cdots, \sigma_p^2))^T \boldsymbol{\alpha},$$

hence

$$\boldsymbol{\mu} = diag(\sigma_1, \cdots, \sigma_p)\mathbf{U}^T\boldsymbol{\alpha}.$$

Conversely, if we replace the μ_k by these values in the right handside of (35), we verify that we obtain the left handside.

It remains to verify that $\Lambda(v_j) = \sigma_j^2 v_j$, $j = 1, \cdots, p$. By definition

$$\Lambda(v_j) = \sum_{i=1}^{N} < v_j, \beta_i - \hat{\beta} > (\beta_i - \hat{\beta}).$$

We replace in the right handside of this equation v_j by its expression (34), use proposition 22, and obtain the desired result. □

The velocities v_k, $k = 1, \cdots, p$ can be interpreted as modes of variation of the shape and the σ_k^2's as variances for these modes. Looking at how the mean shape varies with respect to the kth mode is equivalent to solving the following PDEs:

$$\Gamma_t = \pm v_k\, \mathbf{n} \tag{37}$$

with initial conditions $\Gamma(0, .) = \hat{\Gamma}(.)$. Note that v_k is a function of Γ through Λ which has to be reevaluated at each time t. One usually solves these PDEs until the distance to $\hat{\Gamma}$ becomes equal to σ_k.

An example of this evolution for the case of the fingers is shown in figure 22. Another interesting case, drawn from the example of the eight fish of figure 21, is shown in figure 23 where the first four principal modes of the covariance operator corresponding to those eight sample shapes are displayed.

8 Some remarks about our implementation

Mainly, we have to implement the motion of a curve Γ under a velocity field v: $\Gamma_t = v$. When Γ is composed of one or more closed connected components, we use the Level Set Method introduced by Osher an Sethian in 1987 [36, 41, 35].

8.1 Closed curves: the Level Set Method

The key idea of the Level Set Method is to represent the curve $\Gamma(t)$ implicitly, i.e. as the zero level of some function $u(x, t)$ defined for $x \in D$. Usually, u if negative inside Γ and positive outside. It can be easily proved that, if u evolves according to

$$\frac{\partial u(x, y)}{\partial t} + v\nabla u = 0$$

then, its zero level $\{x | \Gamma(x, t) = 0\}$ evolves according to the required equation $\Gamma_t = v$. Here, v is the desired velocity on Γ and is arbitrary elsewhere (see below).

