Links in edge-colored graphs

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Abstract

A graph is k-linked (k-edge-linked), $k \geq 1$, if for each k pairs of vertices $x_1,\ y_1, \cdots,\ x_k,\ y_k$, there exist k pairwise vertex-disjoint (respectively edge-disjoint) paths, one per pair x_i and y_i, $i = 1, 2, \cdots, k$. Here we deal with the properly-edge-colored version of the k-linked (k-edge-linked) problem in edge-colored graphs. In particular, we give conditions on colored degrees and/or number of edges, sufficient for an edge-colored multigraph to be k-linked (k-edge-linked). Some of the obtained results are the best possible. Related conjectures are proposed.

1 Introduction and notation

The investigation of k-linkings for non colored graphs gave some important and interesting results both from a mathematical and algorithmic point of view [6, 7, 8, 9, 10, 14, 15, 16, 17]. Here we deal with the colored version of the k-linked problem in edge-colored multigraphs. In the case of edge-colored complete graphs, some results of algorithmic nature for the k-linked problem were already obtained in [11]. The study of this type of problems has witnessed significant development during last decades, both from the point of view of its theoretical interest and of its domains of applications. In particular, problems arising in molecular biology are often modeled using colored graphs, i.e., graphs with colored edges and/or vertices [13]. Given such an edge-colored graph, original problems correspond to extract subgraphs colored in a specified pattern. The most natural pattern in such a context is that of a proper coloring, i.e., adjacent edges having different colors. Various applications of properly edge-colored Hamiltonian and Eulerian cycles and paths are studied in [12, 13]. Properly colored paths and cycles have also applications in various other fields, as in VLSI for compacting a programmable logical array [5]. Although a large body of work has already been done [1, 2, 3, 4, 11], in most of that previous work the number of colors was restricted to two. For instance, while it is well known that properly edge-colored hamiltonian cycles can be found efficiently in 2-edge colored complete graphs, it is a long standing question whether there exists a polynomial algorithm for finding such hamiltonian cycles in edge-colored complete graphs with three colors or more [3]. In this paper we consider graphs with edges colored with an arbitrary number of colors. In particular, we study conditions on colored degrees and/or edges sufficient for an edge-colored multigraph to be k-linked (k-edge-linked).

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Formally, let $\{1, 2, \cdots, c\}$ be a set of given $c \geq 2$ colors. Throughout the paper, G^c denotes an edge-colored multigraph so that each edge is colored with some color $i \in \{1, 2, \cdots, c\}$ and no two parallel edges joining the same pair of vertices have the same color. We also suppose that G^c has no isolated components, i.e., the underling non-colored graph is connected. The vertex and edge-sets of G^c are denoted $V(G^c)$ and $E(G^c)$, respectively. The order n of G^c is the number of its vertices. The size m of G^c is the number of its edges. For a given color i, $E^i(G^c)$ denotes the set of edges of G^c on color i. When no confusion arises, we write V, E and E^i instead of $V(G^c), E(G^c)$ and $E^i(G^c)$, respectively. When G^c is not a multigraph, i.e., no parallel edges between any two vertices are allowed, we call it graph, as usual. If H is an induced subgraph of G^c, then $N^i_H(x)$ denotes the set of vertices of H, joined to x with an edge on color i. The colored $i - degree$ of x in H, denoted by $d^i_H(x)$ corresponds to the cardinality $\|N^i_H(x)\|$ of $N^i_H(x)$. Whenever $H \cong G^c$, for simplicity, we write $N^i(x)$ (resp. $d^i(x)$) instead of $N^i_{G^c}(x)$ (resp. $d^i_{G^c}(x)$). For a given vertex x and a given positive integer k, the notation $d^c(x) \geq k$ means that for every $i \in \{1, 2, \cdots, c\}$, $d^i(x) \geq k$. An edge between two vertices x and y is denoted by xy and its color by $c(xy)$. For two given vertices x and y and a given color i, some times, to help reading, we use the notation $x \overset{i}{-} y$ instead of $xy \in E^i(G^c)$. A subgraph of G^c is said to be properly edge-colored, if any two adjacent edges in this subgraph differ in color. A properly edge-colored path does not allow vertex repetitions and any two successive edges on this path differ in colors. The length of a path is the number of its edges. A graph is k-linked (k-edge-linked) whenever for every k disjoint pairs of vertices $x_1, y_1, x_2, y_2, \cdots, x_k, y_k$, there exist k vertex-disjoint (edge-disjoint) properly edge-colored paths, one per pair x_i and y_i, $i = 1,\ 2, \cdots, k$.

The paper is organized as follows: In Section 2, we give conditions on colored degrees, sufficient for the k-linked (k-edge linked) property. In Section 3, we give conditions involving both minimum colored degrees and the number of edges, sufficient for the k-linked (k-edge-linked) property. One of the results of this section is a partial answer to an old question by one of the authors published in [10]. Through both sections, several conjectures are proposed.

2 Degree conditions for k-linked edge-colored multigraphs

Let us start with the following conjecture for k-linked edge-colored multigraphs involving colored degrees.

Conjecture 2.1. *Let G^c be a c-edge-colored multigraph of order n and k a non-zero positive integer. There exists a minimum function $f(n, k)$ such that if for every vertex x, $d^c(x) \geq f(n, k)$, then G^c is k-linked.*

Probably in the above conjecture it suffices to set $f(n, k) = \frac{n}{2} + k - 1$. Indeed, let A, (resp. B, C) be a complete edge-colored multigraph of order $\frac{n-2k+2}{2}$ (respectively, $2k-2$, $\frac{n-2k+2}{2}$). Consider the disjoint union of A, B, C and suppose that each vertex of B is joined to each vertex of $A \cup C$ by c parallel edges all on distinct colors. Although the resulting multigraph has colored degrees at least $\frac{n}{2} + k - 2$, it has no k vertex-disjoint properly edge-colored paths between pairs of vertices x_i and y_i, where x_1 is a vertex in A, y_1 is a vertex in C and the remaining x_i, y_i vertices belong to B, $2 \leq i \leq k$.
Some support to the above conjecture may be obtained from the following theorem.

Theorem 2.2. *Let G^c be a c-edge-colored multigraph of order n, with $n \geq 242k$, k a non-zero positif integer. If for every vertex x, $d^c(x) \geq \frac{n}{2} + k - 1$, then G^c is k-linked.*

Proof. Let G^c be a c-edge-colored multigraph of order n, $n \geq 242k$, such that for every vertex x, $d^c(x) \geq \frac{n}{2} + k - 1$. We are going to prove by contradiction that G^c is k-linked. More precisely, we will prove the stronger result that, given k pairs of vertices $x_1,\ y_1, \cdots, x_k,\ y_k$ of G^c, each pair x_i and y_i is joined by a properly edge-colored path of length at most 8.
Assume therefore that G^c is not k-linked. Then there are $2k$ distinct vertices x_1, y_1, x_2, y_2, ..., x_k, y_k such that there are no k pairwise vertex-disjoint paths, one path per pair $x_i,\ y_i$. Let us consider now a set I of integers, such that there are $\|I\|$ pairwise vertex-disjoint paths of length at most 8 joining the pairs $x_i,\ y_i$, with $i \in I$. We consider I such that its cardinality $\|I\|$ is the maximum possible. Clearly $\|I\| < k$, for otherwise we are finished. In the rest of the proof we are going to show that we can find more than $\|I\|$ pairwise vertex-disjoint paths as long as the cardinality of I is considered strictly smaller than k. This will contradict the maximality property of I and end the proof.

Claim 1. $\|I\| \geq 2$.

Proof. Assume that there exists at least one pair $x_i,\ y_i$, say x_1 and y_1, of vertices such that there is no edge between x_1 and y_1. Otherwise, we are finished, by considering the k paths defined by the k edges x_iy_i, $i = 1, 2 \cdots, k$. Set $S = \{x_i, y_i, 1 \leq i \leq k\}$ and let r (red) and b (blue) be two fixed colors in $\{1, 2, \cdots, c\}$. So $d^r_{G-S}(x_1) \geq \frac{n}{2} + k - 1 - 2(k-1) = \frac{n}{2} - k + 1$. Similarly, $d^b_{G-S}(y_1) \geq \frac{n}{2} - k + 1$. However $\|N^r_{G-S}(x_1) \cup N^b_{G-S}(y_1)\| \leq n - 2k$, so $\|N^r_{G-S}(x_1) \cap N^b_{G-S}(y_1)\| \geq 2$. Consequently, we can find two distinct vertices, say u and v, in $N^r_{G-S}(x_1) \cap N^b_{G-S}(y_1)$. If there is an edge between x_2 and y_2, then this edge x_2y_2 together with the path $x_1 \overset{r}{-} u \overset{b}{-} y_1$ prove that $\|I\| \geq 2$. If not, then there are two distinct vertices u' and v' in $N^r_{G-S}(x_2) \cap N^b_{G-S}(y_2)$. W.l.o.g. we may suppose $u \neq v'$, but then we have found again two paths $x_1 \overset{r}{-} u \overset{b}{-} y_1$ and $x_2 \overset{r}{-} v' \overset{b}{-} y_2$, as desired. $\square$

As $\|I\| < k$, in the sequel, let us suppose w.l.ofg. that $1 \notin I$. In order words, we suppose that there is no properly-edge colored path of length at most 8 between x_1 and y_1.
Let X be the set of vertices which are used in order to build the $\|I\|$ pairwise vertex-disjoint pairwise vertex-disjoint paths of length at most 8, one per pair x_i and y_i, with $i \in I$. Clearly $\|X\| \leq 7k$.
Set $A = N^r(x_1) - (S \cup X)$ and $B = N^b(y_1) - (S \cup X)$. Then $A \cap B = \emptyset$, for otherwise if there is a vertex $z \in A \cap B$, then the path x_1zy_1 is of length two, a contradiction to the choice of x_1 and y_1. Also $\|A\| \geq \frac{n}{2} + k - 1 - \|X\| - \|S\| = \frac{n}{2} - 8k - 1$. Similarly, $\|B\| \geq \frac{n}{2} - 8k - 1$.
Set $C = G - (A \cup B \cup X \cup S)$. We have $\|A\| \geq \frac{n}{2} - k - 1 - \|X\|$ and $\|B\| \geq \frac{n}{2} - k - 1 - \|X\|$. Thus $\|C\| = n - \|A\| - \|B\| - \|S\| - \|X\| \geq 2 + \|X\|$, hence $\|C\| \leq 8k - 1$ and $k \geq 3$.

We distinguish now between two Cases (I) and (II) depending upon A and B.

(I) There is no edge between A and B.

For a color $i \in \{r, b\}$ and for each vertex $x \in A$ and $y \in B$,

$$d^i_{A\cup C}(x) = d^i_{G-(B\cup X\cup S)}(x) = d^i_{G-(X\cup S)}(x) \geq \frac{n}{2} + k - 1 - 7k - 2k = \frac{n}{2} - 8k - 1 \quad (1)$$

and

$$d^i_{B\cup C}(y) = d^i_{G-(A\cup X\cup S)}(y) = d^i_{G-(X\cup S)}(y) \geq \frac{n}{2} - 8k - 1 \quad (2)$$

Claim 2. For every pair $u,\ v$ of distinct vertices in A (repectively in B), there are at least $3k$ distinct red-blue paths between u and v (the order of the colors is important here) and at least $3k$ distinct blue-red paths between u and v. These paths do not go through a vertex of $X \cup S$.

Proof. We will prove only the red-blue case, the blue-red case being similar by intechanging the red-blue colors and applying the same arguments. Let u and v be two distinct vertices of A. We have $d^r_{G-(B\cup X\cup S)}(u) + d^b_{G-(B\cup X\cup S)}(v) \geq n - 16k - 2$. However $\|G - (B \cup X \cup S)\| < \frac{n}{2} - k + 1$ and $n - 16k - 2 - (\frac{n}{2} - k + 1) = \frac{n}{2} - 15k - 3 > 3k$. Therefore there are at least $3k$ distinct red-blue paths between u and v. These paths do not go through a vertex of $X \cup S$. We obtain the same results for the red-blue paths between u and v in B. □

Claim 3. $N^b(x_1) \cap B = \emptyset$ and $N^r(y_1) \cap A = \emptyset$.

