Hierarchical Random Energy Model of a Spin Glass

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We introduce a random energy model on a hierarchical lattice where the interaction strength between variables is a decreasing function of their mutual hierarchical distance, making it a non-mean-field model. Through small coupling series expansion and a direct numerical solution of the model, we provide evidence for a spin-glass condensation transition similar to the one occurring in the usual mean-field random energy model. At variance with the mean field, the high temperature branch of the free-energy is nonanalytic at the transition point.

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Clarifying the nature of glassy states is a fundamental goal of modern statistical physics. Both for spin glasses [1] and for structural glasses [2], the mean-field theory of disordered systems provides a suggestive picture of laboratory glassy phenomena as the reflection of an ideal thermodynamic phase transition. Unfortunately, the development of a first principles theory of glassy systems going beyond mean field has resisted decades of intense research [3–5]. One of the main obstacles towards this goal lies in the lack of reliable real space renormalization group (RG) schemes allowing us to reduce the effective number of degrees of freedom and identify the relevant fixed points describing glassy phases. In ferromagnetic systems, an important role in the understanding of the real space RG transformation has been played by spin systems with power law interactions on hierarchical lattices [6,7]. In these models, the RG equations take the simple form of nonlinear integral equations for an unknown function (as opposed to the functional of statistical field theory), that can be solved with high precision. In this perspective, it is natural to generalize these models to spin glasses [8,9].

In this Letter, we introduce the simplest such spin-glass model, a random energy model (REM) [13,14]. As we shall see, the hierarchical REM is such that the interaction energy between subsystems scales subextensively in the system size. It thus qualifies as a non-mean-field model. We report in what follows the results of a small coupling expansion and of an algorithmic solution of the RG equations for the entropy that, exploring complementary regions of parameter space, provide the first analytic evidence in favor of an ideal glass transition in a non-mean-field model. Interestingly, this transition turns out to have—as in the case of the standard REM—the character of an entropy catastrophe analogous to the one hypothesized long ago for the structural glasses [15,16].

The hierarchical REM can be defined as a system of $N = 2^k$ Ising spins with an energy function defined recursively. The recursion is started at the level of a single spin $k = 0$, with the definition of $H_0[S] = \epsilon_0(S)$, where the

single spin energies are independent identically distributed (i.i.d.) random variables extracted from a distribution $\mu_0(\epsilon)$. At the level $k + 1$, we consider then two independent systems of 2^k spins $S_1 = \{S_{1i}\}$, $i = 1, \ldots, 2^k$ and $S_2 = \{S_{2i}\}$, $i = 1, \ldots, 2^k$ with Hamiltonians $H_{1k}[S_1]$ and $H_{2k}[S_2]$, respectively, and put them in interaction to form a composite system of 2^{k+1} spins and Hamiltonian

$$H_{k+1}[S_1, S_2] = H_{1k}[S_1] + H_{2k}[S_2] + \epsilon_k[S_1, S_2],$$

where the ϵ_k are i.i.d. random variables extracted from a distribution $\mu_{k+1}(\epsilon)$, chosen to have zero mean and variance $\langle \epsilon_k[S_1, S_2]^2 \rangle \sim 2^{(k+1)(1-\sigma)}$. The interaction term $\epsilon_k[S_1, S_2]$ is physically analogous to a surface interaction energy between the two subsystems. For $\sigma \in (0, 1)$, this model qualifies as a non-mean-field system, where the interaction energy between different parts of the system scales with volume to a power smaller than unity. On the contrary, when $\sigma \leq 0$, the interaction energy grows faster than the volume. A rescaling of the energy is then necessary to get a well-defined thermodynamic limit. The system behaves in this case as a mean-field model. Finally, for $\sigma > 1$, the interaction energy decreases with distance and asymptotically the model behaves as a free system. In the following, we focus on the most interesting region $0 < \sigma < 1$.

We have studied this model with two different methods. The first one is a replica study of the quenched free energy, performed through a small coupling perturbative expansion. The second one is a numerical estimate of the microcanonical entropy as a function of the energy. Both methods suggest that a REM-like finite-temperature phase transition occurs for all $\sigma \in (0, 1)$.

