# Paths and Trails in Edge-Colored Graphs

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**Abstract**

This paper deals with the existence and search of properly edge-colored paths/trails between two, not necessarily distinct, vertices $s$ and $t$ in an edge-colored graph from an algorithmic perspective. First we show that several versions of the $s-t$ path/trail problem have polynomial solutions including the shortest path/trail case. We give polynomial algorithms for finding a longest properly edge-colored path/trail

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between $s$ and $t$ for some particular graphs and characterize edge-colored graphs without properly edge-colored closed trails. Next, we prove that deciding whether there exist $k$ pairwise vertex/edge disjoint properly edge-colored $s-t$ paths/trails in a $c$-edge colored graph $G^c$ is NP-complete even for $k=2$ and $c=O(n^2)$, where $n$ denotes the number of vertices in $G^c$. Moreover, we prove that these problems remain NP-complete for $c$-colored graphs containing no properly edge-colored cycles and $c=O(n)$. We obtain some approximation results for those maximization problems together with polynomial results for some particular classes of edge-colored graphs.
**Key words:** Edge colored graphs, connectivity, properly edge-colored paths, trails and cycles.

# 1 Introduction, Notation and Terminology

In the last few years a great number of problems have been dealt with in terms of edge-colored graphs for modelling purposes as well as for theoretical investigation [4, 8, 9, 10, 20, 25]. Previous work on the subject has focused on the determination of particular properly edge-colored subgraphs, such as Hamiltonian or Eulerian configurations, colored in a specified pattern [2, 3, 5, 6, 7, 11, 23, 27, 29], that is, subgraphs such that adjacent edges have different colors. Our first aim in that respect was to extend the graph-theoretic concept of connectivity to colored graphs with a view to gaining some insight into our problem from Menger's Theorem in particular. In other words, we intended to define some sort of local alternating connectivity for edge-colored graphs.

Difficulties arose, however, from local connectivity being not polynomially characterizable in edge-colored graphs, as can easily be seen. Thus, there can be no counterpart to Menger's Theorem as such, and even the notion of a connected component as an equivalence class does not carry over to edge-colored graphs since the concatenation of two properly edge colored paths is not necessarily properly edge colored. We settled then for some practical and theoretical results, herein presented, which deal with the existence of vertex-disjoint paths/trails between given vertices in $c$-edge colored graphs. Most of those path/trail problems happen to be NP-complete, which thwarts all attempts at systematization.

Formally, let $I_c=\{1,2,\cdots,c\}$ be a set of given colors, $c\geq 2$. Throughout the paper, $G^c$ will denote an edge-colored simple graph such that each edge is in some color $i\in I_c$ and no parallel edges linking the same pair of vertices occur. The vertex and edge-sets of $G^c$ are denoted by $V(G^c)$ and $E(G^c)$, respectively. The *order* of $G^c$ is the number $n$ of its vertices. The *size* of $G^c$ is the number $m$ of its edges. For a given color $i$, $E^i(G^c)$ denotes the set of edges of $G^c$ colored $i$. For edge-colored complete graphs, we write $K_n^c$

instead of $G^c$. If $H^c$ is a subgraph of $G^c$, then $N^i_{H^c}(x)$ denotes the set of vertices of $H^c$, linked to $x$ with an edge colored $i$. The colored $i-degree$ of $x$ in $H^c$, denoted by $d^i_{H^c}(x)$, is $|N^i_{H^c}(x)|$, i.e., the *cardinality* of $N^i_{H^c}(x)$. An edge between two vertices $x$ and $y$ is denoted by $xy$, its color by $c(xy)$ and its cost (if any) by $cost(xy)$. The cost of a subgraph is the sum of its edge costs. A subgraph of $G^c$ containing at least two edges is said to be *properly edge-colored* if any two adjacent edges in this subgraph differ in color. A *properly edge-colored path* does not allow vertex repetitions and any two successive edges on this path differ in color. A *properly edge-colored trail* does not allow edge repetitions and any two successive edges on this trail differ in color. The *length* of a path (trail) is the number of its edges. Given two vertices $s$ and $t$ in $G^c$, we define an $s-t$ path (trail) to be a path (trail) with end-vertices $s$ and $t$. Sometimes $s$ will be called the *source*, and $t$ the *destination* of the path (trail). A properly edge-colored path/trail is said to be *closed* if its endpoints coincide, and its first and last edges differ in color. A closed properly edge-colored path (trail) is usually called a *properly edge-colored cycle* (*closed trail*).

Given a digraph $D(V,A)$ and 2 vertices $u,v \in V$, we denote by $\vec{uv}$ an arc of $A$. In addition, we define $N^+_D(x) = \{y \in V : \vec{xy} \in A\}$ the *out-neighbourhood* of $x$ in $D$, and by $N^-_D(x) = \{y \in V : \vec{yx} \in A\}$ the *in-neighbourhood* of $x$ in $D$. Finally, we represent by $N_D(x) = N^+_D(x) \cup N^-_D(x)$ the *in-out-neighbourhood* of $x \in V(D)$ (or just *neighbourhood* for short). Also, given an induced subgraph $Q$ of a non colored graph $G$, a *concatenation* of $Q$ in $G$ consists of replacing $Q$ by a new vertex, say $z_Q$, so that each vertex $x$ in $G-Q$ is connected to $z_Q$ by an edge iff there exists an edge $xy$ in $G$ for some vertex $y$ in $Q$.

This paper is concerned with algorithmic issues regarding various trail/path problems between two given vertices $s$ and $t$ in $G^c$. First, we study the $s-t$ path/trail version problem. Polynomial algorithms are established for such problems as the *Shortest properly edge-colored path/trail*, the *Shortest properly edge-colored path/trail with forbidden pairs*, the *Shortest properly edge-colored cycles/closed trails* and the *Longest properly edge-colored path/trail* for some particular instances. We also characterize edge-colored graphs without properly edge-colored closed trails. Next, we deal with the *Maximum Properly Vertex Disjoint Path* and *Maximum Properly Edge Disjoint Trail* problems, whose objective is to maximize the number of properly edge-colored vertex-disjoint paths (respectively, edge-disjoint trails) between $s$ and $t$. Although these problems can be solved in polynomial time in general non-colored graphs, most of their instances are proved to be NP-complete in the case of edge-colored graphs. In particular we prove that, given an integer $k \geq 2$, deciding whether there exist $k$ properly edge-colored vertex/edge disjoint $s-t$ paths/trails in $G^c$ is NP-complete even for $k = 2$ and $c = O(n^2)$. Moreover, for an arbitrary $k$ we prove that these problems remain NP-complete for $c$-colored graphs containing no properly edge-colored cycles/closed trails and $c = O(n)$. We show a greedy procedure for these maximisation problems, throught the sucessive construction of properly edge-colored shortest $s-t$ paths/trails. This is a straithfoward generalization of the greedy procedure to maximize the number of edge or vertex disjoint paths between $k$ pair of vertices in non-colored graphs (see [22, 19] for details). Similarly, we obtain an approximation performance ratio.

The following two results will be used in this paper. The first result, due to Yeo [29], characterizes edge-colored graphs without properly edge-colored cycles.

**Theorem 1.1.** *(Yeo) Let $G^c$ be a c-edge colored graph, $c \geq 2$, such that every vertex of $G^c$ is incident with at least two edges colored differently. Then either $G^c$ has a properly edge-colored cycle or for some vertex $v$, no component of $G^c - v$ is joined to $v$ by at least two edges in different colors.*

In terms of edge-colored graphs, Szeider's main result [26] on graphs with prescribed general transition systems may be formulated as follows:

**Theorem 1.2.** *(Szeider) Let $s$ and $t$ be two vertices in a c-edge colored graph $G^c$, $c \geq 2$. Then, either we can find a properly edge-colored $s-t$ path or else decide that such a path does not exist in $G^c$ in linear time on the size of the graph.*

Given $G^c$, the main idea of the proof is based on earlier work by Edmonds (see for instance Lemma 1.1 in [23]) and amounts to reducing the properly edge-colored path problem in $G^c$ to a matching problem in a non-colored graph defined appropriately. The latter graph will be called henceforth *the Edmonds-Szeider graph* and is defined as follows. Given two vertices $s$ and $t$ in $G^c$, set $W = V(G^c) \setminus \{s,t\}$. Now, for each $x \in W$, we first define a subgraph $G_x$ with vertex- and edge-sets, respectively:

$V(G_x) = \bigcup_{i \in I_c} \{x_i, x'_i | N^i_{G^c}(x) \neq \emptyset\} \cup \{x''_a, x''_b\}$ and
$E(G_x) = \{x''_a x''_b\} \cup \left( \bigcup_{\{i \in I_c | x'_i \in V(G_x)\}} (\{x_i x'_i\} \cup (\bigcup_{j=a,b} \{x'_i x''_j\})) \right)$.

Now, the Edmonds-Szeider non-colored graph $G(V,E)$ is constructed as follows:

$G(V) = \{s,t\} \cup (\bigcup_{x \in W} V(G_x))$, and
$G(E) = \left( \bigcup_{i \in I_c} (sx_i | sx \in E^i(G^c)) \cup (x_i t | xt \in E^i(G^c)) \cup (x_i y_i | xy \in E^i(G^c)) \right) \cup \left( \bigcup_{x \in W} E(G_x) \right)$.

The interesting point in the construction is that, given a particular (trivial) perfect matching $M$ in $G \setminus \{s,t\}$, a properly edge-colored $s-t$ path exists in $G^c$ if and only if there exists an augmenting path $P$ relative to $M$ between $s$ and $t$ in $G$. Recall that a path $P$ is augmenting with respect to a given matching $M$ if, for any pair of adjacent edges in $P$, exactly one of them is in $M$, with the further condition that the first and last edges of $P$ are not in $M$. Since augmenting paths in $G$ can be found in $O(|E(G)|)$ linear time (see

[28], p.122), the same execution time holds for finding properly edge-colored paths in $G^c$ as well, since $O(|E(G)|) = O(|E(G^c)|)$.

## 2 The $s - t$ path/trail problem

Given two, not necessarily distinct, vertices $s$ and $t$ in $G^c$, the main question of this section is to give polynomial algorithms for finding (if any) a properly edge-colored $s-t$ path or trail in $G^c$. The $s-t$ path problem was first solved by Edmonds for two colors (see Lemma 1.1 in [23]) and then extended by Szeider [26] to include any number of colors. Here we deal with variations of the properly edge-colored trail/path problem, i.e., the problem of finding an $s - t$ trail, closed trails, the shortest $s - t$ path/trail, the longest $s - t$ path (trail) in graphs with no properly edge-colored cycles (closed trails), $s-t$ paths/trails of size $O(logn)$ and $s-t$ paths/trails with forbidden pairs.