Proof. Assume that $N^b(x_1) \cap B$ is not empty. Let y be a vertex in $N^b(x_1) \cap B$. As $\frac{n}{2} - 8k - 1 > 8k$, we have $d^r_{B\cup C}(y) \geq \frac{n}{2} - 8k - 1 \geq \|C\|$. Consequently there is a vertex $y' \in B$ such that the edge yy' is red. But then we can consider the path $x_1 \overset{b}{-} y \overset{r}{-} y' \overset{b}{-} y_1$, a contradiction to the hypothesis that there is no path between x_1 and y_1 of length less than 8 in G^c. We obtain the same result whenever $N^r(y_1) \cap A \neq \emptyset$. This completes the proof of the claim. □

Let us set now $\Phi = A \cap N^b(x_1)$ and $\Psi = B \cap N^r(y_1)$. Consider two vertices $x \in \Phi$ and $y \in \Psi$. For $i \in \{r, b\}$, we have $d^i_{G-S}(x) = d^i_{G-(B\cup S)}(x) \geq \frac{n}{2} + k - 1 - 2k + 1 = \frac{n}{2} - k$, since $x \notin N^i(y_1)$. Similarly, $d^i_{G-S}(y) = d^i_{G-(A\cup S)}(x) \geq \frac{n}{2} + k - 1 - 2k + 1 = \frac{n}{2} - k$, since $y \notin N^i(x_1)$. Consequently $d^r_{G-S}(x) + d^b_{G-S}(y) \geq n - 2k$. Moreover $(S \cup \{x, y\}) \cap (N^r_{G-S}(x) \cup N^b_{G-S}(y)) = \emptyset$ and so $\|N^r_{G-S}(x) \cup N^b_{G-S}(y)\| \leq n - 2k - 2$. In conclusion, $\|N^r_{G-S}(x) \cap N^b_{G-S}(y)\| \geq 2$ and $\|N^b_{G-S}(x) \cap N^r_{G-S}(y)\| \geq 2$. If $N^r_{G-S}(x) \cap N^b_{G-S}(y)$ is not a subset of X, then by considering some vertex $z \in N^r_{G-S}(x) \cap N^b_{G-S}(y) - X$ we define the path $x_1 \overset{b}{-} x \overset{r}{-} z \overset{b}{-} y \overset{r}{-} y_1$ of length less than 8, a contradiction. So, assume that $N^r_{G-S}(x) \cap N^b_{G-S}(y) \subset X$ and $N^b_{G-S}(x) \cap N^r_{G-S}(y) \subset X$ Let Γ_{xy} a subset of I such that $i \in I$ if and only if there is a vertex $z \in \big(N^b_{G-S}(x) \cap N^r_{G-S}(y)\big) \cup \big(N^r_{G-S}(x) \cap N^b_{G-S}(y)\big)$ and with the property that any path between x_i and y_i goes through z. Clearly, Γ_{xy} is not empty. We define also Γ to be a subset of I such that $i \in \Gamma$ if and only if there are at least two distinct pairs of vertices $x,\ y$ and $x',\ y'$, with $\{x,\ x'\} \subseteq \Phi$ and $\{y,\ y'\} \subseteq \Psi$ and i satisfies $i \in \Gamma_{xy} \cap \Gamma_{x'y'}$.

Let X_Γ be the set of vertices which are used in order to build the $\|\Gamma\|$ pairwise vertex-disjoint paths of length at most 8 joining the pairs x_i, y_i, with $i \in \Gamma$. Thus $X_\Gamma \subset X$.

Claim 4. Γ is not empty.

Proof. We have $\|\Phi\| \geq \frac{n}{2} - 8k - 7k > 3k$, since $\Phi = A \cap N^b(x_1)$ and $\|N_A^b(x_1)\| \geq \frac{n}{2} + k - 1 - \|X\| - \|C\| - \|S\| = \frac{n}{2} - 16k - 1 \geq 3k$. Also $\|\Psi\| > 3k$. Thus there are at least $3k$ pairs of distinct vertices in $\Phi \times \Psi$. However $\|I\| < k$. So Γ is not empty. $\square$

Claim 5. For each $i \in \Gamma$ and any choice of two distinct colors j and l, either $N^j(x_i) \cap \Phi = \emptyset$ and $N^l(y_i) \cap \Psi = \emptyset$ or $N^j(x_i) \cap \Psi = \emptyset$ and $N^l(y_i) \cap \Phi = \emptyset$.

Proof. Since $\|A \cup B\| > n - 16k - 2$, we may consider that there are at least three vertices $u,\ u'$ and u'' of $N^r(x_i)$ which belong either to A or to B. Assume that these three vertices are in A. We must show that there is no edge between y_i and A. Indeed, assume that there is an edge vy_i in G^c, $v \in A$. W.l.o.g. we may suppose that $u \neq v$ and that there are two vertices $x \in \Phi$ and $y \in \Psi$ such that $i \in \Gamma_{xy}$ and $x \neq u$ and $x \neq v$. If $c(vy_i) = b$, then, by Claim 3, there is a vertex w, distinct from $u,\ v,\ x$ such that the path $x_i \overset{r}{-} u \overset{b}{-} w \overset{r}{-} v \overset{b}{-} y_i$ exists in G^c. If $c(vy_i) = r$, then there is a vertex w distinct from $u,\ v,\ x$ in A (see (1)) and a vertex w' distinct from $u,\ v,\ w,\ x$ in A such that the path $x_i \overset{r}{-} u \overset{b}{-} w' \overset{r}{-} w \overset{b}{-} v \overset{r}{-} y_i$ (Claim 3) exists in G^c. But then, we may obtain the paths $x_1 \overset{b}{-} x \overset{r}{-} z \overset{b}{-} y \overset{r}{-} y_1$ or $x_1 \overset{r}{-} x \overset{b}{-} z \overset{r}{-} y \overset{b}{-} y_1$ (where z is a vertex used between x_i and y_i) which is in contradiction with our assumption that there is no path of length less than 8 between x_1 and y_1. This completes the proof of this claim. $\square$

Claim 5 means that there is no vertex in Φ (respectively in Ψ) having both x_i and y_i as neighbors, for any $i \in \Gamma$. As there is no edge between Φ and y_1 and no edge between Ψ and x_1, then for $i \in \{r, b\}$, and for each vertex x of Φ and each vertex y of Ψ we have,

$$d^i_{G-S}(x) = d^i_{A \cup C \cup X}(x) \geq \frac{n}{2} + k - 1 - (2k - 1) + \|\Gamma\| = \frac{n}{2} - k + \|\Gamma\|$$

and

$$d^i_{G-S}(y) = d^i_{B \cup C \cup X}(y) \geq \frac{n}{2} - k + \|\Gamma\|.$$

By summing the above inequalities we obtain $d^r_{G-S}(x) + d^b_{G-S}(y) \geq n - 2k + 2\|\Gamma\|$. Also $(S \cup \{x, y\}) \cap (N^r_{G-S}(x) + N^b_{G-S}(y)) = \emptyset$. Consequently $\|N^r_{G-S}(x) \cup N^b_{G-S}(y)\| \leq n - 2k - 2$. In conclusion, we obtain $\|N^r_{G-S}(x) \cap N^b_{G-S}(y)\| \geq 2 + 2\|\Gamma\|$ and $\|N^b_{G-S}(x) \cap N^r_{G-S}(y)\| \geq 2 + 2\|\Gamma\|$.

From now on, we are going to define $\|I\| + 1$ disjoint paths each of length at most 8. This will contradict the maximality property of I and will permit to complete the proof of case (I). Without loss of generality, let us set $\Gamma = \{2, 3, ..., \|\Gamma\| + 1\}$. Furthermore, since x_i and y_i play a symmetric role, we may suppose that for each $i \in \Gamma$ and for any choice of two colors j and l, we have $N^j(x_i) \cap \Phi = \emptyset$ and $N^l(y_i) \cap \Psi = \emptyset$. Since $\|N^r(x_i) \cap A\| \geq k$ and $\|N^b(y_i) \cap B\| \geq k$, we can find $\|\Gamma\| + 1$ distinct vertices $x', x'_2, x'_3, ..., x'_{\|\Gamma\|+1}$ in A and $\|\Gamma\| + 1$ distinct vertices $y', y'_2, y'_3, ..., y'_{\|\Gamma\|+1}$ in B such that $c(x_i x'_i) = c(x_1 x') = r$ and $c(y_i y'_i) = c(y_1 y') = b$. Recall also that for every pair of vertices $x \in \Phi$, $y \in \Psi$, we have $\|N^b_{G-S}(x) \cap N^r_{G-S}(y)\| \geq 2 + 2\|\Gamma\|$. In addition $\|\Phi\| \geq 3k$ and $\|\Psi\| \geq 3k$. Consequently we can find $\|\Gamma\|$ distinct vertices $x''_2, x''_3, ..., x''_{\|\Gamma\|+1}$ in A, $\|\Gamma\|$ distinct vertices $y''_2, y''_3, ..., y''_{\|\Gamma\|+1}$ in B and $\|\Gamma\| + 1$ distinct vertices $z, z_2, z_3, ..., z_{\|\Gamma\|+1}$ in X_Γ. Indeed, there are at least $2k$ pairs of distinct vertices in $\Phi \times \Psi$ and there are at most $\|I\| - \|\Gamma\|$ pairs of distinct vertices of $\Phi \times \Psi$ which are joined by paths $x_i'' \overset{b}{-} z_i \overset{r}{-} y_i''$ and $x' \overset{b}{-} z \overset{r}{-} y'$ of length two going through $X - X_\Gamma$ according to the definition of Γ. Also all vertices of these paths are distinct from the previous ones. Finally, according to the Claim 2, we can find $\|\Gamma\|$ distinct vertices $x_2^{(3)}, x_3^{(3)}, ..., x_{\|\Gamma\|+1}^{(3)}$ in A and $\|\Gamma\|$ distinct vertices $y_2^{(3)}, y_3^{(3)}, ..., y_{\|\Gamma\|+1}^{(3)}$ in B, such that the paths $x_i' \overset{b}{-} x_i^{(3)} \overset{r}{-} x_i''$ and $y_i'' \overset{b}{-} y_i^{(3)} \overset{r}{-} y_i'$ are in G^c and all vertices of these paths are distinct from the previous ones.

In this way we defined the $\|I\| + 1$ distinct paths

$$x_i \overset{r}{-} x_i' \overset{b}{-} x_i^{(3)} \overset{r}{-} x_i'' \overset{b}{-} z_i \overset{r}{-} y_i'' \overset{b}{-} y_i^{(3)} \overset{r}{-} y_i' \overset{b}{-} y_i$$

and

$$x_1 \overset{r}{-} x' \overset{b}{-} z \overset{r}{-} y' \overset{b}{-} y_1.$$

As each of the above paths has length less than 8, this is a contradiction to the maximality property of I.

(II) There is at least one edge between A and B.

Claim 6. There are two subsets D and E of $V(G^c)$ such that
i) $D \subset A$, $E \subset B$, $\|D\| \geq 3k$, $\|E\| \geq 3k$ and $\|D \cup E\| \geq \frac{n}{2} + k$,
ii) for each vertex $x \in D$, $N^b(x) \cap A = \emptyset$ and for each vertex $y \in E$, $N^r(y) \cap B = \emptyset$ and
iii) for each pair of vertices x, x' of D, $\|N_B^r(x) \cup N_B^b(x')\| \geq 3k$ and for each pair of vertices y, y' of E, $\|N_A^r(y) \cup N_A^b(y')\| \geq 3k$.

Proof. Let xy be an edge between A and B, $x \in A$, $y \in B$. Assume w.l.o.g. that its color is blue. Then, there is no red edge between y and B, for otherwise the path $x_1 \overset{r}{-} x \overset{b}{-} y \overset{b}{-} z \overset{b}{-} y_1$, $z \in B$, has length less than 8, a contradiction to our assumptions. So, $d_A^r(y) = d_{A \cup B}^r(y) \geq \frac{n}{2} + k - 1 - 9k - 8k + 1 = \frac{n}{2} - 16k$ and for each vertex $x \in N_A^r(y)$, $d_B^b(x) \geq \frac{n}{2} - 16k$.