Perturbative computation of the free energy.—In order to make the calculations as simple as possible, we have chosen a Gaussian distribution for the energies ϵ_k. We then considered the perturbative expansion in $g \equiv 2^{1-\sigma}$ of the free energy $f(T) = f^{(m)}(T) + O(g^{m+1})$. Notice that the expansion of f to the mth order takes into account just
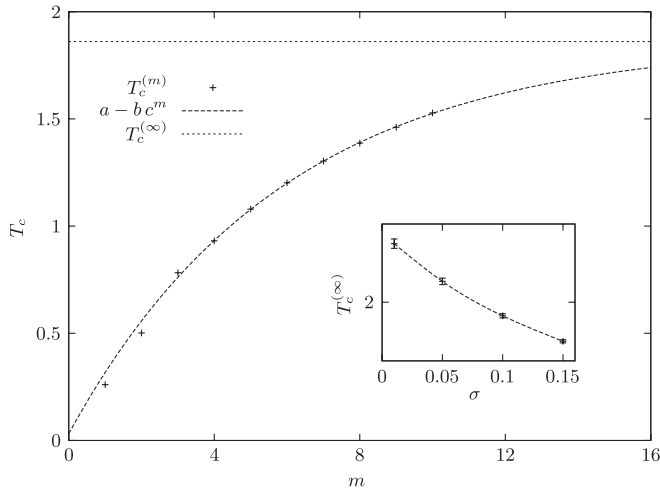
FIG. 1. The temperatures $T_c^{(m)}$ vs m for $\sigma = .1$. Here, $T_c = 1.861 \pm .021$. Inset: T_c vs σ.
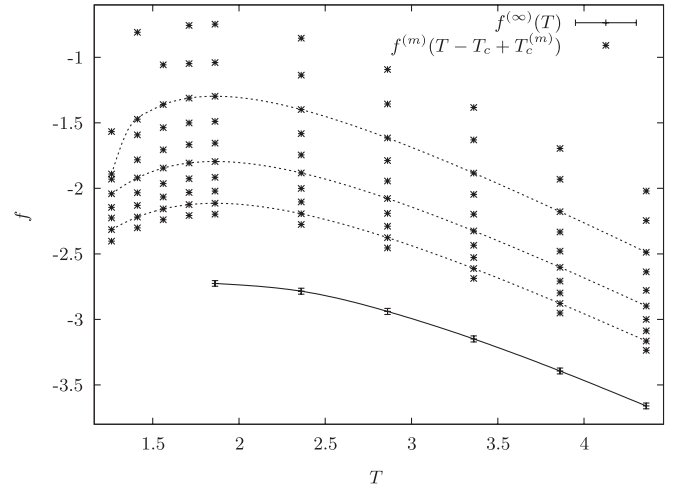

FIG. 2. To get a better convergence for the free energy, we considered the sequence $f^{(m)}(T - T_c + T_c^{(m)})$ instead of $f^{(m)}(T)$. As $T_c^{(m)} \to T_c$ for $m \to \infty$, the two sequences have the same limit $f^{(\infty)}(T)$. Here, we see that for $\sigma = 0.1$, $f^{(m)}(T)$ has negative entropy $s^{(m)}(T)$ for $T < T_c^{(m)}$.

the interactions with range less or equal to 2^m, i.e., the first m hierarchical levels.

The computation of the free energy has been done with the replica method. In this context, it is just a mathematical tool to organize the terms of the series. We considered then the expansion of the average partition function of the system replicated n times

$$\overline{Z^n} = \sum_{S_1 \cdots S_n} \exp\left[\frac{\beta^2}{4} \sum_{j=0}^{k} g^j \sum_{i=1}^{2^{k-j}} \sum_{a,b=1}^{n} \delta_{S_a^{(j,i)} S_b^{(j,i)}} \right] \quad (1)$$

where $\beta \equiv 1/T$, and $S^{(j,i)}$ is the configuration of the ith group of spins at the jth level of the hierarchy. This representation allowed an automated computation of $f^{(m)}$ up to the value of $m = 10$.