### 2.1 Finding a properly edge-colored trail between two vertices

This section is devoted to the $s - t$ trail problem. Among other results, we prove that the $s - t$ trail problem reduces to the $s - t$ path problem. As the latter problem has been proved polynomial [26], it follows that our problem is polynomial as well. Let us start with the following simple, though fundamental, result.

**Lemma 2.1.** *(Fundamental Lemma) Given two vertices $s, t$ of $G^c$, assume that there exists a $s-t$ properly edge-colored trail $T$ in $G^c$. Further, suppose that at least one internal vertex on this trail is visited three times or more. Then, there exists another $s - t$ trail $T'$ in $G^c$ such that no vertex is visited more than twice on $T'$.*

**Proof.** Set $T = e_1e_2 \cdots e_k$, where $e_i$ are the edges of the trail. Let $\{a_1, a_2, \cdots, a_r\}$ denote the set of distinct vertices of $T$. Let now $\lambda_i$ denote the number of times vertex $a_i$ is visited on $T$, for each $i = 1, 2, \cdots, r$. Set $\lambda = max(\lambda_1, \lambda_2, \cdots, \lambda_r)$. Let us choose $T$ to be the shortest such trail so that $\lambda$ is the smallest possible, as is therefore the number of vertices $a_i$ with $\lambda_i = \lambda$. If $\lambda \leq 2$ we are done. Assume therefore $\lambda \geq 3$. Thus, there exist some vertex, say $a_p$, $1 \leq p \leq r$, visited at least three times on $T$. Assume $\lambda = 3$, the proof being almost identical for higher values. Let us rewrite $T = e_1e_2 \cdots e_ie_{i+1} \cdots e_je_{j+1} \cdots e_fe_{f+1}e_{f+2} \cdots e_k$ so that : i) $a_p$ is the vertex common to the pair of edges $e_i, e_{i+1}$, (respectively to $e_j, e_{j+1}$ and to $e_f, e_{f+1}$) and ii)$a_p$ is not a member of the vertex set of the graph induced by the edges of the segment $e_{f+2} \cdots e_k$. Notice that edges $e_i$ and $e_{j+1}$ have the same color, for otherwise, the trail $e_1e_2 \cdots e_ie_{j+1} \cdots e_fe_{f+1}e_{f+2} \cdots e_k$ violates the choice of $T$, since $a_p$ is visited fewer times on this trail than on $T$. Similarly, edges $e_i$ and $e_{f+1}$ have the same color. But then the trail $e_1e_2\cdots e_ie_f, e_{f-1}e_{f-2}\cdots e_{j+2}e_{j+1}e_{f+1}e_{f+2}\cdots e_k$ violates the choice of $T$. This completes the argument and the proof of lemma. $\square$

Thus, we need only consider trails where no vertex is visited more than twice to deal with our problem. As a result, we can transform the trail- to the path-problem as follows. Given $G^c$ and an integer $p \geq 2$, let us consider an edge-colored graph denoted by $p-H^c$ (henceforth called *the trail-path graph*) obtained from $G^c$ as follows. Replace each vertex $x$ of $G^c$ by $p$ new vertices $x_1,\ x_2,\cdots,x_p$. Furthermore for any edge $xy$ of $G^c$ colored, say $j$, add two new vertices $v_{xy}$ and $u_{xy}$, add the edges $x_iv_{xy}, u_{xy}y_i$, for $i = 1,\ 2,\ \cdots,p$ all of them colored $j$, and finally add the edge $v_{xy}u_{xy}$ in the new (unused) color $j' \in \{1, 2, \cdots, c\}$ with $j' \neq j$. For convenience of notation, the edge-colored subgraph of $p - H^c$ induced by the vertices $x_i, v_{xy}, u_{xy}, y_i$ and associated with the edge $xy$ of $G^c$ will be denoted throughout by $H^c_{xy}$. Moreover for $p = 2$, we write $H^c$ instead of $p - H^c$. Then, we have the following relation between $G^c$ and $p - H^c$, for $p = 2$:

**Theorem 2.2.** *Given two vertices $s$ and $t$ in $G^c$, there exists a properly edge-colored $s - t$ trail in $G^c$, if and only if, there exists a properly edge-colored $s_1 - t_1$ path in $H^c$.*

**Proof.** Let $s, t$ be two vertices in $G^c$. Assume first that there exists a properly edge-colored trail, say, $T = e_1, e_2, \cdots, e_k$ between $s$ and $t$ in $G^c$, where $e_i$ are the edges of the trail and $s$ is the left endpoint of $e_1$ while $t$ is the right endpoint of $e_k$. By Lemma 2.1, we may choose $T$ so that no vertex is visited more than twice on $T$. Given $H^c$ as defined above, we show how to construct a properly edge-colored path $P$ between $s_1$ and $t_1$ in $H^c$. For any edge $e_i = xy$ of $T$, we consider the associated subgraph $H^c_{e_i}$ in $H^c$, and then replace the edge $e_i$ by one of the segments $x_1v_{xy}, v_{xy}u_{xy}, u_{xy}y_1$ or $x_1v_{xy}, v_{xy}u_{xy}, u_{xy}y_2$ or $x_2v_{xy}, v_{xy}u_{xy}, u_{xy}y_1$ or $x_2v_{xy}, v_{xy}u_{xy}, u_{xy}y_2$ in $H^c$.

Conversely, any properly edge-colored $s_1-t_1$ path in $H^c$ uses precisely one of the subpaths $x_1v_{xy}, v_{xy}u_{xy}, u_{xy}y_1$ or $x_1v_{xy}, v_{xy}u_{xy}, u_{xy}y_2$ or $x_2v_{xy}, v_{xy}u_{xy}, u_{xy}y_1$ or $x_2v_{xy}, v_{xy}u_{xy}, u_{xy}y_2$ in each subgraph $H^c_{xy}$ of $H^c$. Now it is easy to see that a properly edge-colored $s_1-t_1$ path in $H^c$ will correspond to a properly edge-colored $s-t$ trail $T$ in $G^c$ where no vertices are visited more than twice on $T$. $\square$

The following corollary is a straightforward consequence of Theorem 1.2 and Theorem 2.2.

**Corollary 2.3.** *Consider two distinct vertices $s$ and $t$ in a $c$-edge colored graph $G^c$. Then we can either find a properly edge-colored $s - t$ trail or else decide correctly that such a trail does not exist in $G^c$ in linear time on the size of $G^c$.*

**Proof.** To find a properly edge-colored $s - t$ trail in $G^c$, it suffices to construct $H^c$ as above and then use Theorem 1.2 in order to find a properly edge-colored path between $s_1 - t_1$ in $H^c$, provided that one exists. Clearly graph $H^c$ has $O(km)$ edges where $m$ is the number of edges of $G^c$ and $k$ is a small constant. We obtain the corresponding properly edge-colored trail in $G^c$ by replacing appropriate segments of the path in $H^c$ with the associated edges in $G^c$. □

We conclude the section with some results on closed trails in edge-colored graphs. In particular, we intend to characterize edge-colored graphs without properly edge-colored closed trails. Recall that the problem of checking whether $G^c$ contains no properly edge-colored cycle was initially solved by Grossman and Häggkvist [18] for 2-edge colored graphs and then by Yeo [29] for an arbitrary number of colors (see Theorem 1.1 above). In both studies, the authors used the concept of a cut-vertex separating colors, i.e. a vertex $x$ such that all the edges between each component of $G^c - x$ and $x$ are colored alike. Here, by introducing the concept of bridges separating colors, we obtain the following :

**Theorem 2.4.** *Let $G^c$ be c-edge colored graph, such that every vertex of $G^c$ is incident with at least two edges colored differently. Then either $G^c$ has a bridge or $G^c$ has a properly edge-colored closed trail.*

**Proof:** Given $G^c$, let us consider again the trail-path graph $H^c$, associated with $G^c$ as in the foregoing. Observe that if a vertex $x$ of $G^c$ is incident with two edges colored differently in $G^c$, then both $x_1$ and $x_2$ will be incident with edges of different colors in $H^c$. In addition, for every edge $xy$ of $G^c$, we have by the definition of $H^c$ that both $v_{xy}$ and $u_{xy}$ are incident with edges of two different colors. Therefore, we conclude that if every vertex of $G^c$ is incident with at least two edges in different colors in $G^c$, than every vertex of $H^c$ will be incident with at least two edges of different colors in $H^c$. Then, it follows by Theorem 1.1 that $H^c$ has either a cut-vertex separating colors or a properly edge-colored cycle.

Now, suppose first that $H^c$ has a cut-vertex separating colors. If this cut-vertex is one of $v_{xy} \in H^c_{xy}$, then it is easy to see that $u_{xy}$ is another cut-vertex of $H^c$ separating colors. Therefore, the edge $v_{xy}u_{xy}$ is a bridge in $H^c$. This implies that the edge $xy$ of $G^c$ associated with $H^c_{xy}$ is also a bridge in $G^c$.

Assume now that $H^c$ has a properly edge-colored cycle. Then we conclude that $G^c$ has a properly edge-colored trail if and only if we have an properly edge-colored cycle in $H^c$.

From the above, it follows that if each vertex of $G^c$ is incident with at least two edges colored differently, then $G^c$ has either a bridge or a properly edge-colored trail, as required. □

As for the algorithmic aspects of this problem, it suffices to delete all bridges and all vertices adjacent to edges of the same color in $G^c$ to test for the existence of a properly edge-colored closed trail in polynomial time. Notice that all such edges and vertices may be deleted without any properly edge-colored closed trail being destroyed. Thus, if the resulting graph is non-empty, it will contain a properly edge-colored closed trail.

### 2.2 Shortest properly edge-colored paths/trails

In this section we consider shortest properly edge-colored $s-t$ paths and trails. By associating appropriate costs with the edges of the Edmonds-Szeider non-colored graph $G(V,E)$ defined in the introduction, we first show how to find, if any, a shortest properly edge-colored path between (not necessarelly distinct) $s$ and $t$ in $G^c$. The procedure will then be used to find a shortest properly edge-colored trail between $s$ and $t$ in $G^c$. At the end of the section, we will show how to find a shortest properly edge-colored cycle and closed trail.