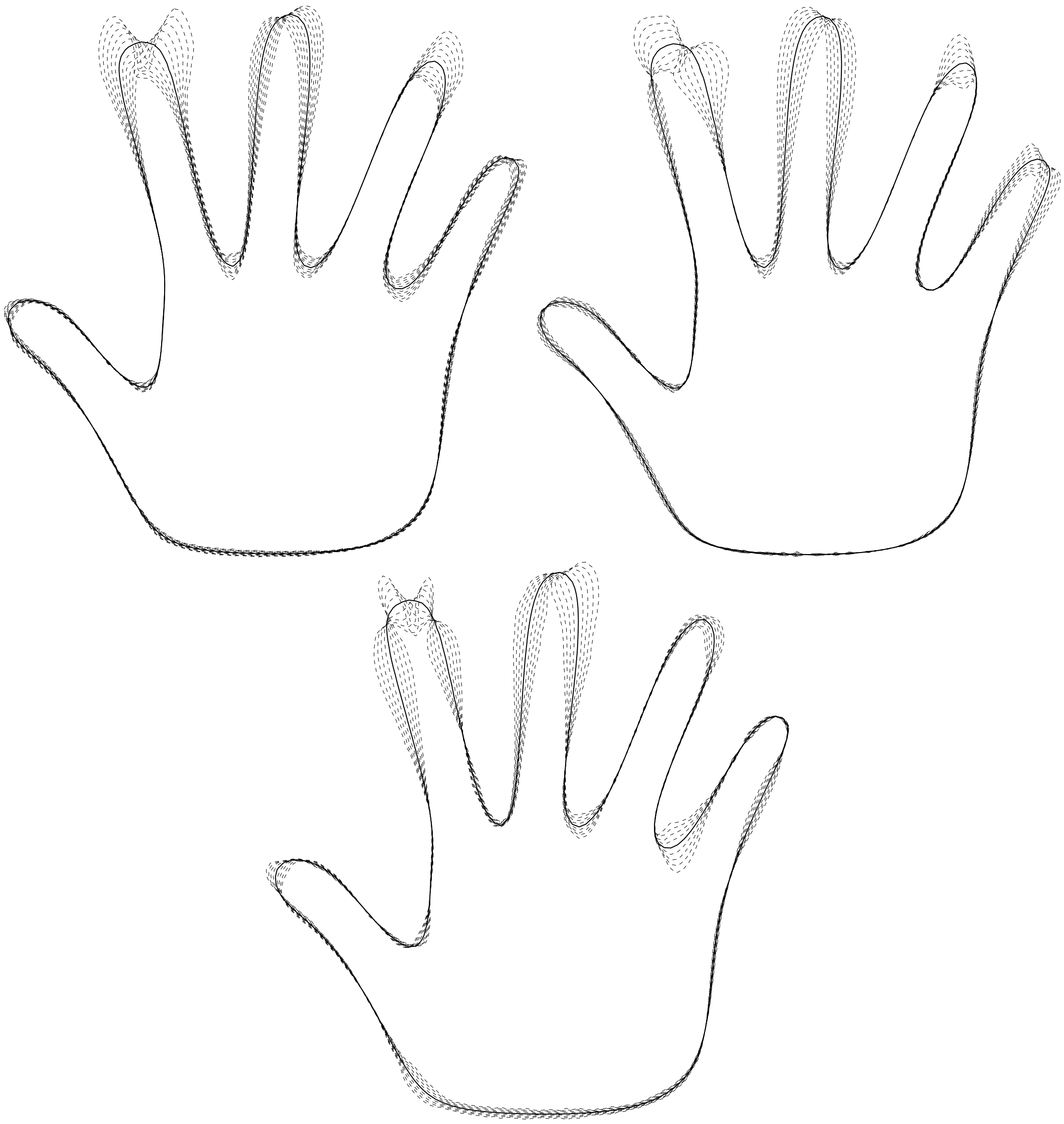

Figure 22: The first three modes of variation for the nine sample shapes and their mean shown in figure 20. The mean is shown in thick continuous line, the solutions of equation (37) for $k = 1, 2, 3$ are represented in dotted lines.

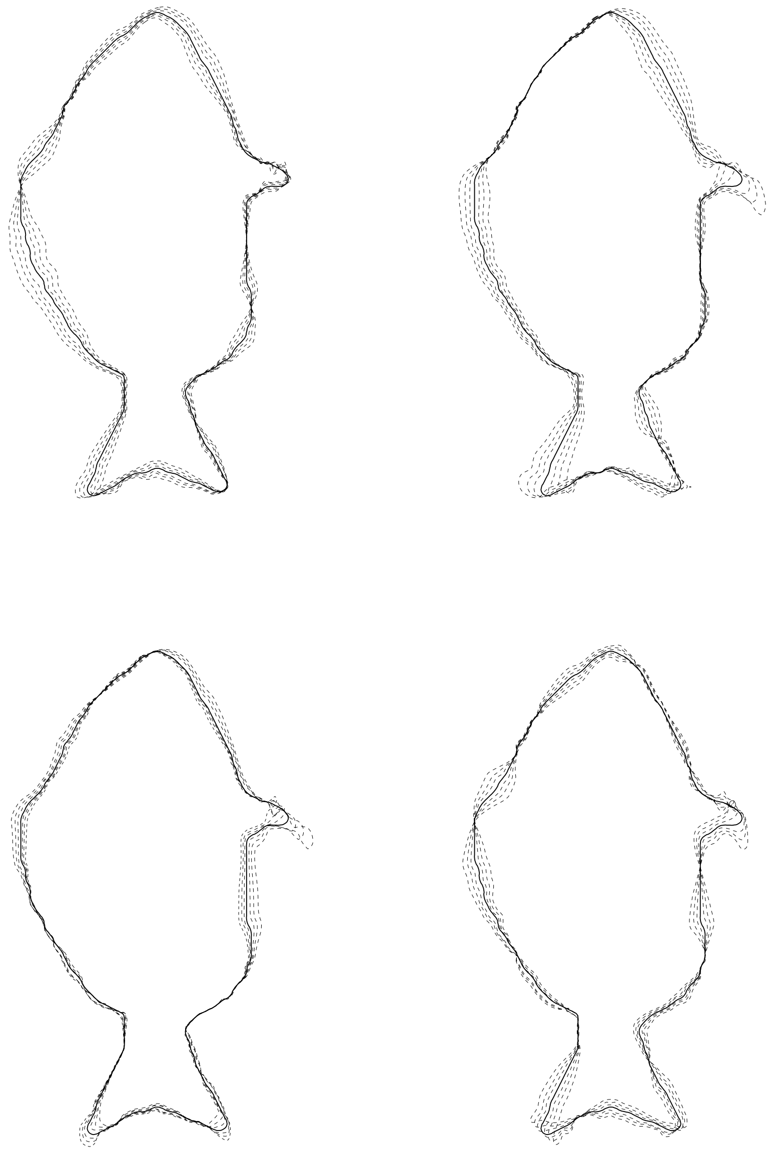

Figure 23: The first four modes of variation for the eight sample shapes and their mean shown in figure 21. The mean is shown in thick continuous line, the solutions of equation (37) for $k = 1, \cdots, 4$ are represented in dotted lines.