A useful check of the method is obtained considering $\sigma < 0$. Since in this case high values of j dominate the energy in (1), correlations between the energy levels can be neglected. After rescaling the energies by $\epsilon_j \to 2^{k\sigma/2} \epsilon_j$, the free energy of the model becomes equal to the one of the standard REM [13,14] with critical temperature $T_c = \sqrt{\sum_j 2^{-\sigma j} / \log 2}$. We found that, when increasing m, $f^{(m)}$ converges to the REM free energy with exponential speed in the whole high temperature regime $\beta < \beta_c$.

We now consider $f(T)$ for $0 < \sigma < 1$. The direct inspection of the curves shows that, for $m \geq 1$, the mth order entropy $s^{(m)}(T) \equiv -df^{(m)}(T)/dT$ while positive at high temperature, becomes negative at some temperature $T_c^{(m)}$. As can be seen in Fig. 1, the sequence $T_c^{(m)}$ exhibits a good exponential convergence to a finite limit T_c for $\sigma \leq 0.15$. The stability of these data for large m clearly suggests that an entropy crisis transition is present in the model at T_c. The inset in Fig. 1 shows that T_c is a decreasing function of σ, consistently with the fact that the larger σ, the weaker the interaction strength. At high temperature also, the free-

energy series has a good exponential convergence in g (see Fig. 2).

This small g expansion gives some evidence for an entropy crisis taking place at temperature T_c. It is important to realize that this T_c cannot be simply computed from the sum of the variances of the ϵ_k: the energy correlations cannot be neglected. An entropy crisis implies the existence of a phase transition at a temperature $\geq T_c$. In a REM scenario, the phase transition would take place exactly at T_c, when the entropy vanishes. An argument in favor of such a result can be found with a one-step replica symmetry breaking ansatz. Consider the partition function (1) and suppose that the n replicas are grouped into n/x groups, so that, for any two replicas a, b in the same group, $S_a^{(j,i)} = S_b^{(j,i)}$ for all i, j. Then perform again the small g expansion, within this ansatz. To each order m, this procedure gives a free energy $f_x^{(m)}(T) = f^{(m)}(T/x)$. The maximization over x then gives $x = 1$ for $T > T_c^{(m)}$, and $x = T/T_c^{(m)}$ for $T < T_c^{(m)}$. This result is in complete analogy with the one found in the REM, so the above replica symmetry breaking Ansatz predicts a REM-like transition at $T = T_c$. In order to get distinct evidence for this scenario, we have done some numerical study.

Numerical computation of the entropy.—We exploit the hierarchical structure of the model to compute the microcanonical entropy $S_k(E)$. In order to make the computations as simple as possible, we have chosen for $\mu_k(\epsilon)$ the binomial distribution [17,18],

$$\mu_k(\epsilon) = \frac{1}{2^{M_k}} \binom{M_k}{\epsilon + \frac{M_k}{2}}.$$

At the level k, M_k is the integer part of $\gamma 2^{k(1-\sigma)}$, to have the

same scaling of the variance as in the Gaussian model. The constant γ is chosen so that for all the values of σ studied, $\lfloor(\gamma 2^{k(1-\sigma)})\rfloor/(\gamma 2^{k(1-\sigma)}) \approx 1$ for every k. Consider the disorder-dependent density of states for a sample a: $\mathcal{N}_k^a(E) = \sum_S \delta_{H_k(S),E}$. The recursion relation that defines the model's Hamiltonian implies that when two samples a and b at the level k are merged to define a sample at the level $k + 1$, the resulting density of states $\mathcal{N}_{k+1}^c(E)$ satisfies

$$\mathcal{N}_{k+1}^c(E) = \sum_{\substack{E_a,E_b,\epsilon \\ E=E_a+E_b+\epsilon}} n_k(E_a, E_b, \epsilon) \qquad (2)$$

where $n_k(E_a, E_b, \epsilon) = \sum_{S_1,S_2} \delta_{H_k^a(S_1),E_a} \times \delta_{H_k^b(S_1),E_b} \delta_{\epsilon_k(S_1,S_e),\epsilon}$ is the number of states in the composite system that have $H_a = E_a$, $H_b = E_b$ and interaction energy equal to ϵ. For given E_a and E_b, the joint distribution of the $n_k(E_a, E_b, \epsilon)$ for the different values of ϵ is multinomial with parameters $q_\epsilon = \langle n_k(E_a, E_b, \epsilon)\rangle = \mathcal{N}_k^a(E_a)\mathcal{N}_k^b(E_b)\mu_k(\epsilon)$, while n_k's with different first or second argument are independent.