For the shortest properly edge-colored path problem, let us consider the following algorithm:

Algorithm 1: SHORTEST PROPERLY EDGE-COLORED PATH
**Input:** A $c$-edge colored graph $G^c$, vertices $s,t \in V(G^c)$.
**Output:** If any, a shortest properly edge-colored $s-t$ path $P$ in $G^c$.
**Begin**
1. Define: $W = V(G^c) \setminus \{s,t\}$;
2. For every $x \in W$ construct $G_x$ as defined in Section 1;
3. Construct the Edmonds-Szeider graph $G$ associated with $G^c$;
4. Define: $E' = \cup_{x\in W} E(G_x)$;
5. **For every** $pq \in E(G) \setminus E'$ **do** $cost(pq) \leftarrow 1$;
6. **For every** $pq \in E'$ **do** $cost(pq) \leftarrow 0$;
7. **If** $G$ contains a perfect matching **then**
   7.1 - Find a Mininum Weighted Matching $M$ in $G$;
   7.2 - Using $M$, return $P$ in $G^c$;
   **end if**;

**End**.

Intuitively, the idea in Algorithm 1 is to penalize all edges of $G$ associated with edges in the original graph $G^c$. In this way, we ensure that a minimal perfect matching $M$ will maximize the number of edges of $E(G_x)$ (with cost 0) associated with $x \in V(G^c) \setminus \{s,t\}$.

To obtain $P$ from $M$ in Step 7.2, we concatenate all subgraphs $G_x$ to a single vertex $x$ and delete the remaining edges not in $M$. Notice that all the vertices not in the associated $s-t$ path in $G^c$ are isolated, otherwise, $M$ would not be a minimum perfect matching.

In addition, observe that the overall complexity of Algorithm 1 is dominated by the complexity of the minimum perfect matching (Step 7.1). Several matching algorithms exist in the literature. Gabow's bound [13] in $O(n(m + nlogn))$, is one of the best in terms of $n$ and $m$, but other bounds are possible when the edge weights are integers. Note that Algortihm 1 may be easily adapted if we deal with arbitrary positive costs associated with colored edges. Gabow and Tarjan [15] proposed an ingenious approach to obtain a bound in $O(mlog(nN)\sqrt{n\alpha(n,m)logn}))$, where $\alpha(n,m)$ is the Tarjan's "inverse" of Ackerman's function and $N$ is the maximum weight of an edge. See also Gerards [16] for a reference on general matchings.

Formally, we have established the following result:

**Theorem 2.5.** *Algorithm 1 always find, if any, a shortest properly edge-colored $s-t$ path in $G^c$.*

**Proof.** Let $M$ be a minimum perfect matching in $G$ and $P$ the associated path in $G^c$ (obtained after Step 7.2). For a contradiction, suppose that $P$ is not a properly edge-colored shortest path in $G^c$. Then, there exists another properly edge-colored $s-t$ path $P'$ in $G^c$ with $cost(P') < cost(P)$. In addition, suppose that all the remaining vertices not in $P'$ are isolated. Now, observe that $cost(pq) = 1$ for every $pq \in E(G) \setminus E'$ and $cost(pq) = 0$ for every $pq \in E'$. Thus, we can easilly construct a new matching $M'$ in $G$ such that all edges with unit costs are associated with edges in the $s-t$ path $P'$. The remaining edges of $M'$ will have cost *zero*. In this way, since $cost(P') < cost(P)$, we obtain $cost(M') < cost(M)$ resulting in a contradiction. Therefore, $P$ is a shortest properly edge-colored path in $G^c$. □

Now, to solve the shortest trail problem, it suffices to use the above algorithm as follows: Given $s$ and $t$ in $G^c$, construct the trail-path graph $H^c$ associated with $G^c$. Next, we find a shortest properly edge-colored $s_1 - t_1$ path $P$ in $H^c$ by the previous algorithm. Then, by using path $P$ of $H^c$, come back and construct a shortet properly edge-colored $s-t$ trail in $G^c$. Hence our algorithm:

Algorithm 2: SHORTEST PROPERLY EDGE-COLORED $s-t$ TRAIL
**Input:** A $c$-edge colored graph $G^c$, vertices $s, t \in V(G^c)$.
**Output:** The shortest properly edge-colored $s-t$ trail $T$ in $G^c$ (provided that one exists).
**Begin**
1. Construct the trail-path graph $H^c$ associated to $G^c$;
2. Using Algorithm 1, find a shortest $s_1 - t_1$ path $P$ in $H^c$;
3. Return trail $T$ associated to path $P$ with $cost(T) = \frac{cost(P)}{3}$;

**End**.

For the correctness of the algorithm, remember that each subgraph $H^c_{xy}$ of $H^c$ is associated with some edge $xy$ of $G^c$. Furthermore, observe that a properly edge-colored path $P_{x_i,x_j}$ between $x_i$ and $y_j$ in $H^c_{xy}$ contains exactly 3 edges. Thus, in order to obtain $T$ in $G^c$ from $P$ in $H^c$, it suffices to replace each segment $P_{x_i,x_j}$ of $P$ with the corresponding edge $xy$ in $G^c$. Thus, the correctness of Algorithm 2 is guaranteed by Theorems 2.2 and 2.5.

We conclude this section with some algorithmic results on shortest properly edge-colored cycles and closed trails. In particular, we adapt the ideas described above to construct such shortest cycles or closed trails in $G^c$ (if any), as follows. For an arbitrary vertex $x$ of $G^c$, construct a graph $G_x^{c+1}$ (with $c+1$ colors) associated with $x$ by appropriately spliting $x$ into vertices, say $s_x$ and $t_x$, and $c$ auxiliary vertices $x_1, .., x_c$. Vertices $s_x$ and $t_x$ will correspond to temporary source and destination of $G_x^{c+1}$, and vertices $x_1, .., x_c$ are defined in such a way that properly edge-colored $s_x - t_x$ paths in $G_x^{c+1}$ will correspond to properly edge-colored cycles in $G^c$ passing through vertex $x \in V(G^c)$. Therefore, it suffices to repeat this process for every vertex $x$ of $G^c$ while saving the minimum cost solution at each iteration. Formally, we define:

$V(G_x^{c+1}) = (V(G^c) \setminus \{x\}) \cup \{s_x, t_x, x_1, ..., x_c\}$ and
$E(G_x^{c+1}) = (E(G^c) \setminus \{xy : y \in N_{G^c}(x)\}) \cup (\bigcup_{i \in I_c} \{x_i y : y \in N^i_{G_c}(x)\} \cup (\{s_x, t_x\} \times \{x_1, .., x_c\})$.

In the construction of $E(G_x^{c+1})$ above we set $c(x_i y) = i$ for every color $i \in I_c$. In addition we color every edge of $\{s_x, t_x\} \times \{x_1, .., x_c\}$ with a new color $c+1$. After this construction, we find a shortest properly edge-colored path between $s_x$ and $t_x$ in $G_x^{c+1}$. This process is repeated for the remaining vertices of $G^c$. Note that a properly edge-colored $s_x - t_x$ path $P_x$ in $G_x^{c+1}$ of length $|P_x|$ is associated with a properly edge-colored cycle $C_x$ in $G^c$ passing through $x$ of length $|C_x| = |P_x| - 2$. Hence the procedure:

Algorithm 3: PROPERLY EDGE COLORED SHORTEST CYCLE
**Input:** A $c$-edge colored graph $G^c$.
**Output:** If any, a smallest properly edge-colored cycle of $G^c$.
**Begin**
1. $N_{edges} \leftarrow \infty$; {minimum number of edges in the $s_x - t_x$ path in $G_x^{c+1}$}
2. **For every** $x \in V$ **do**
   2.1 Construct $G_x^{c+1}$ as above using $s_x$ and $t_x$ as *source* and *destination*;
   2.2 Using Alg. 2, find (if any) a shortest $s_x - t_x$ path $P_x$ in $G_x^{c+1}$;
   2.3 **If** $|P_x| < N_{edges}$ **then** $SP \leftarrow P_x$ and $N_{edges} \leftarrow |P_x|$;
3. **If** $N_{edges} < \infty$ **then**
   3.1 Using $SP$, return a smallest properly edge-colored cycle in $G^c$ passing by $x$ of length $N_{edges} - 2$;

**End**.

Formally we have established the following result:

**Theorem 2.6.** *Given $G^c$, Algorithm 3 always finds a shortest properly edge-colored cycle in $G^c$ or else decides correctly that $G^c$ has no properly edge-colored cycle at all.*

As with Algorithm 2, the correctness of Algorithm 3 is guaranted by Theorem 2.5.

As for shortest closed trails, to exhibit an arbitrary properly edge-colored closed trail, it suffices to replace $G_x^{c+1}$ with a new graph $G_\alpha^{c+1}(x)$ in the following way (represented by $G_\alpha$, for short). Let $G_\beta = G_x^{c+1} \setminus \{s_x, t_x, x_1, .., x_c\}$ a subgraph of $G_x^{c+1}$. Construct the trail-path graph $H_\beta^c$, associated to $G_\beta$. Finally, to obtain $G_\alpha$ add vertices $\{s_x, t_x, x_1, .., x_c\}$ to $G_\beta$ and define $color(pq) = c+1$ for every edge $pq \in \{s_x, t_x\} \times \{x_1, .., x_c\}$. In this case, all edges $x_i y$ (for $i \in I_c$) of color $i$ in $G_x^{c+1}$ are changed by 2 edges $x_i y_1$, $x_i y_2$ in $G_\alpha$ with the same color. Now, it is easy to see that properly edge-colored paths $s_x - t_x$ in $G_\alpha$ corresponds to properly edge-colored closed trails passing by $x$ in $G^c$ and vice-versa. Now, in Algorithm 3, it is sufficient to change $G_x^{c+1}$ by $G_\alpha$ and repeat the same sequence of steps. Again, the correctness of this new procedure is guaranted by Theorems 2.2 and 2.5.

## 2.3 The longest properly edge-colored $s-t$ path/trail problem

The problem of finding the longest properly edge-colored $s-t$ path in arbitrary $c$-edge colored graphs is obviously NP-complete since it generalizes the Hamiltonian Path problem in non-directed graphs. Here, we propose a polynomial time procedure for finding the longest properly edge-colored $s-t$ path (trail) in graphs with no properly edge-colored cycles (closed trails). Finally, in this section, we generalize the color coding technique (introduced by [1]) to find (if it exists) properly edge-colored $s-t$ paths or trails of length $k = O(logn)$.