Let E be a subset of B such that every vertex u of E has at least $\frac{2}{3}(\frac{n}{2} - 24k)$ neighbors v in $N_A^r(y)$, the color of uv is blue and subject to this requirement E is as big as possible. We must first show that such a set E exists and $\|E\| \geq \frac{n}{2} - 60k \geq 3k$. We have $\|B\| \leq n - \|S\| - min(\|A\|) \leq \frac{n}{2} + 6k$. The worst case arrises when each vertex u of $N_A^r(y)$ is joined with monochromatic blue edges to each vertex of E and distribute the rest of the colors on edges (if any) joining u with the remaining vertices of B. In fact we must show that the average of the blue edges between a vertex of $B - E$ and $N_A^r(y)$ is at least $\frac{2}{3}(\frac{n}{2} - 24k)$. In particular, we must prove that, if $\|E\| = \frac{n}{2} - \alpha k$, then

$$\frac{(d_A^r(y) - \|E\|) * d_A^b(x)}{max(\|B\|) - \|E\|} \geq \frac{2}{3}(\frac{n}{2} - 24k)$$

$$\frac{(\alpha - 16)k * (\frac{n}{2} - 16k)}{(\alpha + 6)k} \geq \frac{2}{3}(\frac{n}{2} - 24k)$$

$$\frac{\alpha n}{6} \geq 10n - 352k$$

In particular, for $\|E\| = \frac{n}{2} - 60k$ we obtain $\alpha = 60$ and then the previous equation is true. Also for each $y \in E$, $d_A^b(y) \geq \frac{2}{3}(\frac{n}{2} - 24k)$ and $d_A^r(y) \geq \frac{n}{2} - 16k$. With similar arguments we define the subset D of A, such that for $x \in N^r_A(y)$, every vertex u of D is connected to at least $\frac{2}{3}(\frac{n}{2}-24k)$ vertices v of $N^b_A(x)$ with red edges and D is as maximum as possible. Then $\|D\| \geq \frac{n}{2}-60k \geq 3k$ and for every $x \in D$, $d^r_B(x) \geq \frac{2}{3}(\frac{n}{2}-24k)$ and $d^b_B(x) \geq \frac{n}{2}-16k$. Moreover, for each vertex $x \in D$, $N^b(x) \cap A = \emptyset$ and for each vertex $y \in E$, $N^r(y) \cap B = \emptyset$. In addition, for each pair x, x' of vertices of D, $d^r_B(x) \geq \frac{2}{3}(\frac{n}{2}-24k)$ and $d^b_B(x) \geq \frac{n}{2}-16k$. So $d^r_B(x)+d^b_B(x) \geq \frac{5}{6}n-32k$. However $\|B\| \leq \frac{n}{2}+6k$. Thus $\|N^r_B(x) \cup N^b_B(x')\| \geq \frac{n}{3}-38k \geq 3k$. Similarly, for every pair y, y' of vertices of E, we have $\|N^r_A(y) \cup N^b_A(y')\| \geq 3k$. Then $\|D \cup E\| \geq n+k$ since $\|D \cup E\| \geq n-120k \geq \frac{n}{2}+k$. This completes the proof of the claim. □

Let $x,\ y$ be two vertices of G^c, $x \in D,\ y \in E$. Clearly $\{x_1, y_1\} \cap N^b(x) = \emptyset$, for otherwise one of the paths $x_1 \overset{r}{-} x \overset{b}{-} y_1$ or $x_1 \overset{b}{-} x \overset{r}{-} y' \overset{b}{-} y_1,\ y' \in E)$ exists in G^c, a contradiction to the assumption that there is no path between x_1 and y_1 of length less than 8. Similarly, $\{x_1, y_1\} \cap N^r(y) = \emptyset$. We have $d^b_{G^c-(S\cup\{y\})}(x) \geq \frac{n}{2}+k-1-2(k-1)-1 = \frac{n}{2}-k$. Analogously $d^r_{G^c-(S\cup\{x\})}(y) \geq \frac{n}{2}-k$. We also have $d^b_{G^c-(S\cup\{y\})}(x)+d^r_{G^c-(S\cup\{x\})}(y) \geq n-2k$ and $(S\cup\{x,y\}) \cap (N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})}(y)) = \emptyset$. So $\|N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})}(y)\| \leq n-2k-2$. In conclusion, we obtain $\|N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})}(y)\| \geq 2$. If $N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})}(y)$ is not a subset of X, then we can consider the path $x_1 \overset{b}{-} x \overset{r}{-} z \overset{b}{-} y \overset{r}{-} y_1$, contradicting again our assumptions. So we assume that $N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})}(y) \subset X$. Let Ω_{xy} be a subset of I such that $i \in \Omega_{xy}$ if and only if there is a vertex $z \in N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})}(y)$ such that the path between x_i and y_i goes through z. The set Ω_{xy} is not empty.
We also define a subset Ω of I such that $i \in \Omega$ if and only if there are at least four distinct vertices $x,\ x',\ y,\ y'$ such that $x,\ x' \in D,\ y,\ y' \in E$ and $i \in \Omega_{xy} \cap \Omega_{x'y'}$. Let X_Ω be the set of vertices which are used in order to build the $\|\Omega\|$ pairwise vertex-disjoint paths of length at most 8 joining the pairs $x_i,\ y_i$, with $i \in \Omega$. Hence, $X_\Omega \subset X$.

Claim 7. Ω is not empty.

Proof. Straightforward from the fact that $\|D\| \geq 2k$, $\|E\| \geq 2k$ and $\|I\| \leq k$. □

Claim 8. For every i in Ω, either $(N^r(x_i) \cup N^r(y_i)) \cap E = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap D = \emptyset$ or $(N^r(x_i) \cup N^r(y_i)) \cap D = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap E = \emptyset$.

Proof. Since $\|D \cup E\| > \frac{n}{2}+k$, we can claim that there are at least three vertices u, u' and u'' of $N^r(x_i)$ which belong either to D or to E. Assume that these three vertices are in A. We will show by contradiction that there is no red edge between y_i and E. Assume therefore that there is a red edge vy_i in G^c, with $v \in E$. According to Claim 6, there is a blue edge vw in G^c, $w \in D$. W.l.o.g. let us suppose that $u \neq w$ and that there are two vertices $x \in D,\ y \in E$ such that $i \in \Omega_{xy}$, $x \neq u$, $x \neq w$ and $y \neq v$. Moreover, according to Claim 6, there is a vertex t of E such that $t \neq u$, $t \neq y$, and the path $u \overset{b}{-} t \overset{r}{-} w$ exists in G^c. But then we may define the paths $x_i \overset{r}{-} u \overset{b}{-} t \overset{r}{-} w \overset{b}{-} v \overset{r}{-} y_i$ and $x_1 \overset{r}{-} x \overset{b}{-} z \overset{r}{-} y \overset{b}{-} y_1$, where z is a vertex used by the path between x_i and y_i, a contradiction to our assumptions. So $(N^r(x_i) \cup N^r(y_i)) \cap E = \emptyset$ or $(N^r(x_i) \cup N^r(y_i)) \cap D = \emptyset$. With the same argument, we obtain that $\left(N^b(x_i) \cup N^b(y_i)\right) \cap E = \emptyset$ or $\left(N^b(x_i) \cup N^b(y_i)\right) \cap D = \emptyset$. Assume that $(N^r(x_i) \cup N^r(y_i)) \cap E = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap E = \emptyset$. Then there are two vertices $x \in D,\ y \in E$ such that $i \in \Omega_{xy}$. Similarly there are two distinct vertices u and v of D such that $x \neq u,\ v$ and $u \in N^r(x_i)$ and $v \in N^b(y_i)$. Moreover, according to Claim 6, there is a vertex w of E such that $w \neq y$ and the path $u \overset{b}{-} t \overset{r}{-} v$ exists in G^c. But then we may define the set of paths $x_i \overset{r}{-} u \overset{b}{-} w \overset{r}{-} v \overset{b}{-} y_i$ and $x_1 \overset{r}{-} x \overset{b}{-} z \overset{r}{-} y \overset{b}{-} x_1$, where z is a vertex used by the path between x_i and y_i, again a contradiction to our assumptions. The completes the proof of Claim 8. $\square$

Now we can define two subsets Ω^r and Ω^b of Ω as follows: $i \in \Omega^r$ if and only if $i \in \Omega$, $(N^r(x_i) \cup N^r(y_i)) \cap E = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap D = \emptyset$. Similarly, $i \in \Omega^b$ if and only if $i \in \Omega$, $(N^r(x_i) \cup N^r(y_i)) \cap D = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap E = \emptyset$. According to Claim 8, we have $\Omega^r \cap \Omega^b = \emptyset$ and $\Omega^r \cup \Omega^b = \Omega$. Moreover $\Omega^r \neq \emptyset$, since $\|N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})}(y)\| \geq 2$. With the dfinitions above, Claim 8 means that, if $i \in \Omega^r$, then there are no blue edges between D and $\{x_i, y_i\}$ and no red edges between E and $\{x_i, y_i\}$. Similarly, if $i \in \Omega^b$, then there are no red edges between D and $\{x_i, y_i\}$ and no blue edges between E and $\{x_i, y_i\}$.
Thus, for every vertex x of D,

$$d^b_{G^c-(S\cup\{y\})}(x) \geq \frac{n}{2} + k - 1 - 2(k-1) - 1 + 2\|\Omega^r\| = \frac{n}{2} - k + 2\|\Omega^r\|,$$

$$d^r_{G^c-(S\cup\{y\})}(x) \geq \frac{n}{2} + k - 1 - 2k - 1 + 2\|\Omega^b\| = \frac{n}{2} - k + 2\|\Omega^b\| - 2.$$

Similarly, for every vertex y in E,

$$d^b_{G^c-(S\cup\{x\})}(y) \geq \frac{n}{2} + k - 1 - 2k - 1 + 2\|\Omega^b\| = \frac{n}{2} - k + 2\|\Omega^b\| - 2,$$

$$d^r_{G^c-(S\cup\{x\})}(y) \geq \frac{n}{2} + k - 1 - 2(k-1) - 1 + 2\|\Omega^r\| = \frac{n}{2} - k + 2\|\Omega^r\|.$$

From the above inequalities we obtain $d^b_{G^c-(S\cup\{y\})}(x) + d^r_{G^c-(S\cup\{x\})}(y) \geq n - 2k + 4\|\Omega^r\|$. Furthermore $(S \cup \{x,y\}) \cap (N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})})(y) = \emptyset$. So $\|N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})}(y)\| \leq n - 2k - 2$. In conclusion we obtain $\|N^b_{G^c-(S\cup\{y\})}(x) \cap N^r_{G^c-(S\cup\{x\})}(y)\| \geq 4\|\Omega^r\| + 2$ and $\|N^r_{G^c-(S\cup\{y\})}(x) \cap N^b_{G^c-(S\cup\{x\})}(y)\| \geq 4\|\Omega^b\| - 2$.

We distinguish now between two cases depending upon the cardinality of Ω^b.

Case 1. $\|\Omega^b\| = 0$.
As $\|\Omega^b\| = 0$, it follows that $\|\Omega^r\| = \|\Omega\|$. Now, we are going to define $\|\Omega\| + 1$ pairwise vertex disjoint paths each of length at most 8. As Ω is a subset of I, this will be a contradiction with its maximality property. Set $\Omega^r = \{2, 3, ..., \|\Omega^r\|+1\}$. Since $\|N^r(x_i) \cap D\| \geq k$ and $\|N^b(y_i) \cap E\| \geq k$, we can find $\|\Omega^r\| + 1$ distinct vertices $x', x'_2, x'_3, ..., x'_{\|\Omega^r\|+1}$ of D and $\|\Omega^r\| + 1$ distinct vertices $y', y'_2, y'_3, ..., y'_{\|\Omega^r\|+1}$ of B such that $c(x_i x'_i) = c(x_1 x') = r$ and $c(y_i y'_i) = c(y_1 y') = b$. Recall also that for every two vertices $x \in D$, $y \in E$, we have$\|N^b_{G^c-(S\cup\{y\})}(x) \cap N^r_{G^c-(S\cup\{x\})}(y)\| \geq 4\|\Omega^r\| + 2$ and $\|D\| \geq 3k$ and $\|E\| \geq 3k$. Thus we can find $\|\Omega^r\|$ vertices $x''_2, x''_3, ..., x''_{\|\Omega^r\|+1}$ of D, $\|\Omega^r\|$ vertices $y''_2, y''_3, ..., y''_{\|\Omega^r\|+1}$ of E and $\|\Omega^r\| + 1$ vertices $z, z_2, z_3, ..., z_{\|\Omega^r\|+1}$ of X_Ω such that the distinct paths $x''_i \overset{b}{-} z_i \overset{r}{-} y''_i$ and $x' \overset{b}{-} z \overset{r}{-} y'$ exist in G^c. All the above mentionned sets of vertices exist in G^c, since there are at least $2k$ pairs of distinct vertices of $D \times E$ and there are at most $\|I\| - \|\Omega^r\|$ pairs of distinct vertices of $D \times E$ which are joined by a path of length two

going through $X - X_\Omega$ according to the definition of Ω. Last, according to Claim 6, we can find $\|\Omega^r\|$ vertices $x_2^{(3)}, x_3^{(3)}, ..., x_{\|\Omega^r\|+1}^{(3)}$ of D and $\|\Omega^r\|$ vertices $y_2^{(3)}, y_3^{(3)}, ..., y_{\|\Omega^r\|+1}^{(3)}$ of E, such that the paths $x_i' \overset{b}{-} x_i^{(3)} \overset{r}{-} x_i''$ and $y_i'' \overset{b}{-} y_i^{(3)} \overset{r}{-} y_i'$ exist in G^c. But in that way we may define the following $\|\Omega^r\| + 1$ pairwise vertex-disjoint paths

$$x_i \overset{r}{-} x_i' \overset{b}{-} x_i^{(3)} \overset{r}{-} x_i'' \overset{b}{-} z_i \overset{r}{-} y_i'' \overset{b}{-} y_i^{(3)} \overset{r}{-} y_i' \overset{b}{-} y_i$$

and

$$x_1 \overset{r}{-} x' \overset{b}{-} z \overset{r}{-} y' \overset{b}{-} y_1,$$

a contradiction.