Our algorithmic approach starts from the exact iteration of Eq. (2). Thanks to the use of a discrete interaction energy, the iteration time grows with k proportionally to $2^{k(3-\sigma)}$. This allowed us to reach the level $k = 12$. The results of the iteration shows that the values of the energy can be divided in bulk region of energy density around the origin where the number of states $\mathcal{N}_k(E) = e^{2^k S_k(E/2^k)}$ is exponential in the system size, and an edge region where the number of states is of order one (see Fig. 3).

In order to proceed further, we assume the existence and self-averaging property of the entropy density $S(e)$ in the thermodynamic limit. We then coarse grain our description. We discretize the energy density in the bulk region

and use an approximated iteration for the entropy, where the sum (2) is approximated by its maximum term. We account for the edge region using the exact recursion for the $N_0 = 10\,000$ lowest energy levels. We can in this way iterate many times and obtain a good estimate of the thermodynamic limit behavior.

In Fig. 3, we present the average entropy density as a function of the energy density e for various values of σ. In order to identify the transition, it is more convenient to average the data obtained with a fixed energy difference from the fluctuating ground states. We can get in this way good estimates of the value of the inverse critical temperature of the model $\beta_c = s'(e_0)$. An interesting feature emerging from our analysis is that close to the ground state energy density e_0, the entropy is not analytic and behaves as $S(e) \approx \beta_c(e - e_0) + C(e - e_0)^a$ with a well fitted by the value $a = 2 - \sigma$. This behavior, when translated in the canonical formalism, implies a singularity of the free energy close to T_c, $F(T) = E_0 + \text{const} \times (T - T_c)^{(2-\sigma)/(1-\sigma)}$, corresponding to a specific heat exponent $\alpha = -\frac{\sigma}{1-\sigma}$.

Having found evidence for a thermodynamic phase transition, we turn our attention to the distribution of low-lying energy states. The REM picture suggests that, close to the ground state, the number of energy levels with given energy E are independent Poissonian variables with density $\langle \mathcal{N}_\infty(E)\rangle = e^{\beta_c(E-E_0)}$. A computation using extreme value statistics shows that the probability $Q_\ell(k)$ that the ground state and first $\ell - 1$ excited states are occupied by n levels is given by

$$Q_\ell(n) = [1 - \exp(-\ell\beta_c)]^n/(\ell\beta_c n). \qquad (3)$$

In Fig. 4, we show the $Q_\ell(n)$ obtained numerically together with a fit with the form (3). This procedure confirms the
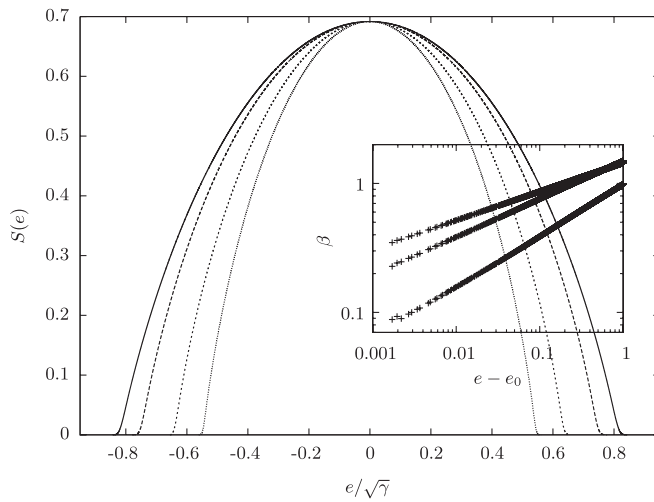