Often, only the normal velocity field $\beta\mathbf{n}$ is important. As $\mathbf{n} = \nabla u/|\nabla u|$, the evolution of u becomes:

$$\frac{\partial u(x,y)}{\partial t} + \beta|\nabla u| = 0$$

The advantages of the Level Set Method are well known: stability, accuracy, convergence to the correct solution, easy extension to higher dimensions, correct handling of topological changes such as breaking and merging.

An important issue is that β is only defined on curve Γ in the partial differential equation $\Gamma_t = \beta\mathbf{n}$. In many cases, β has a natural extension everywhere in domain D, so that equation $u_t + \beta|\nabla u| = 0$ is defined. For instance, when $\beta(x)$ is the curvature of Γ at point $x \in \Gamma$, one could choose, at each point $x \in D$, $\beta(x)$ equal to the curvature of the level set of u going through x. In some other cases, like ours, β can only be computed on Γ and some extension procedure has to be used to get β everywhere in D. This is now classical [37, 1, 20].

It should be noted that the zero level set of function $u(.,t)$ is only extracted to visualize $\Gamma(t)$, usually with the Marching Cubes algorithm [32] which interpolates its position and gives a nice polygonal approximation of it (a triangulated mesh in 3D). For obvious speed and accuracy reasons, it is important not to rely on this approximation to compute the velocity β. Useful quantities can generally be computed directly from u. So it is for the normal and the curvature:

$$\begin{aligned}\mathbf{n} &= \nabla u/|\nabla u| \\ \kappa &= -\nabla \cdot (\nabla u/|\nabla u|)\end{aligned}$$

and for the integral of some quantity $f(x)$ over Γ:

$$\int_\Gamma f(x)d\Gamma(x) = \int_D f(x)\delta(u(x))|\nabla u|dx$$

where $\delta(.)$ is a one dimensional Dirac function. Our β also involves the distance function to the curve Γ which has to be known without extracting the zero level set of function u. Usually, one takes the signed distance to the initial curve $\Gamma(0)$ as an initial value of u. Thus, $u(.,0)$ can be used to compute β at time $t = 0$. Yet, $u(.,t)$ has no reason to remain the distance to $\Gamma(t)$... except in some implementations where β is extended in such a way that the distance function is preserved during the evolution of u (see [20]): this is exactly what we need. And what we use!

As a conclusion, in the case of closed curves, the Hausdorff warping (section 6) and the shape statistics (section 7) are implemented with a Level Set Method with: (i) velocity extension, (ii) distance function preserving, and (iii) no need to extract the zero level set (except for visualization).

The minimization of the $W^{1,2}$ norm (section 5.6 and figure 17) is also implemented with the Level Set Method. As already mentioned, the reprojection on the set of distance functions is a "standard" Level Set technique [44].

8.2 Open curves

For open curves (figure 16), the Level Set Method cannot be used. A straight Lagrangian approach and polygonal approximations of the curves were used as a first step toward more refined methods like the ones described in [3].

9 Further comparison with other approaches and conclusion

We have presented in section 1 the similarities and dissimilarities of our work with that of others. We would like to add to this presentation the fact that ours is an attempt to generalize to a nonlinear setting the work that has been done in a linear one by such scientists as Cootes, Taylor and their collaborators [8] and by Leventon et al. who, like us, proposed to use distance functions to represent shapes in a statistical framework but used a first-order approximation by assuming that the set of distance functions was a linear manifold [31, 30] which of course it is not. Our work shows that dropping the incorrect linearity assumption is possible at reasonable costs, both theoretical and computational. Comparison of results obtained in the two frameworks is a matter of future work.

In this respect we would also like to emphasize that in our framework the process of linear averaging shape representations has been more or less replaced by the linear averaging of the normal deformation fields which are tangent vectors to the manifold of all shapes (see the definition of the covariance operator in section 7.2) and by solving a PDE based on these normal deformation fields (see the definition of a mean in section 7.1 and of the deformation modes in section 7.2).

It is also interesting to recall the fact that our approach can be seen as the opposite of that consisting in first building a Riemannian structure on the set of shapes, i.e. going from an infinitesimal metric structure to a global one. The infinitesimal structure is defined by an inner product in the tangent space (the set of normal deformation fields) and has to vary continuously from point to point, i.e. from shape to shape. As mentioned before, this is mostly dealt with in the work of Miller, Trouvé and Younes [34, 49, 52]. The problem with these approaches, beside that of having to deal with parametrizations of the shapes, is that there exist global metric structures on the set of shapes (see section 2.2) which are useful and relevant to the problem of the comparison of shapes but that do not arise from an infinitesimal structure.