**Theorem 2.7.** *Assume that $G^c$ has no properly edge-colored cycles. Then, we can always find in polynomial time a longest properly edge-colored $s-t$ path or else decide that such a path does not exist in $G^c$.*

**Proof.** Let $W = V(G^c) \setminus \{s, t\}$ and $E' = \cup_{x \in W} E(G_x)$ (see Section 1 for the definition of $G_x$). Now, construct the non-colored Edmonds-Szeider graph $G$ associated to $G^c$ and define $cost(pq) = 1$ for every edge $pq \in E(G) \setminus E'$, and $cost(pq) = 0$ for every $pq \in E'$. Compute (if possible) the maximum perfect matching $M$ in $G$, otherwise, we would not have a properly edge-colored path between $s$ and $t$ (see [17] for the complexity of the maximum perfect matching problem). Now, given $M$, to determine the associated $s-t$ path $P$ in $G^c$, we construct a new non-colored graph $G'$ by just concatenating subgraphs $G_x$ to a single vertex $x$. It is easy to see that $G'$ will contains a $s-t$ path, cycles and isolated vertices, associated respectively to a properly edge-colored $s-t$ path, properly edge-colored cycles and isolated vertices in $G^c$ . However, by hypothesis $G^c$ does not contains properly edge-colored cycles. Therefore, each edge with unitary cost in $M$ it will be associated to an edge in $P$ and vice-versa. Then, since $M$ is a maximum matching, the

associated path $P$ will be the longest properly edge-colored $s-t$ path in $G^c$.
$\square$

Observe in the problem above that, since every vertex is visited at most once and we do not have properly edge-colored cycles, all the vertices not in the longest $s-t$ path will be isolated. However, to find a longest properly edge-colored $s-t$ trail we do not know how many times a given vertex $x \in V(G^c) \setminus \{s,t\}$ will be visited. Note that Lemma 2.1 cannot be applied to this case. Nonetheless, we have the following result concerning the longest properly edge-colored $s-t$ trail.

**Theorem 2.8.** *Let $G^c$ be a c-edge colored graph with no properly edge-colored closed trails and two vertices $s,t \in V(G^c)$. Then, we can always find in polynomial time, a longest properly edge-colored $s-t$ trail in $G^c$, provided that one exists.*

**Proof.** Given $G^c$, construct the associated trail-path graph $p-H^c$ for $p = \lfloor (n-1)/2 \rfloor$ (as described in Subsection 2.1). Note that, no vertices may be visited more than $p$ times in $G^c$. To see that, consider a properly edge-colored $s-t$ trail $T$ of length $n-1$ passing by $x \in G^c$ with every cycle through $x$ in this path of lenght 3.

Now, using the same arguments as in Theorem 2.2, we can easily prove that each properly edge-colored closed trail in $G^c$ is associted with a properly edge-colored cycle in $p-H^c$. Therefore, since $G^c$ does not contain properly edge-colored closed trails (by hypothesis), it follows that $p-H^c$ has no properly edge-colored cycle. In addition, note that $p-H^c$ has $O(n^2)$ vertices. Thus, by Theorem 2.7 we can find (if any) a longest properly edge-colored path, say $P$ between $s_1$ and $t_1$ in $p-H^c$ in polynomial time. Therefore, the associated trail, say $T$ in $G^c$ will be a longest properly edge-colored $s-t$ trail with $cost(T) = \frac{cost(P)}{3}$. $\square$

We conclude this section with a result on longest paths of a randomized flavor. Consider an arbitrary edge-colored graph $G^c$. Now, we show how to construct (if exists) a properly edge-colored $s-t$ path in $G^c$ of length $k = O(logn)$ in randomized polynomial time. Actually, this is a straithforward generalization of the color coding technique [1] for finding arbitrary paths or cycles of length $k$ in non-colored graphs or digraphs. In their work, a random coloring of vertices is performed at each step. Similarly, in our case, a random labelling of the vertices in $W = V(G^c) \setminus \{s,t\}$ using labels of the set $L_{k-1} = \{1,2,..,k-1\}$ is executed at each time.

Here, we say that a properly edge-colored $s-t$ path $P$ is *fully labelled* (instead of colorful, as in [1]) if each vertex on it has a different label. Suppose that labels 0 and $k$ are assigned respectively to $s$ and $t$. In this way, note that $(k-1)!/(k-1)^{(k-1)}$ represents the probability of a properly edge-colored $s-t$ path of length $k$ to become fully labelled. The following result shows how many steps are necessary to find a fully labelled properly edge-colored

$s-t$ path in $G^c$ of length $k$ (provided that one exists).

**Lemma 2.9.** *Consider a c-edge colored graph $G^c$, two vertices $s$ and $t$ of $G^c$, and $l: W \rightarrow L_{k-1}$ a labelling of all vertices of $W$. If any, a fully labered properly edge-colored path between $s$ and $t$ of length $k$ in $G^c$ can be found in $O(c(k-1)2^{(k-1)}m)$ time in the worst case.*

**Proof.** As in [1], the proof is based on a dynamic programming approach. Suppose that at iteration $i$ (for $i = 1, .., k-1$) we have found for each vertex $v \in W$, the possible sets of labels associated with fully labelled properly edge-colored $s-v$ paths of lengh $i$. Let $L_v(i)$ be the collection of all these sets of labels. In addition, also record the color of the last edge in the properly edge colored path associated with some set $L \in L_v(i)$ (represented here by $last(L, s, v)$). At each step $i$, we verify all pairs $(L, last(L, s, v))$ (note that at most $c\binom{k-1}{i}$ of such pairs are possible) and every edge $(v, u) \in E$. Thus, if $l(u) \notin L$ (where $l(u)$ denotes the label of vertex $u$) and $c(vu) \neq last(L, s, v)$, we add label $l(u)$ to the collection of $u$ corresponding to paths of length $i+1$. Finally, in the last step, since $l(t)$ obviouslly belongs to every subset $L$ associated with the collection of some vertex $x \in W$, it suffices to verify if $xt \in E$ and $c(xt) \neq last(L, s, x)$. Therefore, $G^c$ contains a properly edge-colored fully labelled $s-t$ path of length $k$ with respect to labelling $L$, if and only if the final collection associated with vertex $t$ is non-empty. Thus, the maximum number of steps associated with each labelling $L$ will be equal to $cm\sum_{i=1}^{k-1} i\binom{k-1}{i}$ which is clearly $O(cm(k-1)2^{(k-1)})$. $\square$

Therefore, if $\alpha = (k-1)^{(k-1)}/(k-1)!$, we have the following randomized polynomial time algorithm to find a properly edge-colored $s-t$ path of length $k$ in $G^c$.

Algorithm 4: PROPERLY EDGE-COLORED $s-t$ PATH OF LENGTH $k$
**Input:** A $c$-edge colored graph $G^c$ and *two* vertices $s, t$.
**Output:** A properly edge-colored path $s-t$ path of length $k$ in $G^c$ (if any).
**Begin**

1. *count* $\leftarrow 0$;
2. **Repeat**
   - 2.1- Randomly, assign $k-1$ labels to all vertices of $V(G^c) \setminus \{s, t\}$;
   - 2.2- *count* $\leftarrow$ *count* $+ 1$;

   **Until** (A full labered properly edge-colored $s-t$ path of length $k$ is found) or (*count* $= \alpha$);

**End**.

**Theorem 2.10.** *Consider an arbitrary c-edge colored graph $G^c$ and vertices $s, t \in V(G^c)$. Then, Algorithm 4 finds, if any, a properly edge-colored $s-t$ path of length $k = O(logn)$ in randomized polynomial time.*

**Proof.** Initially, observe that $(k-1)!/(k-1)^{(k-1)}$ represents the probability of an $s-t$ path of length $k$ becoming fully labelled at each step. Suppose that this event is represented by $A$ (good labelling) and its complement by $\bar{A}$ (bad labelling). Therefore, the probability of bad labellings in all $\alpha$ trials is nearly $(1-\frac{1}{\alpha})^{\alpha} \leq \frac{1}{e}$. Then, $Pr(A) > 1/e$ after $\alpha$ repetitions.

Now, by the Stirling's approximation it follows that $\alpha < e^{k-1}$. Thus, from the preceeding Lemma, the total complexity will be equal to $O(cm(k-1)2^{(k-1)}e^{k-1})$. Therefore, we have a polynomial time procedure equal to $O(cmn^{O(d)})$ if we consider $k = d.logn$, where $d$ is a constant. □

**Corollary 2.11.** *Consider an arbitrary edge colored graph $G^c$. Then, we can find, if any, a properly edge-colored $s-t$ trail in $G^c$ of length $k = O(logn)$ in randomized polynomial time.*

**Proof.** It suffices to construct the associated trail-path graph $p - H^c$ (for $p = \lfloor (n-1)/2 \rfloor$) and to find, if possible, a properly edge-colored path between $s_1$ and $t_1$ of length $3logn$ in $p - H^c$. □

## 2.4 The forbidden-pair version of the one $s - t$ path/trail problem

Consider a $c$-edge-colored graph $G^c$, a pair a vertices $s, t$ and a set $S = \{\{s_1, t_1\}, \{s_2, t_2\}, \cdots, \{s_k, t_k\}\}$ of $k$ pairs of vertices of $G^c$. In the Properly $s-t$ Path with $k$ Forbidden Pairs problem (PPKFP for short), the objective is to find a properly edge-colored $s-t$ path containing at most one vertex from each pair in $S$. Using a simple transformation attributed to Häggkvist [23], we prove the following result concerning $c$-edge colored graphs:

**Theorem 2.12.** *The* PPKFP *problem is NP-complete.*

**Proof.** The PPKFP obviously belongs to NP. To prove that PPKFP is NP-hard, we construct a reduction from the *Path with Forbidden Pairs* problem - PFP [14]. Given a digraph $D(V, A)$, a pair a vertices $s, t$ and a set $S = \{\{s_1, t_1\}, \{s_2, t_2\}, \cdots, \{s_k, t_k\}\}$ of $k$ pair of vertices, the objective in the PFP problem is to define a $s-t$ directed path in $D$ that contains at most one vertex from each pair in $S$ or else decide that such a path does not exist in $D$. In the present reduction, we construct a $c$-edge colored graph $G^c(V', E)$ with $V' = V \cup \{P^1_{\vec{xy}}, .., P^{c-1}_{\vec{xy}} : \vec{xy} \in A\}$. To simplify the notation, for every $\vec{xy} \in A$ consider $x = P^0_{\vec{xy}}$ and $y = P^c_{\vec{xy}}$. Now, the edge set $E$ is constructed in the following way: every arc $\vec{xy} \in A$ is changed by edges $P^j_{\vec{xy}}P^{j+1}_{\vec{xy}}$ for $j = 0, .., c-1$ with $c(P^j_{\vec{xy}}P^{j+1}_{\vec{xy}}) = j+1$. The set $S$ of forbidden pairs in $G^c$ remains the same. After that, it is easy to see that feasible paths in $D$ corresponds to feasible paths in $G$ and vice-versa. □

In addition, notice that if $k$ is constant, the PPKFP problem can be easily solved in polynomial time. Basically, at each step $i$ of this algorithm, we construct a new graph $G_i = (V_i, E_i)$ with $V_i = V \setminus P_i$ where $P_i = \{p^i_1, .., p^i_k\}$

and $p_j^i = s_j$ or $t_j$ (for $j = 1,..,k$), and $E_i = E(V_i)$. For each subgraph $G_i$ we find an properly edge-colored $s-t$ path (provided that one exists) using the Edmonds-Szeider graph. One can easily check that $2^k$ possible combinations of set $P_i$ are necessary. In this way, the total complexity will be equal to $O(2^k p(n,m))$ where $p(n,m)$ is the complexity of the shortest properly colored $s-t$ path algorithm defined previously. Finally, all results in this section are easily extended to the Properly $s-t$ Trail with $k$ Forbidden Pairs problem since it generalizes the PPKFP problem.