FIG. 3.   The entropy $s(e)$ vs $e/\sqrt{\gamma}$ for $\sigma = 0.9, 0.8, 0.7, 0.6$ (with $\gamma = 30, 10, 5, 5$, respectively), from the outside to the inside. Inset: power law behavior of $\beta - \beta_c$. The slopes are close to $1 - \sigma$, with $\sigma = 0.6, 0.7, 0.8$ from bottom to top.
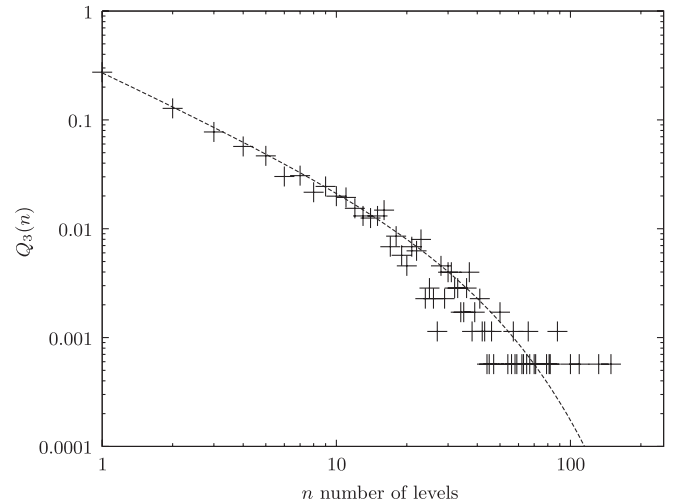

FIG. 4.   Numerical data (cross) and the fitting function $Q_3(n)$ for the statistics of occupation of the ground state and the first two occupied levels. Here, $k = 10$, $\sigma = 0.6$ and $\gamma = 5$. The dashed line is a fit with the form (3) with $\beta_c = 1.20$.
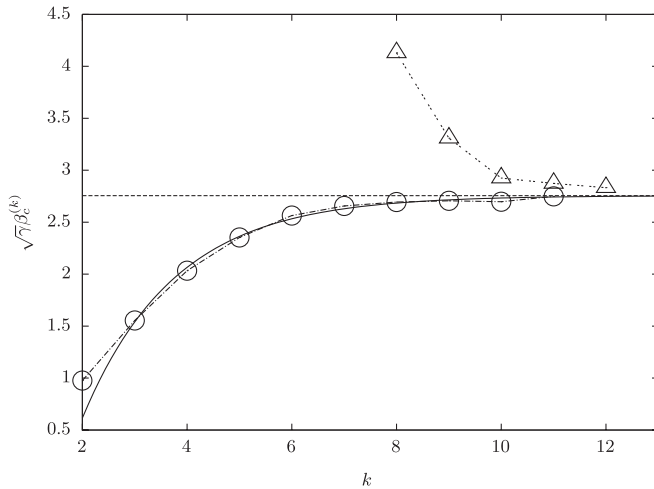
FIG. 5.   Finite volume estimation of the inverse critical temperature, $\sqrt{\gamma}\beta_c^{(k)}$ vs k, for $\sigma = 0.6$ and $\gamma = 5$ determined by (3) (circle) and by $\beta_c = s'(e_0)$ (triangles). The latter have been fitted with a function of the type $\beta_c^{(k)} = \beta_c - BC^k$ (solid line). The asymptotic value β_c is given by the dashed line.

validity of a REM-like transition, and provides an alternative way of estimating the critical temperature. As Fig. 5 shows, the two estimates for different values of k tend to the same limit from opposite directions.

Conclusions.—In this Letter, we have introduced a hierarchical, non-mean-field REM. We have analyzed it through small coupling series, through a 1RSB replica ansatz, and through an algorithmic approach. The two approaches point to the existence of a REM-like phase transition at the temperature where the entropy vanishes. At variance with the mean-field result (which predicts a discontinuity in the specific heat), one finds a nontrivial specific heat exponent at T_c. It will be interesting to study the replica structure of this hierarchical REM in order to explore other possible replica solutions at low temperatures. Another important theme of future research is the study of spin-glass models with p-body interaction [13,19,20]: at the mean-field level, these models display an entropy crisis transition similar to the one of the REM

whenever $p \geq 3$. It will be interesting to study them on hierarchical lattices.

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