Our approach can be seen as taking the problem from exactly the opposite viewpoint from the previous one: we start with a global metric on the set of shapes (ρ_H or the $W^{1,2}$ metric) and build smooth functions (in effect smooth approximations of these metrics) that we use as dissimilarity measure or energy functions and minimize using techniques of the calculus of variation by computing their gradient and performing infinitesimal gradient descent. We have seen that in order to compute the gradients we need to define an inner-product of normal deformation flowss and the choice of this inner-product may influence the way our algorithms evolve from one shape to another. This last point is related to but different from the choice of the Riemaniann metric in the first approach. Its investigation is also a topic of future work.

Another advantage of our viewpoint is that it apparently extends graciously to higher dimensions thanks to the fact that we do not rely on parametrizations of the shapes and work intrinsically with their distance functions (or approximations thereof). This is clearly also worth pursuing in future work.

A Computation of $\nabla \tilde{\rho}_H(\Gamma, \Gamma_0)$

We prove proposition 12.

Proof. We make a few definitions to simplify notations:

$$m^{\varphi,\psi}_{\Gamma_0,\Gamma} = \left\langle \langle d(\cdot,\cdot)\rangle^{\varphi}_{\Gamma_0}\right\rangle^{\psi}_{\Gamma}$$
$$m^{\varphi,\psi}_{\Gamma,\Gamma_0} = \langle \langle d(\cdot,\cdot)\rangle^{\varphi}_{\Gamma}\rangle^{\psi}_{\Gamma_0}$$

We also define the corresponding functions

$$m^{\varphi}_{\Gamma}(x) = \langle d(x,\cdot)\rangle^{\varphi}_{\Gamma} \quad m^{\psi}_{\Gamma}(x) = \langle d(x,\cdot)\rangle^{\psi}_{\Gamma}$$
$$m^{\varphi}_{\Gamma_0}(x) = \langle d(x,\cdot)\rangle^{\varphi}_{\Gamma_0} \quad m^{\psi}_{\Gamma_0}(x) = \langle d(x,\cdot)\rangle^{\psi}_{\Gamma_0}.$$

We then proceed with

$$\mathcal{G}(\tilde{\rho}_H(\Gamma,\Gamma_0),\boldsymbol{\beta}) = \frac{1}{2\theta'(\tilde{\rho}_H(\Gamma,\Gamma_0))}\left[\theta'\left(m^{\varphi,\psi}_{\Gamma_0,\Gamma}\right)\ \mathcal{G}\left(m^{\varphi,\psi}_{\Gamma_0,\Gamma},\boldsymbol{\beta}\right)+\theta'\left(m^{\varphi,\psi}_{\Gamma,\Gamma_0}\right)\ \mathcal{G}\left(m^{\varphi,\psi}_{\Gamma,\Gamma_0},\boldsymbol{\beta}\right)\right],$$

because of (12).

Computation of the first term $\mathcal{G}\left(m^{\varphi,\psi}_{\Gamma_0,\Gamma},\boldsymbol{\beta}\right)$

We apply the chain rule and (11) to obtain

$$\mathcal{G}\left(m^{\varphi,\psi}_{\Gamma_0,\Gamma},\boldsymbol{\beta}\right) = \frac{1}{\psi'\left(m^{\varphi,\psi}_{\Gamma_0,\Gamma}\right)}\left[\frac{1}{|\Gamma|}\ \mathcal{G}\left(\int_{\Gamma}\psi\left(\langle d(\cdot,\cdot)\rangle^{\varphi}_{\Gamma_0}\right),\boldsymbol{\beta}\right) + \left(\int_{\Gamma}\psi\left(\langle d(\cdot,\cdot)\rangle^{\varphi}_{\Gamma_0}\right)\right)\ \mathcal{G}\left(\frac{1}{|\Gamma|},\boldsymbol{\beta}\right)\right].$$

We now compute $\mathcal{G}\left(\frac{1}{|\Gamma|}, \boldsymbol{\beta}\right)$:

$$\begin{aligned}
\mathcal{G}\left(\frac{1}{|\Gamma|}, \boldsymbol{\beta}\right) &= -\frac{1}{|\Gamma|^2}\ \mathcal{G}\left(|\Gamma|, \boldsymbol{\beta}\right) \\
&= -\frac{1}{|\Gamma|^2}\mathcal{G}\left(\int_0^1 |\Gamma'(p)|dp, \boldsymbol{\beta}\right) \\
&= -\frac{1}{|\Gamma|^2}\int_0^1 \frac{\Gamma'(p)}{|\Gamma'(p)|}\cdot \boldsymbol{\beta}'(p)\ dp \\
&= \frac{1}{|\Gamma|^2}\int_0^1 \kappa(p)\mathbf{n}(p)\cdot\boldsymbol{\beta}(p)\ \ |\Gamma'(p)|dp.
\end{aligned}$$