# 3 The k-path/trail problem

Let $k$-PVDP and $k$-PEDT be the decision versions associated respectively with Maximum Properly Vertex Disjoint Path (MPVDP) and the Maximum Properly Edge Disjoint Trail (MPEDT) problems, *i.e.*, given a $c$-edge colored graph $G^c$, two vertices $s,t \in V(G^c)$ and an integer $k \geq 2$, we want to determine if $G^c$ contains at least $k$ properly edge-colored vertex disjoint paths (respectively, edge disjoint trails) between $s$ and $t$. Initially, in next section we show that that both $k$-PVDP and $k$-PEDT are NP-complete even for $k = 2$ and $c = O(n^2)$. In particular, in graphs with no properly colored cycles (respectively, closed trails) and $c = O(n)$ colors, we prove that $k$-PVDP (respectively, $k$-PEDT) is NP-complete for an arbitrary $k \geq 2$. Next, at the end of the section, we stablish some approximation results and polynomial algorithms for special cases for both MPVDP and MPEDT problems.

## 3.1 NP-complete results for general graphs

In Theorem 3.2 stated below we will prove that both 2-PVDP and 2-PEDT are NP-complete for 2-edge-colored graphs. In view of that theorem, let us first consider an auxiliary result concerning directed closed trails in digraphs.

Let $u$ and $v$ be two fixed vertices in a digraph $D$. Deciding if $D$ contains or not a directed cycle containing both $u$ and $v$ is known to be NP-complete [12]. Here, we denote this problem by *Vertex-Disjoint Oriented Cycle* (VDOC). In next theorem we prove that deciding if $D$ contains or not a directed closed trail containing both $u$ and $v$ is also NP-complete. We denote this last problem by *Arc-Disjoint Oriented Closed Trail* (ADOCT).

**Theorem 3.1.** *The* ADOCT *problem is NP-Complete.*

**Proof.** The ADOCT problem obviously belongs to NP. To prove that ADOCT is NP-hard, we define a reduction from the following problem. Given four vertices $p_1, q_1, p_2, q_2$ beloning to a digraph $D$, we wish to determine if there exist 2 arc-disjoint directed trails connecting $p_1 - q_1$ and $p_2 - q_2$ in $D$. Here, this problem will be named 2-*Arc Disjoint Trail* (2-ADT) problem. As proved in [12] the 2-ADT is NP-complete.

In particular, given a digraph $D$, we show how to construct in polynomial time another directed graph $D'$ with a pair of vertices $u, v$ in $D'$ such that there are 2 arc-disjoint trails $p_1 - q_1$ and $p_2 - q_2$ in $D$, if and only if there exists a directed closed trail containing both $u$ and $v$ in $D'$.

Before constructing $D'$ let us set $S = \{p_1, p_2, q_1, q_2\}$, $S' = \{p'_1, p'_2, q'_1, q'_2\}$ and $S'' = \{p''_1, p''_2, q''_1, q''_2\}$. The idea is to split apropriately each vertex $p_i$ ($q_i$) in $S$ into two new vertices $p'_i$ and $p''_i$ ($q'_i$ and $q''_i$) beloning to $S'$ and $S''$, respectively. Thus, we have:

$$V(D') = (V(D) \setminus S) \cup S' \cup S'' \cup \{u, v\},$$

and

$$A(D') = \left(A(D) \setminus \{\bigcup_{x \in S} \{\vec{xy}, \vec{yx} : y \in N_D(x)\}\}\right) \cup \left(\bigcup_{x'' \in S''} \{\vec{x''w} : w \in N_D^+(x)\}\right) \cup$$
$$\cup \left(\bigcup_{x' \in S'} \{\vec{wx'} : w \in N_D^-(x)\}\right) \cup \{\vec{up'_1}, \vec{p'_1p''_1}, \vec{p'_2p''_2}, \vec{q'_1q''_1}, \vec{q''_1v}, \vec{vp'_2}, \vec{q'_2q''_2}, \vec{q''_2u}\}.$$

Given the definitions above, consider two arc-disjoint trails $p_1 - q_1$ and $p_2 - q_2$, say $T_1$ and $T_2$ respectively, in $D$. Then, it is easy to see that the sequence:

$$T = (u, p'_1, p''_1, T_1, q'_1, q''_1, v, p'_2, p''_2, T_2, q'_2, q''_2, u)$$

defines a closed trail containing both $u$ and $v$ in $D'$ (see Figure 1).

Conversely, consider a closed trail containing both vertices $u$ and $v$ in $D'$. Note that, we have exacly one outcoming and one incoming arc incident to $u$ and $v$. It follows that, all closed trails containing $u$ and $v$, also contain all vertices in $S'$ and $S''$ and each pair $(p'_i, p''_i)$ and $(q'_i, q''_i)$, for $i = 1, 2$, must be visited exactly once. This is possible, if and only if we have a trail between $p'_1$ and $q''_1$, and $p'_2$ and $q''_2$ in $D'$. If we delete $u, v \in D'$, and contract all pairs $(p'_i, p''_i)$ to obtain $p_i$, and $(q'_i, q''_i)$ to obtain $q_i$, $i = 1, 2$, we obtain 2 arc-disjoint trails $p_1 - q_1$ and $p_2 - q_2$ in $D$. $\square$

Now, we have the following result:

**Theorem 3.2.** *Let $G^c$ be a 2-edge colored graph and 2 vertices $s, t \in V(G^c)$. Then, both 2-PVDP and 2-PEDT problems are NP-Complete.*

**Proof.** We can easily check in polynomial time that both 2-PVDP and 2-PEDT problems are in NP. To show that they are NP-hard, we propose polynomial time reductions from the VDOC and ADOCT problems, respectively. Consider two vertices $u$ and $v$ in a digraph $D$ . We show how to construct in polynomial time, a 2-edge colored graph $G^c$ and a pair of vertices $a, b \in V(G^c)$, such that there is a cycle (respectively, closed trail) containing

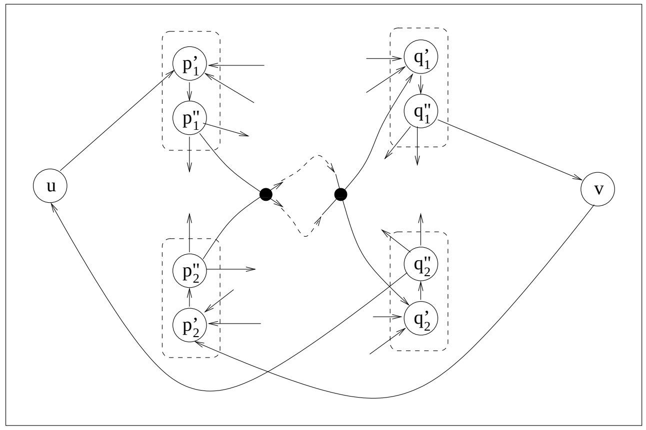


Figure 1: Reduction 2-ADT $\alpha$ ADOC

$u$ and $v$ in $D$, if and only if there are 2 vertex-disjoint properly edge-colored $a-b$ paths (respectively, 2 edge-disjoint properly edge-colored $a-b$ trails) in $G^c$. Let us first define from $D$ another digraph $D'$ by replacing $u$ by two new vertices $s_1, s_2$ with $N^-_{D'}(s_2) = N^-_D(u)$, $N^+_{D'}(s_1) = N^+_D(u)$. Similarly replace $t_1, t_2$ and $N^-_{D'}(t_2) = N^-_D(v)$, $N^+_{D'}(t_1) = N^+_D(v)$. Finally, add the arcs $(s_2, s_1)$ and $(t_2, t_1)$ in $D'$. Now in order to define $G^c$ replace each arc $\vec{xy}$ of $D'$ by a colored segment $xzy$ where $z$ is a new vertex and edges $xz$, $zy$ are on colors red and blue, respectively. Finally, we define $z = a$ for $z$ between $s_1$ and $s_2$, and $z = b$ for $z$ between $t_1$ and $t_2$. Observe now that there is a vertex-disjoint cycle (respectively, arc-disjoint closed trail) containing $u$ and $v$ in $D$ if and only if there are two vertex-disjoint properly edge-colored $a-b$ paths (respectively, properly edge-colored edge-disjoint $a-b$ trails) in $G^c$. □

**Theorem 3.3.** *Both* 2-PVDP *and* 2-PEDT *problems remain NP-complete even for graphs with* $O(n^2)$ *colors.*

**Proof.** Both 2-PVDP and 2-PEDT problems restricted to graphs with $O(n^2)$ colors obviously belongs to NP. Now, given a 2-edge colored graph $G^c$ with $n$ vertices, define a complete graph $K_n^{c'}$ with all edges of different colors and an additional edge $xy$ with $x \in V(K_n^{c'})$, $y \in V(G^c)$ and color $c(xy) = c' + 1$. In this way, the new resulting graph $G_\alpha^{c'+1}$ with edges $E(G_\alpha^{c'+1}) = E(G^c) \cup E(K_n^{c'}) \cup \{xy\}$ will have $n^2+1$ different edge colors and $2n$ vertices. Therefore, 2 properly edge-colored $s-t$ paths/trails in $G^c$ (with 2 colors) will correspond to 2 properly edge-colored paths/trails in $G_\alpha^{c'+1}$ with $c' = O(n^2)$ colors and vice-versa. Thus, from the preceeding theorem (restricted to 2-edge colored graphs), we conclude that both 2-PVDP and 2-PEDT problems in graphs with $O(n^2)$ colors are NP-complete. □

### 3.2 NP-complete results for graphs with no properly edge-colored cycles (closed trails)

Now, we prove that $k$-PVDP and (respectively, $k$-PEDT) for $k \geq 2$, remains NP-complete even for 2-edge colored graphs with no properly edge-colored cycles (respectively, closed trails). We conclude this section generalizing these results for graphs with $O(n)$ colors.