Case 2. $\|\Omega^b\| > 0$ The proof of this second case is based on Claims 9-12 below.

Claim 9. There are at least three disinct pairs of vertices u_j and v_j, $j = 1,\ 2,\ 3$, such that $\Omega_{u_j v_j} \cap \Omega^b \neq \emptyset$.

Proof. Let X_{Ω^r} be the set of vertices which are used in order to define $\|\Omega^r\|$ pairwise vertex-disjoint paths of length at most 8, one per pair x_i, y_i, with $i \in \Omega^r$. Then $X_{\Omega^r} \subset X_\Omega$. Indeed assume there are at most two distinct pairs of vertices u_j, v_j such that $\Omega_{u_j v_j} \cap \Omega^b \neq \emptyset$, $j = 1, 2$. Then, by using arguments similar to those of Case 1, we can define $\|\Omega^r\| + 1$ pairwise vertex disjoint paths between x_1 and y_1 and between x_i and y_i of length at most 8, for every $i \in \Omega^r$. To define these paths, we need to find $\|\Omega^r\|$ vertices $x_2'', x_3'', ..., x_{\|\Omega^r\|+1}''$ in D, $\|\Omega^r\|$ vertices $y_2'', y_3'', ..., y_{\|\Omega^r\|+1}''$ in E and $\|\Omega^r\| + 1$ vertices $z, z_2, z_3, ..., z_{\|\Omega^r\|+1}$ in X_{Ω^r} such that the following paths $x_i'' \overset{b}{-} z_i \overset{r}{-} y_i''$ and $x' \overset{b}{-} z \overset{r}{-} y'$ exist in G^c. Indeed all the above-mentionned sets of vertices exist in G^c because, as there are at least $2k$ pairs of vertices of $D \times E$, then there are at most $\|I\| - \|\Omega^r\|$ pairs of distinct vertices of $D \times E$ which are joined by paths of length two going through $X - X_\Omega$ according to the definition of Ω. Also, as there are at most 2 pairs of distinct vertices of $D \times E$ which are joined by paths of length two going through $X - X_\Omega$, then there are at least $\|\Omega^r\| + 1$ pairs of distinct vertices (u_j', v_j') of $D \times E$ such that $\Omega_{u_j' v_j'} \subset \Omega^r$ for every j. This completes the proof of the claim. □

Claim 10. $N^r_{G^c-(S\cup\{y\})}(x) \cap N^b_{G^c-(S\cup\{x\})}(y) \subset X$.

Proof. Assume that there is a vertex z of $N^r_{G^c-(S\cup\{y\})}(x) \cap N^b_{G^c-(S\cup\{x\})}(y)$ such that $z \in C \cup A \cup B$. Since $\|\Omega^b\| > 0$, there is an integer $t \in \Omega^b$ such that we can find four distinct vertices x', $u' \in D$ and $y', v' \in E$ such that $t \in \Omega_{x'y'}$, $u \in N^b(x_t)$. Also we can find two vertices $v \in N^r(y_t)$ such that x, x', u, z (respectively y, y', v, z) are pairwise distinct. Let z_t be a vertex of the path between x_t and y_t such that $z_t \in N^b_{G^c-(S\cup\{y'\})}(x') \cap N^r_{G^c-(S\cup\{x'\})}(y')$. By using this vertex z_t and according to Claim 6, we can find two vertices u' and w' such that both paths $x_t \overset{b}{-} u \overset{r}{-} u' \overset{b}{-} x \overset{r}{-} z \overset{b}{-} y \overset{r}{-} v' \overset{b}{-} v \overset{r}{-} y_t$ and $x_1 \overset{r}{-} x' \overset{b}{-} z_t \overset{r}{-} y' \overset{b}{-} y_1$ exist in G^c.

Let Θ_{xy} a subset of I such that $i \in \Theta_{xy}$ if and only if there is a vertex $z \in N^r_{G^c-(S\cup\{y\})}(x) \cup N^b_{G^c-(S\cup\{x\})}(y)$ such that the path between x_i and y_i goes through z. Θ_{xy} is not empty, since $\|N^r_{G^c-(S\cup\{y\})}(x) \cap N^b_{G^c-(S\cup\{x\})}(y)\| \geq 4\|\Omega^b\| - 2$ and $\|\Omega^b\| > 0$. We define also a subset Θ of I

such that $i \in \Theta$ if and only if there are at least four vertices $x,\ x' \in \Phi,\ y,\ y' \in \Psi$ with $x \neq x'$ and $y \neq y'$ such that $i \in \Theta_{xy} \cap \Theta_{x'y'}$. Let X_Θ be the set of vertices which are used in order to build the $\|\Theta\|$ pairwise vertex-disjoint paths each of length at most 8 joining the pairs $x_i,\ y_i$, with $i \in \Theta$. Clearly $X_\Theta \subset X$. □

Claim 11. Θ is not empty.

Proof. As $\|D\| \geq 2k$, $\|E\| \geq 2k$ and $\|I\| \leq k$, the conclusion is straightforward. □

Claim 12. For every $i \in \Theta$, either $(N^r(x_i) \cup N^r(y_i)) \cap E = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap D = \emptyset$ or $(N^r(x_i) \cup N^r(y_i)) \cap D = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap E = \emptyset$.

Proof. Let $i \in \Theta$. If $i \in \Omega$ the Claim is true according to Claim 7. Assume therefore that $i \notin \Omega$. Since $\|D \cup E\| > \frac{n}{2} + k$, there are at least three vertices u, u' and u'' of $N^r(x_i)$ which belong either to D or to E. Assume that these three vertices are in D. We must show that there is no red edge between y_i and E. Assume by contradiction that there is a red edge vy_i in G^c with $v \in E$. According to Claim 6, there is a blue edge vw in G^c, $w \in D$. W.l.o.g. we may assume that $u \neq w$ and that there are two vertices $x \in D,\ y \in E$ such that $i \in \Theta_{xy}$, $x \neq u$, $x \neq w$ and $y \neq v$. Since $\|\Omega^b\| > 0$, by Claim 10 we can find an integer $t \in \Omega^b$ and two vertices $x'_t \in D$ and $y'_t \in E$ such that $t \in \Omega_{x'_t y'_t}$. Furthermore x, x'_t, w (respectively y, y'_t, v) are pairwise distinct. In addition, we can also find two distinct vertices x''_t and y''_t such that $x''_t \in N^b(x_t)$ and $y''_t \in N^r(y_t)$. Now, according to Claim 6, there are two vertices w', $x_t^{(3)}$ of E and one vertex $x_t^{(3)}$ of D, distinct from the above-mentionned ones, and such that the paths $u \overset{b}{-} w' \overset{r}{-} w$, $x''_t \overset{r}{-} x_t^{(3)} \overset{b}{-} x$ and $y''_t \overset{r}{-} y_t^{(3)} \overset{b}{-} y$ exist in G^c. However in that way we may define the paths $x_i \overset{r}{-} u \overset{b}{-} w' \overset{r}{-} w \overset{b}{-} v \overset{r}{-} y_i$, $x_1 \overset{r}{-} x'_t \overset{b}{-} z_t \overset{r}{-} y'_t \overset{b}{-} x_1$ and $x_t \overset{b}{-} x''_t \overset{r}{-} x_t^{(3)} \overset{b}{-} x \overset{r}{-} z_i \overset{b}{-} y \overset{r}{-} y_i^{(3)} \overset{b}{-} y''_i \overset{r}{-} y_1$ (z_i is a vertex used between x_i and y_i and z_t a vertex used between x_t and y_t) a contradiction to our assumptions. Thus we may conclude that either $(N^r(x_i) \cup N^r(y_i)) \cap E = \emptyset$ or $(N^r(x_i) \cup N^r(y_i)) \cap D = \emptyset$. By similar arguments we obtain $\left(N^b(x_i) \cup N^b(y_i)\right) \cap E = \emptyset$ or $\left(N^b(x_i) \cup N^b(y_i)\right) \cap D = \emptyset$. In order to complete the prood we need to exclude the case $(N^r(x_i) \cup N^r(y_i)) \cap E = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap E = \emptyset$. Assume therefore that $(N^r(x_i) \cup N^r(y_i)) \cap E = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap E = \emptyset$. There are two vertices $x \in D,\ y \in E$ such that $i \in \Omega_{xy}$. In addition, there are two distinct vertices u and v of D such that $x \neq u,\ v,\ u \in N^r(x_i)$ and $v \in N^b(y_i)$. But, then we may define the paths $x_i \overset{r}{-} u \overset{b}{-} w \overset{r}{-} v \overset{b}{-} y_i$, $x_1 \overset{r}{-} x'_t \overset{b}{-} z_t \overset{r}{-} y'_t \overset{b}{-} x_1$ and $x_t \overset{b}{-} x''_t \overset{r}{-} x_t^{(3)} \overset{b}{-} x \overset{r}{-} z_i \overset{b}{-} y \overset{r}{-} y_i^{(3)} \overset{b}{-} y''_i \overset{r}{-} y_1$, again a contradiction to our assumptions. This completes the proof of Claim 12. □

Let us now set $\Lambda = \Omega \cup \Theta$. We define two new subsets Λ^r and Λ^b of I as follows: We let $i \in \Lambda^r$ if and only if $i \in \Lambda$, $(N^r(x_i) \cup N^r(y_i)) \cap E = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap D = \emptyset$. Similarly, $i \in \Lambda^b$ if and only if $i \in \Lambda$, $(N^r(x_i) \cup N^r(y_i)) \cap D = \emptyset$ and $\left(N^b(x_i) \cup N^b(y_i)\right) \cap E = \emptyset$. According to these definitions and Claim 8, we have $\Lambda^r \cap \Lambda^b = \emptyset$ and $\Lambda^r \cup \Lambda^b = \Lambda$. Now, in terms of Λ^r and Λ^b, Claims 8 and 13 mean that, if $i \in \Lambda^r$, then there are no blue edges between D and $\{x_i, y_i\}$ and no red edges between E and $\{x_i, y_i\}$. Similarly, if $i \in \Lambda^b$, then there are no red edges between D and $\{x_i, y_i\}$ and no blue edges between E and $\{x_i, y_i\}$. We recall that there is no blue edge between D and $\{x_1, y_1\}$ and no red edge between E and $\{x_1, y_1\}$).

Thus for each vertex x of D we have,

$$d^b_{G^c-(S\cup\{y\})}(x) \geq \frac{n}{2}+k-1-2(k-1)-1+2\|\Lambda^r\| = \frac{n}{2}-k+2\|\Lambda^r\|,$$

$$d^r_{G^c-(S\cup\{y\})}(x) \geq \frac{n}{2}+k-1-2k-1+2\|\Lambda^b\| = \frac{n}{2}-k+2\|\Lambda^b\|-2.$$

Similarly, for each vertex y of E,

$$d^b_{G^c-(S\cup\{x\})}(y) \geq \frac{n}{2}+k-1-2k-1+2\|\Lambda^b\| = \frac{n}{2}-k+2\|\Lambda^b\|-2,$$

$$d^r_{G^c-(S\cup\{x\})}(y) \geq \frac{n}{2}+k-1-2(k-1)-1+2\|\Lambda^r\| = \frac{n}{2}-k+2\|\Lambda^r\|.$$