The last line is obtained by integrating by parts and using the hypothesis that $\boldsymbol{\beta}$ is parallel to $\mathbf{n}$. $\kappa(p)$ is the curvature of Γ at the point $\Gamma(p)$. This yields

$$\left(\int_\Gamma \psi\left(\langle d(\cdot,\cdot)\rangle^{\varphi}_{\Gamma_0}\right)\right)\ \ \mathcal{G}\left(\frac{1}{|\Gamma|}, \boldsymbol{\beta}\right) = \frac{1}{|\Gamma|}\psi(m^{\varphi,\psi}_{\Gamma_0,\Gamma})\int_\Gamma \kappa(y)\mathbf{n}(y)\cdot\boldsymbol{\beta}(y)\ d\Gamma(y) \tag{38}$$

We continue with

$$\mathcal{G}\left(\int_\Gamma \psi\left(\langle d(\cdot,\cdot)\rangle^{\varphi}_{\Gamma_0}\right), \boldsymbol{\beta}\right) = \mathcal{G}\left(\int_0^1 \psi\left(\langle d(\Gamma(p),\cdot)\rangle^{\varphi}_{\Gamma_0}\right)\ |\Gamma'(p)|dp, \boldsymbol{\beta}\right) =$$

$$\int_0^1 \frac{\psi'}{\varphi'}\left(\langle d(\Gamma(p),\cdot)\rangle^{\varphi}_{\Gamma_0}\right)\ \frac{1}{|\Gamma_0|}\ \lim_{\tau\to 0}\frac{\int_{\Gamma_0}\left(\varphi(d(\Gamma(p)+\tau\boldsymbol{\beta}(p),\cdot)-\varphi(d(\Gamma(p),\cdot))\right)}{\tau}\ \ |\Gamma'(p)|dp+$$

$$\int_0^1 \psi\left(\langle d(\Gamma(p),\cdot)\rangle^{\varphi}_{\Gamma_0}\right)\ \lim_{\tau\to 0}\frac{|\Gamma'(p)+\tau\boldsymbol{\beta}'(p)|-|\Gamma'(p)|}{\tau}dp.$$

The last term is equal to (using the hypothesis that $\boldsymbol{\beta}(p)$ is parallel to $\mathbf{n}(p)$ for all p's):

$$-\int_0^1 \psi\left(\langle d(\Gamma(p),\cdot)\rangle^{\varphi}_{\Gamma_0}\right)\ \kappa(p)\mathbf{n}(p)\cdot\boldsymbol{\beta}(p)\ |\Gamma'(p)|dp.$$

The first term can be written:

$$\int_0^1 \frac{\psi'}{\varphi'}\left(\langle d(\Gamma(p),\cdot)\rangle^{\varphi}_{\Gamma_0}\right)\frac{1}{|\Gamma_0|}\left(\int_{\Gamma_0}\varphi'(d(\Gamma(p),x))\ \ \frac{\Gamma(p)-x}{d(\Gamma(p),x)}\cdot\boldsymbol{\beta}(p)\ d\Gamma_0(x)\right)\ |\Gamma'(p)|dp$$

Combining them we obtain

$$\int_\Gamma\left(\frac{\psi'}{\varphi'}\left(m^{\varphi}_{\Gamma_0}(y)\right)\langle\varphi'(d(y,\cdot))\ \ \frac{y-\cdot}{d(y,\cdot)}\rangle_{\Gamma_0} - \psi\left(m^{\varphi}_{\Gamma_0}(y)\right)\kappa(y)\mathbf{n}(y)\right)\cdot\boldsymbol{\beta}(y)\ d\Gamma(y) \tag{39}$$

We finally combine (38) and (39)

$$\mathcal{G}\left(m^{\varphi,\psi}_{\Gamma_0,\Gamma},\boldsymbol{\beta}\right)=\frac{1}{\psi'\left(m^{\varphi,\psi}_{\Gamma_0,\Gamma}\right)|\Gamma|}\int_\Gamma\left(\frac{\psi'}{\varphi'}\left(m^{\varphi}_{\Gamma_0}(y)\right)\left\langle\varphi'(d(y,\cdot))\ \frac{y-\cdot}{d(y,\cdot)}\right\rangle_{\Gamma_0}+\left(\psi\left(m^{\varphi,\psi}_{\Gamma_0,\Gamma}\right)-\psi\left(m^{\varphi}_{\Gamma_0}(y)\right)\right)\kappa(y)\mathbf{n}(y)\right)\cdot\boldsymbol{\beta}(y)\ d\Gamma(y) \tag{40}$$

Computation of the second term $\mathcal{G}\left(m^{\varphi,\psi}_{\Gamma,\Gamma_0},\boldsymbol{\beta}\right)$