Recall that, as discussed in previous sections, the existence or not of properly edge-colored cycles or closed trails in edge-colored graphs may be checked in polynomial time. Our proof is based on some ideas similar to those used by Karp [21] for the Discrete Multicommodity Flow problem for non-oriented (and non-colored) graphs (usually known in the literature as the Vertex Disjoint Path problem).

**Theorem 3.4.** *Let $G^c$ be a 2-edge colored graph without properly edge-colored cycles (respectively, closed trails). Given two vertices $s$ and $t$ in $G^c$, to decide if there exist $k$ properly vertex-disjoint $s - t$ paths (respectively, $k$ properly edge-disjoint $s - t$ trails) in $G^c$ is NP-complete.*

**Proof.** Let us first consider the vertex-disjoint version. The problem (shortened as usual to $k$-PVDP) obviously belongs to NP. To show that $k$-PVDP is NP-hard we construct a reduction using the Satisfiability problem. Consider a boolean expression $B = \wedge_{l=1}^{k} C_l$ in the Conjuntive Normal Formula with $k$ clauses and $n$ variables $x_1, .., x_n$. We show how to construct a 2-edge colored graph $G^c(V, E)$ and two vertices $s, t$ and with no properly edge-colored cycles, such that a truth assignment for $B$ corresponds to $k$ properly vertex disjoint $s - t$ paths in $G^c$, and reciprocally, $k$ properly vertex-disjoint $s - t$ paths in $G^c$ define a truth assignment for $B$. Basically, the idea is to construct a set of $k$ auxiliary source-sink pairs $s_l, t_l$ of vertices, each pair corresponding a to clause $C_l$. Each variable $x_j$ is associated to a 2-edge colored *grid* graph $G_j$. Then graph $G^c$ is obtained by apropriately joining all together these grid graphs and then adding two new vertices $s$ and $t$.

Given $B$, consider a boolean variable $x$ occurring in the positive form in clauses $i_1, i_2, .., i_p$ and in the negative form in clauses $j_1, j_2, .., j_q$. Each ocurrence of $x$ in the positive (negative) form is associated to a horizontal path $s_{i_a} - t_{i_a}$ (vertical path $s_{j_b} - t_{j_b}$) in the grid $G_x$ such that all consecutive edges between vertices $s_{i_a}$ and $t_{i_a}$ for $a = 1, .., p$ (respectively, between $s_{j_b}$ and $t_{j_b}$ for $b = 1, .., q$) differ in one color. Every properly edge-colored path $s_{i_a} - t_{i_a}$ has a vertex in common with every properly edge-colored path $s_{j_b} - t_{j_b}$. We say that grid $G_x$ satisfy the *blocking property* if there are no properly edge-colored paths between $s_{i_a}$ and $t_{j_b}$, or respectively, between $s_{j_b}$ and $t_{i_a}$ for some $a = 1, .., p$ and $b = 1, .., q$ (see the example of Figure 2). In the first step, all grids $G_{x_j}$, for $j = 1, .., n$, are constructed satisfying the blocking property. Note that, different colorings of $G_x$ satisfying the blocking property are possible. In this case, we can arbitrally choose any one among

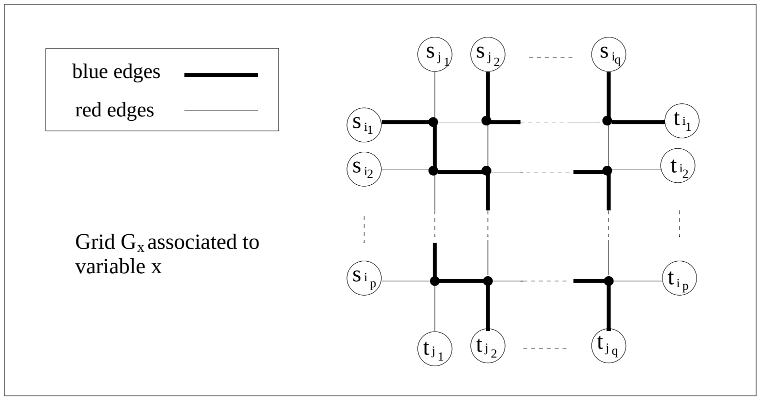


Figure 2: Blocking property

them.

Now, we say that a set of grids satisfies the *color constraint* if all edges incident to $s_l$ and $t_l$, $l = 1, .., k$, in all ocurrences of $s_l$ and $t_l$ in the various grids, have the same color. All grids $G_{x_j}$ for $j = 1, .., n$, must be constructed in order to satisfy both blocking property and color constraint. However, note that the color constraint may be not verified after the first step. To solve this problem, suppose *w.l.o.g.*, that all edges incident to $s_l$ in the various grids must be red if $l$ is odd, and blue if $l$ is even. Similarly, suppose that all edges incident to $t_l$ (in the various grids) must be blue if $l$ is odd, and red if $l$ is even.

Therefore, suppose that edge $s_l w$ (for $w \in N_{G_{x_j}}(s_l)$) must be *blue*. If $c(s_l w) = blue$ after the first step, we are done. Otherwise, we add a new vertex $p$ between $s_l$ and $w$ and fix $c(s_l p) = blue$ and $c(pw) = red$. We apply this procedure for every edge incident to $s_l$ (for $l = 1, ..k$) in the various subgraphs $G_{x_j}$ for $j = 1, .., n$. Finally, we repeat the same transformation for every $t_l$ and $G_{x_j}$ for $l = 1, .., k$ and $j = 1, ..n$. Note that, at the end of this process, we have all grids satisfing both blocking property and color constraint (see Figure 3(a)).

Now, the overall construction of graph $G^c$ is done in two steps. Initially, we identify all occurrences of $s_l$ (respectively, $t_l$) beloning to the various grids $G_{x_j}$, as a single vertex $s'_l$ (respectively, $t'_l$). We repeat this process for each $l = 1, .., k$. Let $G'$ be this new graph. Note that, due to the color constraint, all edges incident to $s'_l$ (respectively $t'_l$) in $G'$ must have the same color.

In the second step, we add a source $s$ and destination $t$, and new edges $ss'_l$ and $t'_l t$ for $l = 1, .., k$. Therefore, to construct $k$ properly edge-colored paths between $s$ and $t$ in this new graph, all edges $ss'_l$ (respectively $t'_l t$) must be colored with a different color, other than those incident to $s_l$ or $t_l$ in $G'$ (see Figure 3(b)). Let $G''$ this new graph.

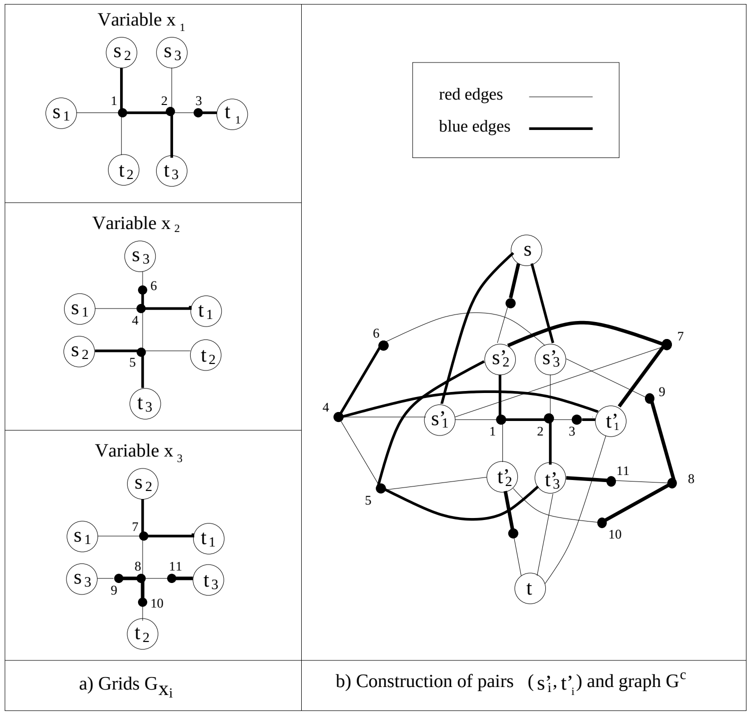


Figure 3: Redution using $B = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee x_2 \vee \bar{x_3}) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_3)$. (a) To satisfy the color constraint, we colored all edges incident to $s_1, s_3, t_2$ and $t_1, t_3, x_2$, respectively, with *red* and *blue* colors. (b) To construct $G^c$ we add $s$ and $t$, and 2 auxiliary vertices. All edges incident to $s$ and $t$ are *blue* and *red* respectively.

Now, note that we can have $c(ss'_a) \neq c(ss'_b)$ (analgously $c(t'_a t) \neq c(t'_b t)$) for some $a, b \in \{1, .., k\}$ with $a \neq b$. In addition, by construction of our grids, we can have a properly edge-colored path between $s'_a$ and $s'_b$ in some grid $G_{x_j}$ for some $j \in \{1, .., n\}$. Therefore, in this case, we can have a properly edge-colored cycle through $s$ (or $t$) in $G''$ (what is not allowed by hypothesis). To avoid that in the construction of $G^c$, it suffices to add auxiliary vertices $p_i$ between $s$ and $s'_i$ (respectively, auxiliary vertices $q_i$ between $t$ and $t'_i$) and conveniently change the colors of edges $sp_i$ (respectively $q_i t$) such that all edges incident to $s$ (respectively $t$), have the same color. In this way, the new resulting graph $G^c$ will contains no properly edge-colored cycles.

Thus, given a truth assignment for $B$, we obtain a set of $k$ properly edge-colored vertex disjoint $s - t$ paths in the following manner. If variable $x_j$ is true, we select the horizontal paths in the grid $G_{x_j}$ between vertices $s_{i_a}$ and $t_{i_a}$ (for $a = 1, .., p$); if $x_j$ is false, we select the vertical paths between $s_{j_b}$ and $t_{j_b}$ (for $b = 1, .., q$). Note that, if either $x_j$ or $\bar{x_j}$ occurs in clause $C_l$, and is true in the assignment, we have a path between vertices $s'_l$ and $t'_l$ in $G'$ (and consequently, between $s$ and $t$ in $G^c$). Therefore, if $B$ is true, we will

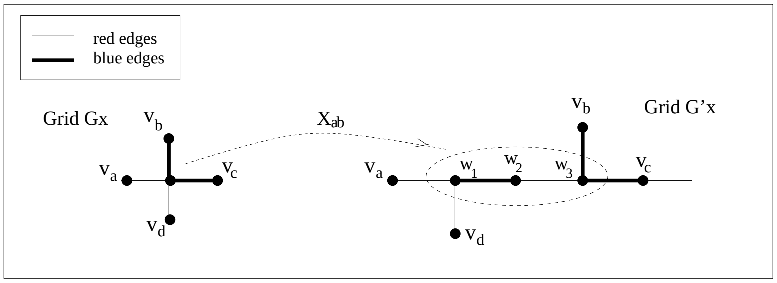


Figure 4: Tranformation from the $k$-PVDP to the $k$-PEDT problem

have $k$ properly vertex-disjoint paths between $s$ and $t$ in $G^c$, each of them passing by $s'_l$ and $t'_l$ for $l = 1, .., k$.