By summing the above inequalities we obtain, $d^b_{G^c-(S\cup\{y\})}(x) + d^r_{G^c-(S\cup\{x\})}(y) \geq n-2k+4\|\Lambda^r\|$. Also, $(S\cup\{x,y\}) \cap (N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})})(y) = \emptyset$. So $\|N^b_{G^c-(S\cup\{y\})}(x) \cup N^r_{G^c-(S\cup\{x\})}(y)\| \leq n-2k-2$. We conclude that $\|N^b_{G^c-(S\cup\{y\})}(x) \cap N^r_{G^c-(S\cup\{x\})}(y)\| \geq 4\|\Lambda^r\|+2$. Similarly, $\|N^r_{G^c-(S\cup\{y\})}(x) \cap N^b_{G^c-(S\cup\{x\})}(y)\| \geq 4\|\Lambda^b\|-2$.
Let X_Λ be the set of vertices which are used in order to define the $\|\Lambda\|$ pairwise vertex-disjoint paths of length at most 8, joining the pairs x_i, y_i, for each $i\in\Lambda$. We have $X_\Lambda \subset X$. In the sequel, we shall define $\|\Lambda\|+1$ distinct paths each of length at most 8. This will contradict the maximality property of Λ, and will permit to complete the proof of the theorem. Set $\Lambda=\{2,3,...,\|\Lambda\|+1\}$. Recall that for each $i\in\Lambda^r$, $\|N^r(x_i)\cap D\| \geq k$ and $\|N^b(y_i)\cap E\| \geq k$. Respectively, for each $i\in\Lambda^b$, $\|N^b(x_i)\cap D\| \geq k$ and $\|N^r(y_i)\cap E\| \geq k$. Thus, we can find $\|\Lambda\|+1$ distinct vertices $x', x'_2, x'_3, ..., x'_{\|\Omega^r\|+1}$ in D and $\|\Lambda\|+1$ vertices $y', y'_2, y'_3, ..., y'_{\|\Omega^r\|+1}$ in B. Furthermore, for every $i\in\Lambda^r$, $c(x_1x')=r$, $c(y_1y')=b$, $c(x_ix'_i)=r$. Also for every $i\in\Lambda^b$, $c(x_ix'_i)=b$. In addition, for every $i\in\Lambda^r$, $c(y_iy'_i)=b$. Finally, for every $i\in\Lambda^b$, $c(y_iy'_i)=r$. Moreover, we can find $\|\Lambda\|$ vertices $x''_2, x''_3, ..., x''_{\|\Lambda\|+1}$ of D, $\|\Lambda\|$ vertices $y''_2, y''_3, ..., y''_{\|\Lambda\|+1}$ of E and $\|\Lambda\|+1$ vertices $z, z_2, z_3, ..., z_{\|\Lambda\|+1}$ of X_Λ such that the paths $x''_i \overset{b}{-} z_i \overset{r}{-} y''_i$ if $i\in\Lambda^r$ or $x''_i \overset{r}{-} z_i \overset{b}{-} y''_i$ if $i\in\Lambda^b$ and $x' \overset{b}{-} z \overset{r}{-} y'$ exist in G^c. Indeed, there are at least $2k$ pairs of distinct vertices of $D\times E$ and there are at most $\|I\|-\|\Lambda\|$ pairs of distinct vertices of $D\times E$ which are joined by a path of length two going through $X-X_\Lambda$ according to the definition of Λ. Assume first $\|\Lambda^r\| \geq \|\Lambda^b\|$. First, we consider the vertices x''_i, y''_i, z_i with $i\in\Lambda^b$ (recall that for $x\in D,\ y\in E$, we have $\|N^r_{G^c-(S\cup\{y\})}(x) \cap N^b_{G^c-(S\cup\{x\})}(y)\| \geq 4\|\Lambda^b\|-2 \geq \Lambda^b$). Next we consider the vertices x''_i, y''_i, z_i with $i\in\Lambda^r$ (again for $x\in D,\ y\in E$, it holds $\|N^b_{G^c-(S\cup\{y\})}(x) \cap N^r_{G^c-(S\cup\{x\})}(y)\| \geq 4\|\Lambda^r\|+2 \geq \Lambda^b+\Lambda^r$). Assume next $\|\Lambda^r\| < \|\Lambda^b\|$. In that case, we consider first the vertices x''_i, y''_i, z_i with $i\in\Lambda^r$ (recall again that for $x\in D,\ y\in E$, $\|N^b_{G^c-(S\cup\{y\})}(x) \cap N^r_{G^c-(S\cup\{x\})}(y)\| \geq 4\|\Lambda^r\|+2 \geq \Lambda^r$). Next we consider the vertices x''_i, y''_i, z_i with $i\in\Lambda^b$ (as for $x\in D,\ y\in E$, it holds $\|N^r_{G^c-(S\cup\{y\})}(x) \cap N^b_{G^c-(S\cup\{x\})}(y)\| \geq 4\|\Lambda^b\|-2 \geq \Lambda^b+\Lambda^r$). Finally, according to Claim 6, we can find $\|\Lambda\|$ vertices $x^{(3)}_2, x^{(3)}_3, ..., x^{(3)}_{\|\Lambda\|+1}$ of D and $\|\Lambda\|$ vertices $y^{(3)}_2, y^{(3)}_3, ..., y^{(3)}_{\|\Lambda\|+1}$ of E. By using these vertices we may define a set of paths as follows: If $i\in\Lambda^r$ we define the paths $x'_i \overset{b}{-} x^{(3)}_i \overset{r}{-} x''_i$ or if $i\in\Lambda^b r$, then $x'_i \overset{r}{-} x^{(3)}_i \overset{b}{-} x''_i$. Furtehrmore, if $i\in\Lambda^r$, we define $y''_i \overset{b}{-} y^{(3)}_i \overset{r}{-} y'_i$ or if $i\in\Lambda^r$ we define $y''_i \overset{r}{-} y^{(3)}_i \overset{b}{-} y'_i$.

In that way we may define $\|\Lambda\|+1$ distinct paths as follows:
For every $i \in \Lambda^r$,

$$x_i \overset{r}{-} x_i' \overset{b}{-} x_i^{(3)} \overset{r}{-} x_i'' \overset{b}{-} z_i \overset{r}{-} y_i'' \overset{b}{-} y_i^{(3)} \overset{r}{-} y_i' \overset{b}{-} y_i.$$

For every $i \in \Lambda^r$,

$$x_i \overset{b}{-} x_i' \overset{r}{-} x_i^{(3)} \overset{b}{-} x_i'' \overset{r}{-} z_i \overset{b}{-} y_i'' \overset{r}{-} y_i^{(3)} \overset{b}{-} y_i' \overset{r}{-} y_i.$$

Finally, for $i = 1$

$$x_1 \overset{r}{-} x' \overset{b}{-} z \overset{r}{-} y' \overset{b}{-} y_1.$$

This contradicts the maximality property of Λ and completes the proof of the theorem. $\square$

Theorem 2.3. *Let G^c be a c-edge-colored multigraph of order n and k a non-zero positive integer. If for every vertex x, $d^c(x) \geq \frac{n}{2}$, then G^c is k-edge-linked.*

Proof. Let $x_i,\ y_i,\ 1 \leq i \leq k$, be $2k$ distinct vertices of G^c. We shall prove a stronger result, namely, that we can find k pairwise edge-disjoint paths of length at most two, one per pair x_i and y_i.
Assume first that for some i, the edge x_iy_i exists in G^c. Then this edge defines a path between x_i and y_i. This choice will not affect the rest of the proof, as any path between another pair of vertices x_j, y_j, $1 \leq j \neq i \leq k$ going through the edge x_iy_i has length at least three.
In the sequel, we can therefore assume that there are no edges x_iy_i in $E(G^c)$, for each $i = 1,\ 2, \cdots, k$. Let us choose two colors, say r (red) and b (blue). As $d^r(x_i) \geq \frac{n}{2}$ and $d^b(y_i) \geq \frac{n}{2}$, we obtain $d^r(x_i) + d^b(y_i) \geq n$, for each $i = 1, ..., k$. As there is no edge x_iy_i, we can find two distinct vertices, say a_i and b_i, in G^c such that $a_i \in N^r(x_i) \cap N^b(y_i)$ and $b_i \in N^r(x_i) \cap N^b(y_i)$. If $a_i \notin \{x_j, y_j, 1 \leq j \neq i \leq k\}$, then we consider the path $x_i \overset{r}{-} a_i \overset{b}{-} y_i$. These two edges x_ia_i and a_iy_i are not used by paths joining other pairs of vertices x_j, y_j, $j \neq i$, since we claim that the length of these other paths is at most two. After the choice of such paths, it suffices to construct pairwise edge-disjoint paths with the remaining pairs of vertices x_i, y_i such that $\{a_i, b_i, 1 \leq i \leq k\} \subset \{x_i, y_i, 1 \leq i \leq k\}$.
We shall complete the proof by showing that, in the worst case (which is $\{a_i, b_i, 1 \leq i \leq k\} \subset \{x_i, y_i, 1 \leq i \leq k\}$), we can construct k pairwise edge-disjoint paths of length at most two, one per pair x_i, y_i, for each $i = 1, ..., k$. Assume therefore that for any i and j, $1 \leq j \neq i \leq k$, we have $x_j \in N^r(x_i) \cap N^b(y_i)$ or $y_j \in N^r(x_i) \cap N^b(y_i)$. Now, let us choose and group together the properly edge-colored paths of the form $x_i - x_{i+1} - y_i$ which are pairwise edge-disjoint ones. We change the order of the pairs x_q, y_q and we swap, if necessary, x_q and y_q and a_q with b_q in order to maximize the cardinality of each group. Let d be the cardinality of a maximal group. W.l.o.g., this group can be considered as the one defined by $x_1 - x_2 - y_1$,..., $x_d - x_{d+1} - y_d$ (d is considered modulo k). If $d = k$, then the proof has done since there are k pairwise edge-disjoint paths of length at most two, one per pair x_i, y_i, $i = 1, ..., k$, as claimed. Otherwise, we use the same process in order to find the next maximal group of pairwise edge-disjoint paths for the remaining pairs of vertices x_i, y_i, $i = d+1, ..., k$. This is possible, since if $a_i = x_j$, with $i > d$ and $j \leq d$, then we can consider the path $x_i \overset{r}{-} x_j \overset{b}{-} y_i$, which uses new edges not already used be previously defined groups of paths. This process is finite, since at each step the number of the remaining pairs not linked yet decreases strictly. Hence, at the end of the process, we have

found k pairwise edge-disjoint paths of length at most two, one per pair x_i, y_i, $i = 1, ..., k$. This completes the proof of the theorem. $\square$

3 Minimum colored degrees and number of edges sufficient for the k-linked property in edge-colored multigraphs

Let us start with the following theorem involving minimum number of edges sufficient for the k-linked property.

Theorem 3.1. *Let G^c be a c-edge-colored multigraph of order n and k a non-zero positive integer, $n \geq 2k$. If $m \geq c\frac{n(n-1)}{2} - c(n-2k+1) + 1$, then G^c is k-linked.*

Proof. By induction on n. For $n = 2k$ the statement is true. Indeed, in this particular case, it is an easy exercice to see that all edges x_iy_i are present in G^c, $i = 1, 2, \cdots, k$. The theorem is also true for small values of n and k. Let us fix $2k$ distinct vertices $x_1, y_1, \cdots, x_k, y_k$ in G^c.

Assume first that for some i, say $i = 1$, there exists a path of length one between x_1 and y_1 in G^c. Consider the graph $G' = G^c \setminus \{x_1, y_1\}$ on $n-2$ vertices. It has at least $m - (2nc - 3c) = c\left(\dfrac{n^2 - 7n}{2} + 2k + 2\right) + 1$ edges. Hence G' is $(k-1)$-linked by induction. Therefore we can find $k-1$ pairwise vertex-disjoint paths between each pair x_i, y_i in G', $i = 2, 3, \cdots, k$. These $k-1$ paths of G' together with the edge x_1y_1 define the k pairwise vertex-disjoint paths in G^c, as desired.

Assume next that for some i, say $i = 1$, there exists a properly edge-colored path of length two between x_1 and y_1 in G^c. Let z denote the indermediate vertex of this path. In this case consider the graph $G' = G^c \setminus \{x_1, z, y_1\}$ on $n-3$ vertices. It has at least $m - [3c(n-3) + 2c] = c\left[\dfrac{n^2 - 9n}{2} + 2k + 6\right] + 1$ edges, thus it is $(k-1)$-linked by induction. The $k-1$ paths of G' together with the path between x_1 and y_1 through z define again the k pairwise-vertex-disjoint paths in G^c, as desired.

Assume finally that for each $i = 1, 2, \cdots, k$, there exist no path between x_i and y_i of length at most two in G^c. Thus c edges are missing in G^c between x_i and y_i, for otherwise a path of length one could be defined between x_i and y_i in G^c. Furthermore, for each vertex $z \notin \{x_1, x_2, \cdots, x_k\}$, at least c edges are missing between z and $\{x_i, y_i\}$ in G^c, $i = 1, 2, \cdots, k$. By summing the missing edges for a given pair x_i and y_i we obtain $c(n-2k) + c$. Therefore for the k pairs, we conclude that at least $kc(n-2k+1)$ edges are missing in G^c. But then, the number of edges of G^c is at most $c\frac{n(n-1)}{2} - kc(n-2k+1)$. For $k \geq 1$, we obtain $c\frac{n(n-1)}{2} - kc(n-2k+1) < c\frac{n(n-1)}{2} - c(n-2k+1) + 1$, a contradiction. This completes the proof of the theorem. $\square$

Above theorem is the best possible. Indeed, let us consider a c-edge-colored multigraph on n

vertices, $n \geq 2k \geq 2$, obtained as follows: Consider the disjoint union of an isolated vertex x_1 and a c-edge-colored complete multigraph on $n-1$ vertices. Then add all possible edges on all possible colors between x_1 and $2k-2$ fixed vertices, say $x_2,\ y_2, \cdots,\ x_k,\ y_k$, of the complete graph. The resulting graph, although it has $c\frac{n(n-1)}{2} - c(n-2k+1)$ edges, it is not k-linked. In fact, let y_1 be a vertex of the complete graph, other than $x_2,\ y_2, \cdots,\ x_k,\ y_k$. Then there are no k pairwise vertex-disjoint paths, one per pair x_i and y_i, $1 \leq i \leq k$, as any path between x_1 and y_1 go through the rest of vertices $x_2,\ y_2, \cdots,\ x_k, y_k$.