Because of (11) we can write

$$\mathcal{G}\left(m^{\varphi,\psi}_{\Gamma,\Gamma_0},\boldsymbol{\beta}\right)=\frac{1}{\psi'\left(m^{\varphi,\psi}_{\Gamma,\Gamma_0}\right)}\frac{1}{|\Gamma_0|}\times\int_{\Gamma_0}\frac{\psi'}{\varphi'}\left(\langle d(x,\cdot)\rangle^{\varphi}_{\Gamma}\right)\ \mathcal{G}\left(\frac{1}{|\Gamma|}\int_0^1\varphi\left(d(x,\Gamma(p))\right)\ |\Gamma'(p)|dp,\boldsymbol{\beta}\right)\ d\Gamma_0(x).$$

Using the chain rule

$$\mathcal{G}\left(\frac{1}{|\Gamma|}\int_0^1\varphi\left(d(x,\Gamma(p))\right)\ |\Gamma'(p)|dp,\boldsymbol{\beta}\right)=\left(\int_\Gamma\varphi\circ d(x,\cdot)\right)\frac{1}{|\Gamma|^2}\int_0^1\kappa(p)\mathbf{n}(p)\cdot\boldsymbol{\beta}(p)\ \ |\Gamma'(p)|dp+\frac{1}{|\Gamma|}\int_0^1\varphi'(d(\Gamma(p),x))\ \frac{\Gamma(p)-x}{d(\Gamma(p),x)}\cdot\boldsymbol{\beta}(p)\ |\Gamma'(p)|dp+\frac{1}{|\Gamma|}\int_0^1\varphi\left(d(x,\Gamma(p))\right)\ \ \frac{\Gamma'(p)}{|\Gamma'(p)|}\cdot\boldsymbol{\beta}'(p)dp\ .$$

Under the same hypothesis for $\boldsymbol{\beta}$, the last term is equal to:

$$-\frac{1}{|\Gamma|}\int_0^1\varphi\left(d(\Gamma(p),x)\right)\ \kappa(p)\mathbf{n}(p)\cdot\boldsymbol{\beta}(p)\ |\Gamma'(p)|dp$$

The expression of $\mathcal{G}(\tilde\rho_H(\Gamma,\Gamma_0),\boldsymbol{\beta})$ is obtained by reordering these terms. This yields

$$\mathcal{G}(\tilde\rho_H(\Gamma,\Gamma_0),\boldsymbol{\beta})=\frac{1}{2\theta'(\tilde\rho_H(\Gamma,\Gamma_0))}\int_\Gamma\Bigg[\Bigg[\nu\kappa(p)\int_{\Gamma_0}\frac{\psi'}{\varphi'}\left(\langle d(x,\cdot)\rangle^{\varphi}_{\Gamma}\right)\ \ \left[\ \varphi\circ\langle d(x,\cdot)\rangle^{\varphi}_{\Gamma}-\varphi\circ d(x,y)\ \right]d\Gamma_0(x)+|\Gamma_0|\eta\kappa(p)\ \left[\psi\left(\left\langle\langle d(\cdot,\cdot)\rangle^{\varphi}_{\Gamma_0}\right\rangle^{\psi}_{\Gamma}\right)-\psi\left(\langle d(\cdot,y)\rangle^{\varphi}_{\Gamma_0}\right)\ \right]\Bigg]\mathbf{n}(p)+\int_{\Gamma_0}\frac{\varphi'\circ d(x,y)}{d(x,y)}\ \left[\nu\frac{\psi'}{\varphi'}\left(\langle d(x,\cdot)\rangle^{\varphi}_{\Gamma}\right)+\eta\frac{\psi'}{\varphi'}\left(\langle d(\cdot,y)\rangle^{\varphi}_{\Gamma_0}\right)\right]\ (y-x)\ d\Gamma_0(x)\Bigg]\cdot\boldsymbol{\beta}(p)d\Gamma(y)$$

where $\nu = \dfrac{1}{|\Gamma|\,|\Gamma_0|}\dfrac{\theta'}{\psi'}\left(\langle\langle d(\cdot,\cdot)\rangle_\Gamma^\varphi\rangle_{\Gamma_0}^\psi\right)$ and $\eta = \dfrac{1}{|\Gamma|\,|\Gamma_0|}\dfrac{\theta'}{\psi'}\left(\langle\langle d(\cdot,\cdot)\rangle_{\Gamma_0}^\varphi\rangle_\Gamma^\psi\right)$.
The gradient $\nabla\,\tilde\rho_H(\Gamma,\,\Gamma_0)$ is obtained by identifying the previous expression as an inner product of normal deformation flows $\int_\Gamma \nabla\,\tilde\rho_H(\Gamma,\,\Gamma_0)(y)\beta(y)\,d\Gamma(y)$

