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Projet CoCoGro: Computational and Combinatorial aspects of Symbolic Dynamics on Groups.

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This research project investigates interplay between computer science and mathematics. It deals with symbolic dynamics on groups, using tools and results in group theory as well as computability theory. For computer scientists, mathematical notions based on group theory provide a new and deeper understanding of subshifts as computational model; likewise, this computer science approach offers an innovative point of view on groups that will interest mathematicians, for instance by providing new invariants. It thus perfectly illustrates how mathematics and theoretical computer science can benefit from each other.

Symbolic dynamics is the study of subshifts, i.e. sets of colourings of a group G by a finite alphabet A that respect local constraints given by forbidden patterns. Given a set of forbidden patterns F, the subshift X_F it defines is the set of configurations $x \in A^G$that avoids all patterns from F. This combinatorial definition has a dynamical equivalent. The set of configurations A^G , endowed with the product topology, is a compact space on which we define the shift transformations: for every $g \in G$, the shift σ^g translates a configuration $x \in A^G$ through $\sigma^g(x)_h = x_{g^{-1}\cdot h}$ for every $h \in G$. In this framework, subshifts are exactly subsets of A^G that are both shift-invariant and closed for the product topology. Subshifts can fruitfully be seen as a computational model, as well as a discrete model for dynamical systems. Subshifts of finite type (SFT) constitute an interesting class of subshifts since they are described by a finite amount of information, and can model real-world phenomena. Classical symbolic dynamics, originally defined in the highly influential article by Morse and Hedlund [5] in order to study the discretization of dynamical systems, has historically focused on the one-dimensional case $G = \mathbb{Z}$ [8] and was later generalized to higher dimensions $G = \mathbb{Z}^d$ with $d \geq 2$ []. But increasing the dimension has a strong impact on the decidability and combinatorial properties of SFTs. Even the simplest question one could ask about SFTs – decide whether a finite set of forbidden patterns defines a non-empty subshift or not, known as the Domino problem – is decidable in dimension 1 but undecidable when the dimension increases. This problem is strongly linked to the existence of periodic configurations in SFT: again whereas every one-dimensional non-empty SFT admits a periodic configuration, there are two-dimensional non-empty SFTs that dot not contain any periodic configuration. Similarly, the entropy of an SFT – roughly speaking a measure of the growth rate of allowed patterns of size n – is easily computable in 1D but becomes non-computable in higher dimensions. These problems are only three examples among many others that reflect the existence of a gap between 1D and higher dimensions. Moreover, several results obtained on finitely generated groups as alternative structures to $\mathbb{Z}$ and $\mathbb{Z}^d$ show us that understanding where undecidability and complexity generally come from is a non-trivial and fertile question.

The objective of this project is to explore computational and combinatorial aspects of symbolic dynamics on finitely generated groups. This new point of view has two advantages: first, it unifies previous examples and will lead to results not specific to one group. Second, it defines a worthwhile computational model, relatively unexplored until now: subshifts on groups. For computer scientists, mathematical notions based on group theory provide a new and deeper understanding of subshifts as computational model. On the other hand, this computer science approach offers an innovative point of view on groups that will interest mathematicians, for instance by providing new invariants. We strive to identify which dynamical, combinatorial or geometric properties are the most significant in different results for particular groups, thus allowing to generalize them to larger classes of groups. To do so we will borrow, improve and generalize techniques from various domains, such as geometric group theory, computability theory, discrete probability theory, combinatorics on words, linear algebra and algebraic geometry, and statistical mechanics.

We first describe different methods we intend to work with. To start with, the following tools and techniques have been chosen both for their innovative aspect and their potential to generate new results:

1. Understanding how the geometry of the group controls the computability properties of the subshifts defined on this group, that is to say identifying geometric properties of groups that are relevant for problems tackled in this project. A particular attention will be paid to Gromov-hyperbolic groups, since their algorithmic properties are well-understood.
2. Encoding computational models inside SFTs, such as modular machines that have been used to simplify proofs of various unsolvability results in group theory.
3. Understanding the links between different techniques to prove non-emptiness of non-SFT subshifts: lower bound method in 1D by Kolpakov [7], entropy compression [9, 10, 4] and different variants of Lovász local lemma [1].
4. Exploiting the representation of 2D configurations as multivariate formal power series over integers [6]: express dynamical or computational properties of SFTs with this formalism, and prove results using tools from linear algebra and algebraic geometry.
5. Generalizing techniques from statistical mechanics – corner transfer matrix method or recent improvements [2] – to compute or give sharp bounds on the entropy of the appropriate class of 2D SFTs.
6. Implementing computer aided proofs: some methods mentioned above – Kolpakov lower bounds method, or the transfer matrix method – are highly computational. Other methods may also take benefit from computed assisted case analysis, like recent generalizations of entropy compression [3].

We aim to tackle three difficult questions: first, characterizing finitely generated groups with decidable Domino problem; second, understanding which groups admit aperiodic SFTs ; third, defining powerful tools to compute the entropy of 2D subshifts.

References

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