In the rest of the section we deal with conditions involving both minimum colored degrees and number of arcs, sufficient for the k-linked (k-edge linked) property. More precisely, let r, k, be two fixed positive non-zero integers. We are looking for functions $f(n,r,k)$ and $g(n,r,k)$ such that if a colored multigraph G^c on n vertices has colored degrees at least r and if the number of its edges is at least $f(n,r,k)$ (respectively, at least $g(n,r,k)$), then it is k-linked (respectively k-edge-linked). In order to state our conjectures later, let us first define the extremal graph $H^c(t_1,t_2,t_3,t_4,t_5)$ (or shortly H^c) as follows: Let t_1,t_2,t_3,t_4,t_5 be given non-zero positive integers. Let now $A_1,\ A_2,\ A_3,\ A_4,\ A_5$ be five c-edge-colored multigraphs on $t_1,\ t_2,\ t_3,\ t_4$ and t_5 vertices, respectively. We define A_1 (respectively $A_2,\ A_5$) to be a complete edge-colored multigraph, so that between each two vertices of A_1 (respectively of $A_2,\ A_5$) there are all possible multicolored edges, one edge per each available color. The graph A_3 (respectively A_4) is complete and monochromatic on one fixed color, say red, (respectively, say blue). We define now H^c to be the disjoint union of $A_1,\ A_2,\ A_3,\ A_4,\ A_5$ by adding all edges on all possible colors between A_5 and $A_1 \cup A_2 \cup A_3 \cup A_4$, all blue and red edges between A_3 and A_4, all red edges between $A_1 \cup A_2$ and A_3 and all blue edges between $A_1 \cup A_2$ and A_4. The graph H^c has the interesting property that any path between a vertex x of A_1 and a vertex y of A_2 go through the set A_5.
The extremal graph H^c helps to state the conjecture below for k-linked multigraphs.

Conjecture 3.2. *Let G^c be a c-edge-colored multigraph of order n and $k,\ r$ be two non-zero positive integers. Assume that for every vertex x, $d^c(x) \geq r,\ 2k-1 \leq r \leq \frac{n}{2}+k-2$.*
i) if $c=2,\ n \geq 6r-10k+14$, and $m \geq f_1(n,r,k) = n^2 - n(2r-4k+7) + (r-2k+2)(3r-2k+3) + 2(2r-2k+3)+1$,
ii) if $c=2,\ n \leq 6r-10k+14$, and $m \geq f_2(n,r,k) = \frac{3n^2}{4} + n(k-\frac{5}{2}) - k(k-3) + 11$,
iii) if $c \geq 3$ and $m \geq f_3(n,r,k,c) = \frac{c}{2}\left[n^2 - n(2r-4k+7) + 2(r-2k+3)(r+1)\right] + 1$,
then G^c is k-linked.

If true, Conjecture 3.2 is the best possible. Indeed, let us consider the following extremal graphs:
For Case (i), we consider the graph $H^c(1,\ n+2k-2r-3,\ r-2k+2,\ r-2k+2,\ 2k-2)$ with $f_1(n,r,k)-1$ edges. Choose now k pairs of vertices, $x_1 \in A_1,\ y_1 \in A_2$ and $x_i,\ y_i \in A_5$, $2 \leq i \leq k$. Then there are no k pairwise vertex-disjoint paths one per pair $x_i,\ y_i$ since any path between x_1 and y_1 goes through vertices of A_5. However all vertices of A_5 are already used by the paths joining the other pairs of vertices $x_i,\ y_i,\ i=2,\cdots,k$.
For Case (ii) we consider the graph $H^c(1,1,\frac{n}{2}-k,\frac{n}{2}-k,2k-2)$. It has $f_2(n,r,k)-1$ edges. However, as in the previous case, there are no k pairwise vertex-disjoint properly edge-colored paths, for $x_1 \in A_1,\ y_1 \in A_2$ and $x_i,\ y_i \in A_5$, with $i=2,\cdots,k$.
Finally, for Case (iii) we consider the graph $H^c(r-2k+3, n-r-1, 0, 0, 2k-2)$ with $f_3(n,r,k,c)-1$ edges. Again, there are no k pairwise vertex-disjoint properly edge-colored paths $x_i,\ y_i$ for

$x_1 \in A_1$, $y_1 \in A_2$ and $x_i,\ y_i \in A_5$, $i = 2, \cdots, k$.

In the sequel, we shall prove Conjecture 3.2 for $k = 1$ and r, c non fixed. But for convenient reasons we will prove the cases $c = 2$, $c = 3$ and $c \geq 4$ separately, in Theorems 3.3, 3.4, 3.5, respectively.

Theorem 3.3. *Let G^c be a 2-edge-colored multigraph of order n and r a non-zero positive integer. Assume that for every vertex x, $d^c(x) \geq r$, $r \leq \frac{n}{2} - 1$.*
i) if $n \geq 6r + 4$ and $m \geq n^2 - n(2r+3) + 3r^2 + 5r + 3$,
ii) if $n \leq 6r + 4$ and $m \geq \frac{3n^2}{4} - \frac{3n}{2} + 1$,
then G^c is linked.

Proof. The proof is by contradiction. Assume that, although conditions of theorem are fullfilled, there is no path between two given vertices x and y of G^c. Let R be a function denoting the number of edges of the complement of G^c. In other words, R denotes the number of edges to be added to G^c in order to become a complete 2-edge colored multigraph of order n. Clearly a 2-edge colored multigraph on n vertices has $n(n-1)$ edges. Under the hypothesis that there is no path between x and y, it will be enough to show that if $n \geq 6r+4$ (respectively $n \leq 6r+4$), then R is at least $n(n-1) - [n^2 - n(2r+3) + 3r^2 + 5r + 2] = n(2r+2) - 3r^2 - 5r - 2$ (respectively $n(n-1) - [\frac{3n^2}{4} - \frac{3n}{2}] = \frac{n^2}{4} + \frac{n}{2}$). This will be a contradiction with the number of edges of G^c. Let A^r, A^b, C, D be four subsets of $V(G^c)$ such that:

- for each $z \in A^r$, there is a path from x to z ending by a red edge and there is no path from x to z ending by a blue edge in G^c.
- for each $z \in A^b$, there is a path from x to z ending by a blue edge and there is no path from x to z ending by a red edge in G^c.
- for every $z \in C$, there are at least two (not necessarily disjoint) paths from x to z in G^c, the first path ending by a red edge and the second one by a blue edge.
- $D = V(G^c) - \big(A^r \cup A^b \cup C \cup \{x\}\big)$

According to previous definitions, the following two Claims 1 and 2 are obvious.

Claim 1. There is no blue edge between x and A^r and no red edge between x and A^b.

Claim 2. There is no edge between D and $C \cup \{x\}$, no blue edge between D and A^r and no red edge between D and A^b.

Claim 3. There is no blue edge in A^r and by symmetry there is no red edge in A^b.

Proof. Assume that there is a blue edge, say uv, in A^r. Let P denote a properly edge-colored path from x to u such that the last edge of this path is red. We may suppose that v is not an internal vertex of P, for otherwise we can consider v instead of u and then consider the segment of P between x and v instead of P. Then the path $P \cup uv$ joins x to v in G^c and its last edge uv is blue. Thus we conclude that $v \in C$, a contradiction since A^r and C are vertex-disjoint by definition. $\square$

Claim 4. For every $z \in C$, there is at most one blue edge, say zu, between z and A^r and if this unique edge zu exists in G^c, then there is no red edge zx in G^c.

Proof. Assume by contradiction that there are at least two blue edges, say zu and zv in G^c $u,\ v \in A^r$. Consider a path P from x to z whose last edge is red. Clearly such a path exists in G^c, by the definition of C. If u is not on this path, then $u \in C$, since $P \cup zu$ defines a path from x to u whose last edge is blue, a contradiction to the definitions of A^r and C. Similar arguments hold if we consider v instead of u. Consequently, we conclude that both u and v belong to P. Let u^- (respectively u^+) denote the predecessor (respectively the sucessor) of u, when we go from x to z along P. Analogously we define v^- and v^+ and z^-. As u and v are both vertices of A^r, the edges u^-u and v^-v are both red. Furtehrmore, as P is a properly edge-colored path, it follows that both edges uu^+ and vv^+ are blue. Now by considering the path $x \cdots u^-uzz^- \cdots v^-v$ between x and v, we conclude that $v \in C$, a contradiction to the definitions of A^r and C. This proves that there exists at most one edge between each vertex $z \in C$ and A^r. Remains to prove that if for some vertex $z \in C$, this unique edge, say zu, $u \in A^r$ exists in G^c, then the edge zx (if any) is not a red one. Assume therefore that a red edge xz exists in G^c. But then the path $x \overset{r}{-} z \overset{b}{-} u$ exists well in G^c and its last edge is a blue one. Thus $u \in C$, again a contradiction, since A^r and C are vertex disjoint by definition. This completes the proof of the claim. □

Claim 5. For every $z \in C$, there is at most one red edge between z and A^b and if this edge exists in G^c, then the edge zx (if any), is not a blue one.

Proof. Similar to that of previous claim. □

Now we are ready to determine R. Set $\|A^r\| = a_r$, $\|A^b\| = a_b$, $\|C\| = c$ and $\|D\| = d$. Clearly $a_r + a_b + c + d = n - 1$. Then,
$R = a_r + a_b + 2d(c+1) + d(a_r + a_b) + \frac{a_r(a_r-1)}{2} + \frac{a_b(a_b-1)}{2} + c(a_r + a_b)$
$= 2d(c+1) + (d + c + 1)(a_r + a_b) + \frac{a_r^2 + a_b^2}{2} - \frac{a_r + a_b}{2}$
$= 2d(c+1) + (d + c + \frac{1}{2})(a_r + a_b) + \frac{(a_r + a_b)^2}{2} - a_r a_b.$
We need to minimize R. Let us first fix $a_r + a_b$. Set $a_r + a_b = a$ and consider $a_r = a_b = \frac{a}{2}$. Then,

$$R = 2d(c+1) + a(d + c + \frac{1}{2}) + \frac{a^2}{4}$$

As a is fixed, then $d + c$ is also fixed, since $a + c + d + 1 = n$. We distinguish now between two cases depending upon a and r.

First case $a \geq 2r$
For $c = 0$ and $d = n - 1 - a$, we obtain

$$R = 2(n - 1 - a) + a(n - a - \frac{1}{2}) + \frac{a^2}{4} = -\frac{3a^2}{4} + a(n - \frac{5}{2}) + 2(n-1)$$

If we consider R as a function of a, then the minimum values of R are obtained for $a = 2r$ or for $a = n - 2$. In particular, $R(2r) = 2n(r+1) - 3r^2 - 5r - 2$ and $R(n-2) = \frac{n^2}{4} + \frac{n}{2}$. Now by comparing $R(2r)$ and $R(n-2)$ we may see that for $n \geq 6r + 4$, $R(2r) \geq R(n-2)$. Otherwise, if $n \leq 6r + 4$, then $R(n-2) \geq R(2r)$. This is in contradiction with the number of edges of G^c

and completes the proof of that case.

Second case $a < 2r$
In this second case we consider $c = r - \frac{a}{2}$ and $d = n - r - \frac{a}{2} - 1$. For these particular values of c and d we obtain,

$$R = 2(n - r - \frac{a}{2} - 1)(r - \frac{a}{2} + 1) + a(n - a - \frac{1}{2}) + \frac{a^2}{4} = -\frac{a^2}{4} - \frac{a}{2} + 2n(r+1) - 2(r+1)^2$$

If we consider R as a function of a, we can see that the minimum values of R are obtained for $a = 2r - 1$ or $a = 2$. Furthermore $R(2r-1) = 2n(r+1) - 2(r+1)^2 - 2$ and $R(2) = 2n(r+1) - 3r^2 - 4r - \frac{7}{4}$. As each of these two values is greater than $n(2r+2) - 3r^2 - 5r - 2$, this contradicts the hypothesis on the number of edges of G^c and completes the proof of the case and of the theorem. □

Theorem 3.4. *Let G^c be a 3-edge-colored multigraph of order n and r a non-zero positive integer. Assume that for every vertex x, $d^c(x) \geq r$, $r \leq \frac{n}{2} - 1$. If $m \geq \frac{3}{2}[n^2 - n(2r+3) + 2r(r+2) + 2] + 1$, then G^c is linked.*

Proof. The proof is by contradiction. Let x and y be two vertices of G^c. Assume that there is no properly edge-colored path between x and y. As in previous theorem, let R be a function denoting the number of edges of the complement of G^c. In other words, R denotes the number of edges to be added to G^c in order to become a complete 3-edge colored multigraph of order n. Clearly the number of edges of a complete 3-edge colored multigraph of order n is $3\frac{n(n-1)}{2}$. Set $\lambda = 3\frac{n(n-1)}{2} - \frac{3}{2}[n^2 - n(2r+3) + 2r(r+2) + 2] = 3n(r+1) - 3r(r+2) - 3$. In order to obtain a contradiction, under the hypothesis that there is no path between x and y, it will be enough to show that $R \geq \lambda$. Let A^i, D, E be five subsets of $V(G^c)$, $1 \leq i \leq 3$, such that :

- For every $z \in A^i$, $1 \leq i \leq 3$, there is a path from x to z ending by an edge on color i and there is no path from x to z ending by an edge on a color different than i.
- For every $z \in D$, there are at least two (not necessarily disjoint) paths from x to z, the first one ending by an edge on color i and the second on ending by an edge on color j, $1 \leq j \neq i \leq 3$.
- $E = V(G^c) - \left(\left(\cup_{1 \leq i \leq 3} A^i\right) \cup D \cup \{x\}\right)$

According to previous definitions, the three Claims 1, 2, and 3 below are obvious.