$$\begin{aligned}
&\nabla\,\tilde\rho_H(\Gamma,\,\Gamma_0)(y) = \\
&\quad \frac{1}{\theta'(\tilde\rho_H(\Gamma,\,\Gamma_0))}\Bigg[\nu\kappa(p)\int_{\Gamma_0}\frac{\psi'}{\varphi'}\left(\langle d(x,\cdot)\rangle_\Gamma^\varphi\right)\ \left[\,\varphi\circ\langle d(x,\cdot)\rangle_\Gamma^\varphi - \varphi\circ d(x,y)\,\right]d\Gamma_0(x) \\
&\qquad\qquad + |\Gamma_0|\eta\kappa(p)\ \left[\psi\left(\langle\langle d(\cdot,\cdot)\rangle_{\Gamma_0}^\varphi\rangle_\Gamma^\psi\right) - \psi\left(\langle d(\cdot,y)\rangle_{\Gamma_0}^\varphi\right)\,\right] \\
&\qquad + \int_{\Gamma_0}\frac{\varphi'\circ d(x,y)}{d(x,y)}\ \left[\nu\frac{\psi'}{\varphi'}\left(\langle d(x,\cdot)\rangle_\Gamma^\varphi\right) + \eta\frac{\psi'}{\varphi'}\left(\langle d(\cdot,y)\rangle_{\Gamma_0}^\varphi\right)\right]\ (y-x)\cdot\mathbf{n}(p)\ d\Gamma_0(x)\Bigg]
\end{aligned}$$

We should note that all these results hold independently of the fact that the curves are open or closed since we only used in the integration by parts the fact that the field $\boldsymbol\beta$ was parallel to the normal field $\mathbf{n}$. □

B Computation of $\nabla\tilde\rho_D(\Gamma,\,\Gamma_0)$

We prove proposition 14

Proof. From the definitions

$$\mathcal{G}(\tilde\rho_D(\Gamma,\,\Gamma_0),\boldsymbol\beta) = \mathcal{G}(\|\tilde d_\Gamma - \tilde d_{\Gamma_0}\|_{L^2(D)},\,\boldsymbol\beta) + \mathcal{G}(\|\nabla(\tilde d_\Gamma - \tilde d_{\Gamma_0}\|_{\mathbf{L}^2(D)},\,\boldsymbol\beta),$$

and

$$\mathcal{G}(\|\tilde d_\Gamma - \tilde d_{\Gamma_0}\|_{L^2(D)},\,\boldsymbol\beta) = \frac{1}{\|\tilde d_\Gamma - \tilde d_{\Gamma_0}\|_{L^2(D)}}\int_D |\tilde d_\Gamma(x) - \tilde d_{\Gamma_0}(x)|\mathcal{G}(\tilde d_\Gamma(x),\,\boldsymbol\beta)\ dx,$$

and

$$\mathcal{G}(\|\nabla(\tilde d_\Gamma - \tilde d_{\Gamma_0})\|_{\mathbf{L}^2(D)},\,\boldsymbol\beta) = \frac{1}{\|\nabla(\tilde d_\Gamma - \tilde d_{\Gamma_0})\|_{\mathbf{L}^2(D)}}\int_D \nabla(\tilde d_\Gamma(x) - \tilde d_{\Gamma_0}(x))\cdot\nabla(\mathcal{G}(\tilde d_\Gamma(x),\,\boldsymbol\beta))\ dx.$$

We now compute $\mathcal{G}(\tilde{d}_\Gamma(x),\, \boldsymbol{\beta})$ and its gradient. Starting with (15), we readily obtain

$$\mathcal{G}(\tilde{d}_\Gamma(x),\, \boldsymbol{\beta}) = \frac{1}{|\Gamma|\; \varphi'(\tilde{d}_\Gamma(x))}\Big[\left(\int_\Gamma \kappa(y)\mathbf{n}(y)\cdot\boldsymbol{\beta}(y)\; d\Gamma(y)\right) \langle \varphi\circ d(x,\cdot)\rangle_\Gamma + \int_\Gamma \varphi'(d(x,y))\frac{y-x}{d(x,y)}\cdot\boldsymbol{\beta}(y)\; d\Gamma(y) - \int_\Gamma \varphi(d(x,y))\kappa(y)\mathbf{n}(y)\cdot\boldsymbol{\beta}(y)\; d\Gamma(y)\Big]$$

According to our initial hypothesis, $\boldsymbol{\beta}(y) = \beta(y)\,\mathbf{n}(y)$. We define

$$B(x,y) = \kappa(y)\left(\langle\varphi\circ d(x,\cdot)\rangle_\Gamma - \varphi\circ d(x,y)\right) + \varphi'(d(x,y))\frac{y-x}{d(x,y)}\cdot\mathbf{n}(y),$$

so that

$$\mathcal{G}(\tilde{d}_\Gamma(x),\, \boldsymbol{\beta}) = \frac{1}{|\Gamma|\; \varphi'(\tilde{d}_\Gamma(x))}\int_\Gamma B(x,y)\beta(y)\; d\Gamma(y).$$