Conversely, consider a set of $k$ properly vertex disjoint $s-t$ paths in $G^c$. Observe in the grid $G_{x_j}$ that, if we have a properly edge-colored path between vertices $s_{i_a}$ and $t_{i_{a'}}$ for $a \in \{1, .., p\}$ and $a' \leq a$, the clause $C_{i_a}$ and variable $x_j$ will be true. Analogously, if we have a path between $s_{j_b}$ and $t_{j_{b'}}$ for $b \in \{1, .., q\}$ and $b' \leq b$, the clause $C_{j_b}$ will be true and variable $x_j$ will be false. Thus, $k$ properly vertex disjoint $s-t$ paths will correspond to $k$ true clauses in $B$. Therefore, for an arbitrary $k \geq 2$, we proved that $k$-PVDP problem is NP-complete in 2-edge colored graphs with no properly edge-colored cycles.

Now, we turn to the edge-disjoint version ($k$-PEDT) of this problem. To prove that $k$-PEDT is NP-Complete, we cannot use the same arguments as above. Note that, we can have 2-edge-disjoint paths between $s$ and $t$ in $G^c$ corresponding to vertical and horizontal paths in some grid $G_x$. In another words, we can have a vertex in the intersection of both paths. If this happens, we cannot determine the value of $x$ in $B$. To solve this problem, it suffices to change each vertex (represented by $X_{ab}$) in the intersection of paths $s_{i_a} - t_{i_a}$ for $a = 1, .., p$ (horizontal path) and $s_{j_b} - t_{j_b}$ for $b = 1, .., q$ (vertical path) in the grid $G_x$ by 3 new verticess $w_1, w_2$ and $w_3$ as described in Figure 4.

In addition, suppose that verticess $v_a, X_{ab}$ and $v_c$ belongs to path $s_{i_a} - t_{i_a}$ and verticess $v_b, X_{ab}$ and $v_d$ belongs to path $s_{j_b} - t_{j_b}$ (in $G_x$). Further, *w.l.o.g.* consider $c(v_a X_{ab}) = c(X_{ab} v_d) = red$ and $c(v_b X_{ab}) = c(X_{ab} v_c) = blue$. In this case, we split $X_{ab}$ into vertices $w_1, w_2, w_3$ and fix $c(w_1 w2) = blue$ and $c(w_1 w2) = red$ (see Figure 4). Note that this new graph with grids, say $G'_x$, also satisfy both blocking property and color constraint. Further, in $G'_x$, if we have a path between $s_{i_a}$ and $t_{i_a}$ (for some $a \in \{1, .., p\}$) passing by $v_a$ and $v_c$, we cannot have a path between $s_{j_b}$ and $t_{j_b}$ (for some $b \in \{1, .., q\}$) passing by $v_b$ and $v_d$ (otherwise, both paths would not be edge disjoint). If we repeat this construction at every grid $G_x$ in $G^c$ (to obtain new grids $G'_x$), we conclude that $k$-PEDT problem is NP-complete in 2-edge colored graphs with no properly edge-colored cycles.

Finally, to extend this result to 2-edge colored graphs with no properly edge-colored closed trails, it suffices to repeat the construction above and replace one or more arbitrary edges $xy \in G^c$ with color $i \in \{red, blue\}$ by a colored segment $xzy$ where $z$ is a new vertex between $x$ and $y$, and 2 additional vertices $p, q$ with edges $zp$, $pq$ and $qz$. These edges are colored in the following way: $c(xz) = c(zy) = c(pq) = i$ and $c(zp) = c(qz) \neq i$. In this way, $k$ properly edge disjoint $s - t$ trails in this new graph $G^c$ (with colors $red$ and $blue$) will be associated to a true assignment for $B$ and vice versa. $\square$

**Theorem 3.5.** *The $k$-PVDP (respectively, $k$-PEDT) problem remains NP-complete even for graphs with $O(n)$ colors and no properly edge-colored cycles (respectively, closed trails).*

**Proof.** The $k$-PVDP ($k$-PEDT) problem in graphs with $n$ colors and no properly edge-colored cycles (closed trails) is obviously in NP. Let $G^c$ a 2-edge colored graph with no properly edge-colored cycles (closed trails) and 2 vertices $s, t \in V(G^c)$. Using $G^c$, we construct a new graph $G_\alpha^{c'}$ with $c' = n$ colors and no properly edge-colored cycles (closed trails) such that $k$ properly vertex (edge) disjoint $s - t$ paths (trails) in $G^c$, corresponds to $k$ properly vertex (edge) disjoint $s - t$ paths (trails) in $G_\alpha^{c'}$ and vice versa.

First, consider a non-colored complete graph $G_1 = K_n$. Choose an arbitrary $x \in V(G_1)$ and color $c(xy) = 1$ for every $y \in N_{G_1}(x)$. Let $G_2 = G_1 \setminus \{x\}$ be the new resulting non-colored graph. Choose a new vertex $x \in V(G_2)$ and color $c(xy) = 2$ for every $y \in N_{G_1}(x)$. Repeat the process above for every non-colored graph $G_i$ for $i = 1, .., n-1$. Let $K_n^{c'}$ for $c' = n - 1$, the resulting colored complete graph. Obviously, $K_n^{c'}$ contains no properly edge-colored cycles (closed trails). Finally, add a new edge $pq$ with $p \in V(G^c)$, $q \in V(K_n^{c'})$ and a new color $c(pq) = n$. Note in this way, that the new graph $G_\alpha^{c'}$ with edges $E(G_\alpha^{c'}) = E(G^c) \cup E(K_n^{c'}) \cup \{xy\}$ contains no alternating cycles (closed trails) and will have $n$ different colors. Therefore, it follows from the preceeding theorem (restricted to 2-edge colored graphs) that the $k$-PVDP ($k$-PEDT) problem in graphs with $n$ colors and no properly edge-colored cycles (closed trails) is NP-complete. $\square$

### 3.3 Some Approximation and Polynomial results

Given two vertices $s$ and $t$ in an edge-colored graph $G^c$ we consider the problem of finding the Maximum number of Properly Edge-Disjoint $s - t$ Trails - MPEDT (respectively, Maximum number of Properly Vertex-Disjoint $s - t$ Paths - MPEDT) in $G^c$. In the sequel, we describe a greedy procedure for the MPEDT, based in the determination of shortest properly edge-colored $s - t$ trails. Its performance ratio is based on the same arguments used for the Edge Disjoint Path problem between $k$ pairs of vertices in non-directed graphs [19, 22]. We conclude this section by presenting some polynomial results for some particular instances of both problems.

Algorithm 5: GREEDY PROCEDURE FOR MPEDT PROBLEM
**Input:** A $c$-edge colored graph $G^c$ and *two* vertices $s,\ t$ of $G^c$.
**Output:** A set $X$ of edges corresponding to the maximum possible number of properly edge-colored edge-disjoint $s-t$ trails.
**Begin**
1. $X \leftarrow \oslash$; $E \leftarrow E(G^c)$
2. **Repeat**
   2.1 Using Alg. 2, find a shortest properly $s-t$ trail $T$ in $G^c$;
   2.2 $X \leftarrow X \cup E(T)$;
   2.3 $E(G^c) \leftarrow E(G^c) \setminus E(T)$;

**Until** (no properly edge-colored $s-t$ trails are found);
**End**.

Now consider the following definitions: we say that a trail $T_1$ *hits* a trail $T_2$, or equivalently, that $T_2$ is *hitted* by $T_1$, if and only if $T_1$ and $T_2$ share a common edge. If $\Gamma$ denotes the set of all properly edge-colored $s-t$ trails, we define $I \subseteq \Gamma$ as the subset of trails obtained by the greedy procedure and $J \subseteq \Gamma$ the subset of trails associated to the optimal solution.

**Theorem 3.6.** *Algorithm* 5 *has performance ratio equal to* $O(1/\sqrt{m})$.

**Proof.** Let $T \in \Gamma$ be an arbitrary trail in $G^c$. We say that a trail $T \in \Gamma$ is *short* if $|E(T)| \leq \sqrt{m}$, and *long* otherwise. Therefore, for a trail $T \in J_{long}$ we have $|E(T)| \geq (\sqrt{m}+1)$ and $|J_{long}|(\sqrt{m}+1) \leq m$. Thus, *w.l.o.g.*, if we consider $|I| \geq 1$, it follows that $|J_{long}| < \sqrt{m} < |I|\sqrt{m}$.

Additionally, we can say that every trail $T_j \in J_{short} \setminus I$ is hit by a trail $T_i \in I_{short}$, otherwise (if $T_i \in I_{long}$) at the point when $T_i$ was picked, $T_j$ was available and shorter than $T_i$ and should have been taken by the greedy procedure. Thus, if $T_i$ is the shortest trail that hits $T_j$ we have $|E(T_i)| \leq |E(T_j)| \leq \sqrt{m}$.

Now, observe that all trails in $I_{short}$ have at most $|I_{short}|\sqrt{m}$ edges and each $P_j \in J_{short} \setminus I$ is hitted by at least one edge of $I_{short}$. Futhermore, since all trails $T_j$ are edge-disjoint it follows that one edge in $I_{short}$ cannot hit more then one trail $T_j$. Thus, $|J_{short} \setminus I| \leq |I_{short}|\sqrt{m} \leq |I|\sqrt{m}$.