Claim 1. For each $i = 1,\ 2,\ 3$, there is no edge on color j between x and A^i, $1 \leq j \neq i \leq 3$.

Claim 2. There is no edge between E and $D \cup \{x\}$.

Claim 3. For each $i = 1,\ 2,\ 3$, there is no edge of color j between E and A^i, $1 \leq j \neq i \leq 3$.

Claim 4. For each $i = 1,\ 2,\ 3$, there is no edge on color j in A^i, $1 \leq j \neq i \leq 3$.

Proof. Assume that there is an edge uv in A^i on some color j, $1 \leq j \neq i \leq 3$. Let P denote a properly edge-colored path from x to u in G^c. We may suppose that v does not belong to P, for otherwise, we may exchange u and v and, instead of P, consider the segment of P from x to v. Moreover the color of the last edge of P is i. But then we may conclude that the path $P \cup uv$ exists in G^c and its last edge is on color j. Thus we obtain that $v \in C$, a contradiction to the definitions of A and C. $\square$

Claim 5. For each i and j, $1 \leq i \neq j \leq 3$, there is no edge on color l between A^i and A^j, $l \neq i$ and $l \neq j$.

Proof. Assume that there is an edge, say uv, on color l between A and A^j, $u \in A^i$ and $v \in A^j$, $l \neq i,\ j$. Let P denote a properly edge-colored path between x and u. W.l.o.g. we may suppose that v does not belong to P, for otherwise, as in previous claim, we can exchange u and v. Moreover the color of the last edge of P is i. In that case, by considering the path $P \cup uv$, we conclude that $v \in C$. This is a contradiction with the definitions of A^j and C. $\square$

Claim 6. For every $z \in D$ and for each $i = \ 1,\ 2,\ 3$, there is at most one edge on color i, between z and $\cup_{j \neq i} A^j$. Furthermore if this unique edge between z and $\cup_{j \neq i} A^j$ exists, then there exist no edge zx on color j in G^c, $j \neq i$.

Proof. Assume by contradiction that there are two distinct edges, say zu and zv, both on color i between z and $\cup_{j \neq i} A^j$, $u,\ v \in \cup_{j \neq i} A^j$. By the definition of D, there is a path, say P, from x to z whose last edge is on color l, $l \neq i$. If the vertex u is not on P, then $u \in D$, since $P \cup zu$ is a path of G^c joining x to u. Similar arguments hold for v. Consequently, in what follows we may suppose that both u and v belong to P. Suppose w.l.ofg. that u is before v when we walk from x to z along P. Let u^- (respectively u^+) denote the predecessor (respectively the sucessor) of u. Analogously, we define v^- and v^+ and z^-. Let q (respectively q') be the color of edge u^-u (respectively v^-v) As u and v are both vertices of $\cup_{j \neq i} A^j$, $q \neq i$ and $q' \neq i$. Furthermore, as P is properly edge-colored, the color of edge vv^+ is different fom q'. Now by considering the path $x \cdots u^- u z z^- \cdots v^- v$ between x and v, we conclude that $v \in D$, a contradiction to the definitions of A^j, $j \neq i$ and D. This proves that there exists at most one edge between each vertex $z \in D$ and $\cup_{j \neq i} A^j$. Remains to prove that if for some vertex $z \in D$, this unique edge, say zu, $u \in \cup_{j \neq i} A^j$ exists in G^c, then the edge zx (if any) is not on color j, $j \neq i$. Assume therefore that an edge xz on color j exists in G^c. But then the path $x \overset{j}{-} z \overset{i}{-} u$ exists well in G^c and its last edge is on color i. Thus $u \in D$, again a contradiction, since $\cup_{j \neq i} A^j$ and D are vertex disjoint by definition. This completes the proof of the claim. $\square$

Now we are ready to determine R. Set $\|A^i\| = a_i$,$\|D\| = d$ and $\|E\| = e$. Clearly $\sum_{i=1}^{3} a_i + d + e + 1 = n$. Then,

$$R = 2\sum_{i=1}^{3} a_i + 3e(d+1) + 2e\sum_{i=1}^{3} a_i + 2\sum_{i=1}^{3} \frac{a_i(a_i-1)}{2} + 2d\sum_{i=1}^{3} a_i + \frac{1}{2}\sum_{i \neq j} a_i a_j,$$

that is,

$$R = 3e(d+1) + 2(e + d + \frac{1}{2})\sum_{i=1}^{3} a_i + 2\sum_{i=1}^{3} \frac{a_i^2}{2} + \frac{1}{2}\sum_{i \neq j} a_i a_j.$$

Set $\sum_{i=1}^{3} a_i = a$. To minimize R, with a fixed, we must consider $a_i = \frac{a}{3}$, for each $i = 1,\ 2,\ 3$. For these particular values of a_i we obtain,

$$R = 3e(d+1) + 2a(e+d+\frac{1}{2}) + \frac{2}{3}a^2$$

As a is fixed, we may suppose that $d+e$ is also fixed, since $a+d+e+1 = n$. Now we distinguish between three cases depending upon n, r and a.

First case $n \geq 3r+2$ and $a \geq 3r$.
By taking $d = 0$ and $e = n-1-a$, we obtain,

$$R = 3(n-a-1) + 2a(n-a-\frac{1}{2}) + \frac{2}{3}a^2 = -a^2\frac{4}{3} + a(2(n-\frac{1}{2})-3) + 3(n-1)$$

If we consider R as a function of a, we can see that the minimum values of R are obtained for $a = 3r$ or $a = n-2$. In particular, $R(n-2) = \frac{2}{3}n^2 + \frac{1}{3}n - \frac{1}{3}$ and $R(3r) = -12r^2 + 3r\left(2(n-\frac{1}{2})-3\right) + 3(n-1) = 3n(2r+1) - 12r\,(r+1) - 3$. It suffices to show that $R(n-2) - \lambda \geq 0$ and $R(3r) - \lambda \geq 0$. However, $R(n-2) - \lambda = \frac{2}{3}n^2 - (3r+\frac{8}{3})n + 3r(r+2) + \frac{8}{3}$. We see easily that $R(n-2) - \lambda \geq 0$, if $n \geq 3r+2$. Similarly, for $n \geq 3r+2$ $R(a = 3r) - \lambda = 3nr - 9r^2 - 6r \geq 0$. This completes the proof of this case.

Second case $n \geq 3r+2$ and $a < 3r$
For $d = r - \frac{a}{3}$ and $e = n - r - \frac{2a}{3} - 1$, we obtain

$$R = 3(n-r-a\frac{2}{3}-1)(r-\frac{a}{3}+1) + 2a(n-a-\frac{1}{2}) + \frac{2}{3}a^2 = -\frac{2}{3}a^2 + a(n-r-2) + 3n(r+1) - 3(r+1)^2$$

We can see that R is minimum for $a = 0$ or $a = 3r$. Furthermore $R(0) = \lambda = 3(n-r-1)(r+1) = 3n(r+1) - 3r(r+2) - 3$ Now we can verify that $R(3r) \geq R(0)$. Indeed,

$$R(3r) - R(0) = (n-2)(\frac{n}{3} - r - \frac{2}{3})$$

But $n \geq 3r+2 \geq 2$, so $R(3r) - R(0) \geq 0$. This completes the proof of this second case.

Third case $n < 3r+2$.
By the hypothesis of the theorem, $n \geq 2r+2$. Set $n = 3r+2-\epsilon$ where ϵ is an integer, $0 < \epsilon \leq r$. Clearly $a+d+e+1 = 3r+2-\epsilon = n$. To maximize a, we take $a = 3(r-\epsilon)$, $d = \epsilon$, $e = \epsilon+1$. However for $d = r - \frac{a}{3}$, $e = n - r - \frac{2a}{3} - 1$ and for any $a < 3(r-\epsilon)$, we have

$$R = 3(2r-\epsilon-\frac{2a}{3}+1)(r-\frac{a}{3}+1) + 2a(3r-\epsilon-a+\frac{3}{2}) + \frac{2}{3}a^2$$

$$R = -\frac{2}{3}a^2 + (2r-\epsilon)a + 3(2r^2+3r-\epsilon r-\epsilon+1)\ f(\epsilon,r) + g(\epsilon,r)$$

The minimum values of R are obtained for $a = 0$ or $a = 3(r-\epsilon)$. In particular, $R(0) = 3(2r-\epsilon+1)(r+1) = 3\left[2r^2+(3-\epsilon)r+1-\epsilon\right]$ and $R(3r-3\epsilon) = 3(2r^2+3r-\epsilon+1-\epsilon^2)$ But $R(3r-3\epsilon) - R(0) = 3\epsilon(r-\epsilon) \geq 0$.
This completes the proof of this last case and of the theorem. □

The previous results deal within at most 3 colors. For more than 3 colors, we have the following theorem.

Theorem 3.5. *Let G^c be a c-edge-colored multigraph of order n, $c > 3$ and r an integer. Assume that for every vertex x, $d^c(x) \geq r$, $1 \leq r \leq \frac{n}{2} - 1$. If $m \geq \frac{c}{2}\left[n^2 - n(2r+3) + 2r(r+2) + 2\right] + 1$, then G^c is linked.*

Proof. By contradiction. Let x and y two vertices of G^c. Assume that there is no path between x and y in G^c. Let R be a function counting the number of edges in the complement of G^c. The main purpose is to show that $R \geq c(n(r+1) - r(r+2) - 1)$ which will be in contardiction with m. Since there is no path between x and y in G^c, we may suppose that there is no path either between x and y in the subgraph G_i^c of G^c containing the edges of color i, $i+1$ and $i+2$ (*modulo* c) of G^c, for every fixed color $i = 1, 2, \cdots, c$. As for R, in a similar way let us define R_i for each such subgraph G_i^c. Now $R_i \geq \frac{3}{2}(n(r+1) - r(r+2) - 1)$. Then $R = \frac{c}{3}R_i \geq c(n(r+1) - r(r+2) - 1)$, since every color is used three times. This completes the proof of the theorem. □

Let us turn now our attention to sufficient conditions involving minimum degrees and number of arcs guarantying the k-edge-linked property. More precisely, let us formulate the following conjecture.

Conjecture 3.6. *Let G^c be a c-edge-colored multigraph of order n and k, r be two non-zero positive integers. Assume that for every vertex x, $d^c(x) \geq r$, $r \leq \frac{n}{2} - 1$.*
i) if $c = 2$, $n \geq 6r + 4$ and $m \geq g_1(n,k,r) = n^2 - n(2r+3) + 3r^2 + 5r + 3$,
ii) If $c = 2$, $n \leq 6r + 4$ and $m \geq g_2(n,k,r) = \frac{3n^2}{4} - \frac{3n}{2} + 1$,
iii) if $c \geq 3$ and $m \geq g_3(n,k,r,c) = \frac{c}{2}\left[n^2 - n(2r+3) + 2r(r+2) + 2\right] + min(k-1, r) + 1$,
then G^c is k-edge-linked.

If true, this conjecture should be the best possible. Indeed :
For (i), we consider the 2-edge-colored multigraph $H^c(1, n-2r-1, r, r, 0)$. Although it has $g_1(n,k,r) - 1$ edges, it is no 1-edge linked. In particular, there is no properly edge-colored path between x_1 and y_1, for any choice of vertices $x_1 \in A_1$ and $y_1 \in A_2$.
For (ii), we consider the the 2-edge-colored multigraph $H^c(1, 1, \frac{n}{2}-1, \frac{n}{2}-1, 0)$ having $g_2(n,k,r) - 1$ edges. As in previous Case (i) there is no properly edge-colored path between any pair of vertices $x_1 \in A_1$, $y_1 \in A_2$.
Finally for (iii), we consider the c-edge-colored multigraph $H^c(r+1, n-r-1, 0, 0, 0)$. If $r+1 \geq k$ then we add $k-1$ edges between A_1 and A_2. Now if we consider $x_i \in A_1$ and $y_i \in A_2$, $1 \leq i \leq k$, then we can not find k pairwise edge-disjoint paths one per pair x_i, y_i, since there are at most $k-1$ edges between A_1 and A_2. Thus H^c is not k-edge-linked. Otherwise, if $r \leq k$ then we add r edges between A_1 and A_2. If we select $x_i \in A_1$ and $y_i \in A_2$, $1 \leq i \leq r+1$, then, again, we can not find $r+1$ pairwise edge-disjoint paths, one per pair x_i, y_i, since there are at most r edges between A_1 and A_2. Thus, although H^c has $\frac{c\left[n^2 - n(2r+3) + 2r(r+2) + 2\right]}{2} + min(k-1, r)$ edges, it is not $(r+1)$-edge-linked for $r+1 < k$.