Let us compute the gradient of this expression with respect to the x variable:

$$\nabla\mathcal{G}(\tilde{d}_\Gamma(x),\, \boldsymbol{\beta}) = -\frac{\varphi"(\tilde{d}_\Gamma(x))}{|\Gamma|\; \varphi'^2(\tilde{d}_\Gamma(x))}\left(\int_\Gamma B(x,y)\beta(y)\; d\Gamma(y)\right)\; \nabla\tilde{d}_\Gamma(x) + \frac{1}{|\Gamma|\; \varphi'(\tilde{d}_\Gamma(x))}\int_\Gamma \nabla B(x,y)\beta(y)\; d\Gamma(y)$$

After some manipulation, we find that

$$\nabla B(x,y) = \kappa(y)\left(\left\langle \varphi'\circ d(x,\cdot)\;\frac{x-\cdot}{d(x,\cdot)}\right\rangle_\Gamma - \varphi'\circ d(x,y)\;\frac{x-y}{d(x,y)}\right) + \left(\varphi"(d(x,y)) - \frac{\varphi'(d(x,y))}{d(x,y)}\right)\mathbf{n}(y),$$

where we have used in particular the fact that

$$\nabla\tilde{d}_\Gamma(x) = \frac{1}{\varphi'(\tilde{d}_\Gamma(x))}\left\langle \varphi'\circ d(x,\cdot)\;\frac{x-\cdot}{d(x,\cdot)}\right\rangle_\Gamma.$$

We also define

$$C_1(x) = \frac{1}{|\Gamma|\; \varphi'(\tilde{d}_\Gamma(x))}\|\tilde{d}_\Gamma - \tilde{d}_{\Gamma_0}\|^{-1}_{L^2(D)}\;\left(\tilde{d}_\Gamma(x) - \tilde{d}_{\Gamma_0})(x)\right),$$

so that

$$\mathcal{G}(\|\tilde{d}_\Gamma - \tilde{d}_{\Gamma_0}\|_{L^2(D)},\, \boldsymbol{\beta}) = \int_D\int_\Gamma B(x,y)C_1(x)\beta(y)\; d\Gamma(y)\; dx = \int_\Gamma\left(\int_D B(x,y)C_1(x)\; dx\right)\beta(y)\; d\Gamma(y). \tag{41}$$

We then define the vector quantity

$$\mathbf{C}_2(x) = \frac{1}{|\Gamma|\ \varphi'(\tilde{d}_\Gamma(x))} \|\nabla(\tilde{d}_\Gamma - \tilde{d}_{\Gamma_0})\|^{-1}_{\mathbf{L}^2(D)}\ \ \nabla(\tilde{d}_\Gamma - \tilde{d}_{\Gamma_0})(x),$$

so that

$$\begin{aligned}
(42)\quad & \mathcal{G}(\|\nabla(\tilde{d}_\Gamma - \tilde{d}_{\Gamma_0})\|_{\mathbf{L}^2(D)}, \boldsymbol{\beta}) = \\
& - \int_D \int_\Gamma \frac{\varphi"}{\varphi'}(\tilde{d}_\Gamma(x)) \left(\mathbf{C}_2(x) \cdot \nabla \tilde{d}_\Gamma(x)\right) B(x,y)\beta(y)\ d\Gamma(y)\ dx + \\
& \int_D \int_\Gamma \left(\mathbf{C}_2(x) \cdot \nabla B(x,y)\right) \beta(y)\ d\Gamma(y)\ dx = \\
& \int_\Gamma \left(\int_D \left(\mathbf{C}_2(x) \cdot \nabla B(x,y) - \frac{\varphi"}{\varphi'}(\tilde{d}_\Gamma(x)) \left(\mathbf{C}_2(x) \cdot \nabla d(x,\Gamma)\right) B(x,y) \right)\ dx \right) \beta(y)\ d\Gamma(y)
\end{aligned}$$

Combining equations (41) and (42) we obtain the corresponding gradient $\nabla\tilde{\rho}_D(\Gamma, \Gamma_0)$:

$$\begin{aligned}
& \nabla\tilde{\rho}_D(\Gamma, \Gamma_0)(y) = \\
& \int_D \left[B(x,y) \left(C_1(x) - \frac{\varphi"}{\varphi'}(\tilde{d}_\Gamma(x)) \left(\mathbf{C}_2(x) \cdot \nabla \tilde{d}_\Gamma(x) \right) \right) + \mathbf{C}_2(x) \cdot \nabla B(x,y) \right]\ dx,
\end{aligned}$$

and this completes the proof. □

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