Finally, we have $|J| = |J_{short}| + |J_{long}| < |(J_{short} \setminus I) \cup I| + |I|\sqrt{m} \leq (2\sqrt{m}+1)|I|$ which guarantees a $O(1/\sqrt{m})$ performance ratio for the MPEDT problem. □

To give some idea about the determination of the value $\sqrt{m}$ above, suppose that a trail $T_1$ hits $k$ paths of $J \setminus I_1$ at the first step of Algortihm 5. Note that, one edge of $T_1$ can hit at most one other path of $J$ and therefore $T_1$ have length at least $k$. Since $T_1$ is a shortest $s-t$ trail, all other trails in $J \setminus I_1$ also have at least $k$ edges. Therefore, $k^2 \leq m$, so $k = \sqrt{m}$. This idea

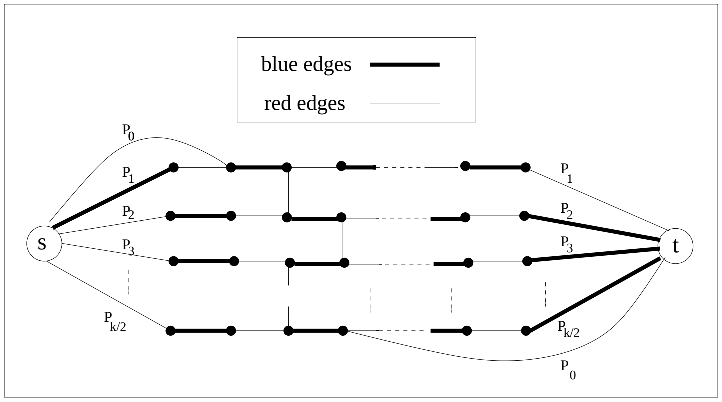


Figure 5: Let $G^c$ be a 2-edge colored graph. Suppose $|E(T_i)| = k+2$ for $i = 1,..,k/2$. The ratio between Algorithm 3 and the optimal solution is $2/k$.

may be inductively applied for the remaining steps of the greedy procedure.

In the Figure 5, we consider a 2-edge colored graph graph $G^c$ with $|E(T_i)| = k+2$ for $i = 1,..,k/2$. In this case, since $|E(T_0)| = k+1$ (the shortest $s-t$ trail), Algorithm 5 will first consider $T_0$, hitting $k/2$ properly edge-colored $s-t$ trails. Clearly the optimal solution is obtained by choosing trails $T_1,..,T_{k/2}$. Thus $2/k$ is ratio between greedy and optimal solution where $k \leq \sqrt{m}$.

We turn now to the vertex-disjoint version of the above problem, namely, the Maximum number of Properly Vertex-Disjoint $s-t$ paths in $G^c$. We can easily modify Algorithm 5 to solve MPVDP. In this case, after the determination of a shortest $s-t$ path $P$ (instead of trail $T$), it suffices to remove all vertices beloning to $P \setminus \{s,t\}$. We call this new procedure GREEDY-VD. Using the same ideas as described in Theorem 3.6, we proof the following result:

**Theorem 3.7.** *The Greedy-VD procedure has performance ratio equal to $O(1/\sqrt{n})$ for the* MPVDP *problem.*

We end this section with some polynomial results for some specific families of graphs. To begin with, we introduce the following definition: given an edge colored graph $G^c$, we say that a cycle $C_x : xa_1 \cdots a_j x$ with $x \neq a_i$ for $i = 1,..,j$ is an *almost properly colored cycle (closed trail) through $x$ in $G^c$*, if and only if $c(xa_1) = c(xa_j)$ and both paths (respectively trails) $x - a_1$ and $x - a_j$ are properly colored. If $c(xa_1) \neq c(xa_j)$, then $C_x$ define a *properly edge-colored cycle (closed trail) through $x$*. In the sequel, we show how to solve the MPVDP (respectively, MPEDT) problem in polynomial time for graphs containing no properly or almost properly colored cycles (respectively, closed trails) through $s$ or $t$. Notice that to check if a colored graph $G^c$ contains or not a properly or an almost properly cycle (closed trail) through $x$, it suffices to define an auxiliary graph $G^c_x$ obtained from $G^c$ by replacing $x$ with two new vertices $x_a$ and $x_b$ and setting $N_{G^c_x}(x_a) = N_{G^c}(x)$ and $N_{G^c_x}(x_b) = N_{G^c}(x)$. Now, using Theorem 1.2 (respectively, Corollary 2.3) we compute, if any, a properly edge-colored $x_a - x_b$ path (trail) in $G^c_x$. Clearly if no such $x_a - x_b$ path (trail) exists in $G^c_x$, then there exists no properly or almost properly edge colored cycle (closed trail) through $x$ in $G^c$.

Initially, consider the following decision version associated with MPVDP problem. Given some constant $k \geq 1$, we show how to construct a polynomial time procedure for the $k$-PVDP in graphs with no (almost) properly colored cycles through $s$ or $t$.

**Theorem 3.8.** *Consider a constant $k \geq 1$ and a c-edge colored graph $G^c$ with no (almost) properly colored cycles through $s$ or $t$. Then, the $k$-PVDP problem may be solved in polynomial time.*

**Proof.** Suppose, *w.l.o.g.*, that we do not have (almost) properly colored cycles through vertex $s$ in $G^c$. Observe in this case that (almost) properly colored closed trails (with vertex repetitions) through $s$ are allowed.

For $k = 1$, the problem is polynomially solved by Edmonds-Szeider's Algorithm. For $k \geq 2$, we construct an auxiliary non-colored graph $G'$ in the following way. As discussed in Section 1, we first define $W = V(G^c)\backslash\{s, t\}$, and non-colored graphs $G_x$ for every $x \in W$ (see the first part in the definition of the Edmonds-Szeider's graph). Now, define $S_k = \{s_1, .., s_k\}$, $T_k = \{t_1, .., t_k\}$ and proceed as follows:

$V(G') = S_k \cup T_k \cup (\bigcup_{x\in W} V(G_x))$, and
$E(G') = \bigcup_{j=1,..,k} \left(\bigcup_{i\in I_c}\{(s_j x_i | sx \in E^i(G^c)) \cup (x_i t_j | xt \in E^i(G^c))\}\right) \cup \left(\bigcup_{i\in I_c}(x_i y_i | xy \in E^i(G^c))\right) \cup \left(\bigcup_{x\in W} E(G_x)\right)$.

Now, find a perfect matching $M$ (if any) in $G'$ and concatenate each subgraph $G_x$ into a single vertex $x$. Let $G''$ this new graph. Observe that all paths in $G''$ are defined by edges belonging to $M \cap E(G'')$. In addition, we cannot have a path between $s_i$ and $s_j$ in $G''$ (otherwise, we would have a (an almost) properly cycle though $s$ in $G^c$). In this way, all paths in $G''$ begins at vertex $s_i \in S_k$ and finish at some vertex $t_j \in T_k$. Finally, we construct a non-colored graph $G'''$ by concatenating $S_k$ and $T_k$ respectively to vertices $s$ and $t$. In this way, note that $s - t$ paths in $G'''$ are associated to properly edge-colored $s - t$ paths in $G^c$ and vice-versa. Therefore, if the construction of a perfect matching $M$ in $G'$ is possible (what is done in polynomial time), we obtain $k$ properly edge colored $s - t$ paths in $G^c$. □

Since the perfect matching problem is solved in polynomial time, we can easily construct a polynomial time procedure for the MPVDP in graphs with no (almost) properly colored cycles through $s$ or $t$. To do that, it suffices to repeat all the steps described in Theorem 3.8 for $k = 1, .., n - 2$ until some non-colored graph $G'$ containing no perfect matchings is found.

The ideas above may be generalized for the MPEDT in graphs with no (almost) properly colored closed trails through $s$ or $t$. First, consider its associated decision version.

**Theorem 3.9.** *Consider a constant $k \geq 1$ and a c-edge colored graph $G^c$ with no (almost) properly edge-colored closed trails through $s$ or $t$. Then, the $k$-PEDT problem may be solved in polynomial time.*

**Proof.** Given $G^c$, construct the associated trail-graph $p - H^c$ (as described in Section 2) for $p = \lfloor (n-1)/2 \rfloor$. Note that, no vertices may be visited more than $p$ times in $G^c$ even if they share different properly edge-colored $s - t$ trails. To see that, condider a vertex $x \in G^c$ and a properly edge-colored $s-t$ trail of length 2 through $x$, all other properly edge-colored trails through $x$ will have at least 4 edges (each of them containing 2 new vertices in $G^c$).

Suppose, *w.l.o.g.*, that we do not have (almost) properly colored closed trails through vertex $s$ in $G^c$. Now, using Theorem 2.2, we can easily prove that $G^c$ contains a (an almost) properly colored closed trail through $s$, if and only if, $H^c$ contains a (an almost) properly colored cycle through $s_1$. As a consequence of that, we have no (almost) properly edge-colored cycles through $s_1$ in $H^c$. Thus, by Theorem 3.8 we can find in polynomial time (if any) $k$ properly edge-colored paths between $s_1$ and $t_1$ in the graph $H^c$. Now, by concatenating every subgraph $H^c_{xy}$ in $H^c$ to edge $xy$ in $G^c$ we obtain $k$ properly edge-colored $s - t$ trails in $G^c$. $\square$

Similarly to the MPVDP problem, to construct a polynomial procedure for the MPEDT, it suffices to repeat all the steps above (in Theorem 3.9 for $k = 1, .., n-2$ until some non-colored graph associated to $H^c$ and containing no perfect matching is found.

# 4 Conclusions and open problems

In this work, we have considered path problems in edge-colored graphs. We generalized some previous results concerning properly edge-colored paths and cycles in colored graphs, which allowed us to devise efficient algorithms for finding them. On the negative side, we proved that finding $k$ properly vertex/edge disjoint $s-t$ paths/trails is NP-complete even for $k = 2$ and $c = O(n^2)$. In addition, we showed that both problems remain NP-complete for arbitrary $k \geq 2$ in graphs with no properly edge-colored cycles (closed trails) and $c = O(n)$, which led us to investigate approximation. For that purpose, a procedure for MAEDP, which greedily builds shortest properly edge-colored $s - t$ paths, was shown to have a respectable $O(1/\sqrt{m})$ performance ratio. Similarly, we obtained an approximation ratio in $O(1/\sqrt{n})$ for the MAVDP. Finally, we showed that both MAVDP (MAEDP) are solved in polynomial time when restricted to graphs with no properly edge-colored cycles (closed trails) through $s$ or $t$. However, the following questions are left open.

Is the following problem NP-complete?

**Problem 4.1.** *Input: Given a 2-edge colored graph $G^c(V, E)$ with no properly edge-colored cycles, two vertices $s, t \in V$ and a fixed constant $k \geq 2$. Question: Does $G$ constain $k$ properly edge-colored vertex/edge disjoint paths between $s$ and $t$?*

As a future direction, another important question is to consider the approximation performance ratio (as well as inapproximability results) for both MPVDP and MPEDT for general colored graphs or for graphs with no properly edge-colored cycles (closed trails). Finally, another interesting topic of research is to generalize our results on properly edge-colored walks in $c$-edge colored graphs with edge capacities.

We conclude our paper by recalling the following open problem from [23].

**Problem 4.2.** *Let $s$ and $t$ be two fixed vertices in an edge-colored complete graph $K_n^c$. Does there exist a polynomial algorithm for finding the maximum number of pairwise edge-disjoint $s - t$ trails in $K_n^c$?*

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