By Theorem 3.5, Conjecture 3.6 above is true for $k = 1$ and r, c non fixed. Also in Theorem 3.8 stated later we prove that this conjecture remains true for $r = 1$, $c = 2$ and k non fixed.
In view of Theorem 3.8, let us prove the following lemma.

Lemma 3.7. *Let G^c be a 2-edge-colored multigraph of order $n \geq 5$. Assume that for every vertex x, $d^c(x) \geq 1$.*
i) If $n \geq 10$, and $m \geq n^2 - 5n + 11$,

ii) If $n < 10$, and $m \geq \frac{3n^2}{4} - \frac{3n}{2} + 1$,
then there exists a properly edge-colored path of length at most four joining any two given vertices x and y in G^c.

Proof. We prove this lemma by contradiction. Let x and y be two fixed vertices of G^c. Assume that there is no properly edge-colored path of length at most four between x and y. As in previous theorems, let R denote the number of missing edges of G^c. The fact that we cannot find paths between x and y of length one, two, three and four, gives an idea of how to determine R. Set $\lambda_1 = n(n-1) - [\frac{3n^2}{4} - \frac{3n}{2} + 1] = \frac{n^2}{4} + \frac{n}{2} - 1$ and $\lambda_2 = n(n-1) - [n^2 - 5n + 11] = 4n - 11$. Set also $\alpha = \|N^b(x) \cap N^b(y)\|$ and $\beta = \|N^r(x) \cap N^r(y)\|$. Then,
$$\begin{aligned} R &= 2 + (n-2-d^b(x)) + (n-2-d^r(x)) + (n-2-d^b(y)) + (n-2-d^r(y)) \\ &\quad + (d^b(x)-\alpha)(d^b(y)-\alpha) + \alpha(d^b(x)-\alpha) + \alpha(d^b(y)-\alpha) + \tfrac{\alpha(\alpha-1)}{2} \\ &\quad + (d^r(x)-\beta)(d^r(y)-\beta) + \beta(d^r(x)-\beta) + \beta(d^r(y)-\beta) + \tfrac{\beta(\beta-1)}{2} \\ &= 4n - 6 + f(d^b(x), d^b(y), \alpha) + g(d^r(x), d^r(y), \beta). \end{aligned}$$

We distinguish now between two cases dependening upon if α, β are zero or not.

Case 1. $\alpha \neq 0$ ***and*** $\beta \neq 0$
In order to minimize R, we set $d^b(x) = d^b(y) = \alpha$ and $d^r(x) = d^r(y) = \beta$. So

$$f(d^b(x), d^b(y), \alpha) = -2\alpha + \frac{\alpha(\alpha-1)}{2} = \frac{\alpha(\alpha-5)}{2}$$

As $\alpha + \beta \leq n - 2$ we obtain $R = 4n - 6 + \frac{\alpha(\alpha-5)}{2} + \frac{\beta(\beta-5)}{2} = 4n - 6 + \frac{(\alpha+\beta)(\alpha+\beta-5)}{2} - \alpha\beta$.
Assume first $\alpha + \beta \leq 5$. We can easily see that $R \geq 4n - 12$ by using all possible different values of α, β, namely, $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,2)$, $(2,3)$. Set $R_{min} = 4n - 12$. The function $\lambda_1 - R_{min} = \frac{n^2}{4} + \frac{7n}{2} + 11$ has two roots, namely, $n_1 = 7 - \sqrt{5}$ and $n_2 = 7 + \sqrt{5}$. Hence, for $5 < n \leq 10$ we have $\lambda_1 < R_{min}$, a contradiction, since R_{min} should be smaller than λ_1.
Assume next $\alpha + \beta > 5$. By studying the function $R(\alpha, \beta)$, we deduce that $R(\alpha, \beta) \geq R(3, 3)$. It means that $R(\alpha, \beta) \geq 4n - 12$. Since $n \geq 10$, there are at least two different vertices z and z' not in $N^b(y) \cup N^r(x) \cup N^b(x) \cup N^r(y) \cup \{x, y\}$. As there does not exist a properly edge-colored path of length four between x and y in G^c, we deduce that for every $u \in N^b(x)$ and for every $v \in N^r(y)$ the red edge uz (or the blue edge zv) is missing in G^c, for otherwise the path $x \overset{b}{-} u \overset{r}{-} z \overset{b}{-} v \overset{r}{-} y$ (respectively for z') between x and y has length four. This implies that $R(\alpha, \beta) \geq 4n - 10$. As $\lambda_2 = 4n - 11 < 4n - 10 \leq R(\alpha, \beta)$, we conclude that the number of edges of G^c is at most $n^2 - 5n + 10$, a contradiction with the hypothesis of Case (i).

Case 2. $\alpha = 0$ ***or*** $\beta = 0$
Assume $\beta = 0$. Then $R(\alpha) = 4n - 6 + \frac{\alpha(\alpha-5)}{2}$. One can easily see that $R(\alpha) \geq R(3) = 4n - 9$. Since $R(3) \geq \lambda_1$ and $R(3) \geq \lambda_2$, this is a contradiction for both Cases (i) and (ii). This completes the proof of the Lemma. □

Theorem 3.8. *Let G^c be a 2-edge-colored multigraph and k an integer, $n \geq 2k \geq 10$. Assume that for every vertex x, $d^c(x) \geq 1$. If $m \geq n^2 - 5n + 11$, then G^c is k-edge-linked.*

Proof. Let x_i and y_i be $2k$ vertices of G^c , $1 \leq i \leq k$. Let us try to find k pairwise edge-disjoint paths, one per pair $x_i,\ y_i$, $1 \leq i \leq k$. By previous lemma, there exists a properly edge-colored path of length at most four between each pair x_i and y_i.

Claim. There exists at most one pair x_i, y_i of vertices such that the length of any path between x_i and y_i is greater than two in G^c.

Proof. Assume by contradiction that there are at least two pairs of vertices, say $x_1,\ y_1$ and $x_2,\ y_2$ such that the length of any path between x_1 and y_1 (respectively x_2 and y_2) is greater than two. Then, for every $z \in G^c - \{x_1, y_1\}$, either the blue edge x_1z or the red edge y_1z is missing in G^c, for otherwise the path $x_1 \overset{b}{-} z \overset{r}{-} y_1$ is of length two between x_1 and y_1. Analogously we conclude that either the red edge x_1z or the blue edge y_1z is missing in G^c. Similar arguments hold for x_2 and y_2. Furthermore, as there is no path of length one between x_1 and y_1 (respectively between x_1 and y_1) there is no edge x_1y_1 (x_2y_2) in G^c. Now by summing all these missing edges we conclude that G^c has less than $n(n-1)-[2(n-2)+2(n-4)+4] < n^2-5n+11$ edges , a contradiction. This completes the proof of the claim. $\square$

According to the previous claim, for at most one pair $x_i,\ y_i$, there is a path between x_i and y_i of length greater than two. In addition, according to Lemma 3.7, the length of such a path is three or four.

Assume first that there exists a path of length at most two between x_i and y_i in G^c, for each $i = 1, \cdots, k$. For every i, let $Z^i = \{z_1^i, z_2^i, ..., z_{p_i}^i\}$ denote the set vertices of G^c such that there is a path between x_i and y_i going through a vertex of Z^i. Then using arguments almost identical to those of the proof of Theorem 2.3 (only replace $\{a_i, b_i, 1 \leq i \leq k\}$ by $Z^i = \{z_1^i, z_2^i, ..., z_{p_i}^i\}$), we may find k pairwise edge-disjoint paths as desired.

Assume next that for some $1 \leq i \leq k$, say $i = k$, any path between x_k and y_k has length 3 or 4. According to arguments used in previous case and in the proof of Theorem 2.3, we can define $k-1$ pairwise edge-disjoint paths for the pairs of vertices $x_i, y_i, 1 \leq i \leq k-1$. In order to complete the proof, it should be enough to show that whenever a path between x_k and y_k shares a common edge with some path between pairs x_i and $y_i, 1 \leq i \leq k-1$, then one can choose either another apropriate path between x_i and y_i or another apropriate path between x_k and y_k, in order to obtain the desired k pairwise edge-disjoint paths.

Assume next that any path between x_k and y_k has length 3. As there is no path between x_k and y_k of length at most two in G^c, as in the proof of claim above, we may conclude that there are at least $2(n-2)+2$ missing edges in G^c. Assume first that x_iy_i is the shared edge between the path joining x_k to y_k and the rest of the paths joining the pairs $x_i,\ y_i$, $1 \leq k \leq k-1$. Then $\forall z \in G^c - \{x_k, x_i, x_{i-1}, y_k, y_i, y_{i-1}\}$, noone of the $(n-6)$ paths $x_i \overset{b}{-} z \overset{r}{-} y_i$ or $x_i \overset{r}{-} z \overset{b}{-} y_i$ of length two is present in G^c. Moreover, an edge x_iy_i, an edge $x_{i-1}y_{i-1}$ and at least 2 edges for the possible existence of an alternating cycle $x_ix_{i-1}y_iy_{i-1}x_i$ are missing in G^c. The sum of above-mentionned missing edges is at least $4n-10$. It follows that G^c has at most $n^2-5n+10$, a contradiction. Assume next that only x_i or y_i, say x_i, is adjacent with a common edge of the path joining x_k and y_k and some of the paths joining the rest of the pairs $x_i,\ y_i$, $1 \leq k \leq k-1$. Then for each z in $G^c - \left\{x_k, x_i, x_{i-1}, y_k, y_i, y_{i-1}, z_l^k\right\}$,where $z_l^k \notin x_k, y_k, x_i$, we may count $2(n-7)$

missing edges, as noone of the paths $x_i \overset{b}{-} z \overset{r}{-} y_i$ or $x_i \overset{r}{-} z \overset{b}{-} y_i$ is present in G^c. Then there are at least 2 missing edges for the pairs x_i, y_i and x_{i-1}, y_{i-1}, an edge linking x_i and y_i through z_l^k and finally 2 edges between $x_i x_{i-1} y_i y_{i-1} x_i$. Hence we deduce here that at least $4n-9$ edges are missing in G^c, a contradiction to the fact that G^c has at least $n^2-5n+11$ edges.

Assume finally that the length of any path between x_k and y_k is 4. We shall complete the proof by taking in account the three following subcases:

- $x_i y_i$ is a common edge between a path joining x_k to y_k and some path joining the rest of pairs $x_i,\ y_i$, $1 \leq k \leq k-1$.
- for some $1 \leq i \leq k-1$, both edges of the path between x_i and y_i of length two belong to the path between x_k and y_k.
- Only x_i (or y_i) is adjacent with a common edge of the path joining x_k to y_k and some path joining the rest of pairs $x_i,\ y_i$, $1 \leq k \leq k-1$.

As there is no path of length at most two between x_k and y_k, we may count at least $2(n-2)+2$ missing edges in G^c. There is also no path of length three between x_k and y_k. Hence we can add at least two more edges on the missing ones. Thus up to now, at least $2(n-2)+2+2=2n$ edges are missing in G^c.
Now, in the first subcase, let $z_l^k,\ z_l^k \neq x_i,\ y_i$ be the fourth vertex which completes the path between x_k and y_k. Then for all z of $G^c - \{x_k, x_i, x_{i-1}, y_k, y_i, y_{i-1}, z_l^k\}$, we may count a total of $2(n-7)$ missing edges. There are also at least 4 missing edges between $x_i x_{i-1} y_i y_{i-1} x_i$ and $x_{i-1} x_{i-2} y_{i-1} y_{i-2} x_{i-1}$.
In the second subcase, besides the counted number of the first cas, we can add 2 missing edges $x_i y_i$.
In the last subcase, instead of $2(n-7)$, we have $2(n-8)$ missing edges and have also all other missing edges mentionned in the second subcase. We also add at least 2 missing edges which could define a path $x_i - x_k - y_i$ and 2 edges for the path $x_i - y_k - y_i$.
For each of the above cases, by summing all missing edges, we find that there are at least $4n-11$ missing edges in G^c. It follows that G^c has at most $n^2-5n+10$ edges, a contradiction. This completes the proof of the theorem. □

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