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MULTIDIMENSIONAL EFFECTIVE S-ADIC SUBSHIFTS ARE SOFIC.

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ABSTRACT. In this article we prove that multidimensional effective S-adic systems, obtained by applying an effective sequence of substitutions chosen among a finite set of substitutions, are sofic subshifts.

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Introduction

Let $\mathcal{A}$ be a finite alphabet. A d-dimensional subshift $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ is a closed and shift-invariant set of configurations, where the shift is the natural action of $\mathbb{Z}^d$ on the configurations space $\mathcal{A}^{\mathbb{Z}^d}$. With a combinatorial point of view, one can equivalently define subshifts by excluding configurations that contain some forbidden finite patterns. Depending on the conditions imposed on this set of forbidden patterns, it is possible to define several classes of subshifts. The simplest one is the class of subshifts of finite type (also called SFT), where the set of forbidden finite patterns may be chosen finite. A larger class is the one of sofic subshifts, which are images of SFT under a factor map. These two classes are defined locally and they are well understood in dimension 1.

A way to construct minimal aperiodic subshifts is to consider subshifts generated by a fix point of substitution, introduced in dimension one by Thue [Thu06] and generalized to higher dimensions. These subshifts constitute the class of the substitutive subshifts. More precisely, for a substitution s one can consider the

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subshift $\mathbf{T}_{\{s\}}$ where the allowed patterns are given by iterations of the substitution s on a letter of $\mathcal{A}$, or the set $\mathbf{T}'_{\{s\}}$ of configurations which have pre-images by arbitrarily many iterations of s. Of course $\mathbf{T}_{\{s\}} \subset \mathbf{T}'_{\{s\}}$. In dimension 1 the class of substitutive subshifts and the class of sofic subshifts are disjoint except for trivial cases: substitutive subshifts have low complexity [Pan84], while the only sofic subshifts with low complexity are periodic. In the multidimensional framework the situation is different since all substitutive subshifts are sofic. This result is a generalization to any substitution satisfying some weak condition (rectangular 2-dimensional substitution satisfying property A, see Theorem 4.5 of [Moz89]) of the original construction of aperiodic tilings [Rob71].

A possible generalization of the construction of substitutive subshifts is to consider S-adic subshifts, which were introduced by S. Ferenczi in the one-dimensional setting [Fer96]. Given a finite set of substitutions $\mathscr{S}$, and a sequence $\mathrm{S} \in \mathscr{S}^{\mathbb{N}}$, we define the subshifts $\mathbf{T}_{\mathrm{S}}$ and $\mathbf{T}'_{\mathrm{S}}$ where the iterations of the different substitutions are given by the sequence S. This class of subshifts is studied in dimension 1 and, under some conditions on the set $\mathscr{S}$, it is shown that the complexity is low [Fer96, Dur00]. It is thus natural to wonder if there exist sofic S-adic subshifts in higher dimensions. If the set of substitutions $\mathscr{S}$ has the unique derivation property, an argument of cardinality shows that the class of S-adic multidimensional subshifts is not included in the class of sofic subshifts. Indeed the class of sofic subshifts is countable, since there are countably many SFT and countably many factor maps. There are uncountably many ways to choose an infinite sequence of $\mathscr{S}$, but each class of conjugacy is countable since there are countably many conjugateness. Thus there are uncountably many different non-conjugate S-adic multidimensional subshifts. The purpose of this article is to show that S-adic subshifts which are sofic are exactly those for which the sequence S is effective. More generally we characterize the set $\boldsymbol{\mathfrak{S}} \subset \mathscr{S}^{\mathbb{N}}$ such that $\mathbf{T}_{\boldsymbol{\mathfrak{S}}} = \bigcup_{\mathrm{S}\in\boldsymbol{\mathfrak{S}}} \mathbf{T}_{\mathrm{S}}$ is a sofic subshift. The set $\boldsymbol{\mathfrak{S}} \subset \mathscr{S}^{\mathbb{N}}$ must be effectively closed, that is to say there exists a recursively enumerable sequence $(w_k)_{k\in\mathbb{N}}$ of elements of $\mathscr{S}^*$ such that $\mathrm{S} \in \boldsymbol{\mathfrak{S}}$ if and only if $S_{[0,|w_k|-1]} \neq w_k$ for all $k \in \mathbb{N}$.

The main idea of the proof is to use the result by S. Mozes which proves that a substitutive subshift is sofic in the case where the substitution is not deterministic and satisfies property A (Theorem 4.5 of [Moz89]). This means that each time one wants to use a substitution, it is possible to choose a rule among a set of substitutions $\mathscr{S}$. However, contrary to the S-adic subshifts, at each level of iteration different substitutions of $\mathscr{S}$ may appear. The aim of the proof is to synchronize these substitutions, and in that purpose we need to introduce a one dimensional effective subshift which codes the sequence of substitutions. This effective subshift can be realized by a 3-dimensional sofic subshift thanks to the result of M. Hochman [Hoc09] or by a 2-dimensional sofic subshift thanks to the improvement obtained by [DRS09] or [AS13].

1. Definition and classical properties

1.1. Notion of subshift

Let $\mathcal{A}$ be a finite alphabet and d be a positive integer. A *configuration* x is an element of $\mathcal{A}^{\mathbb{Z}^d}$. Let $\mathbb{U}$ be a finite subset of $\mathbb{Z}^d$, we denote by $x_{\mathbb{U}}$ the *restriction* of x to $\mathbb{U}$. A $\mathbb{Z}^d$*-dimensional pattern* is an element $p \in \mathcal{A}^{\mathbb{U}}$ where $\mathbb{U} \subset \mathbb{Z}^d$ is finite, $\mathbb{U}$ is the *support* of p, which is denoted by $\mathrm{supp}(p)$. A pattern p of support $\mathbb{U} \subset \mathbb{Z}^d$ *appears* in a configuration x if there exists $i \in \mathbb{Z}^d$ such that $p = x_{i+\mathbb{U}}$, and in this case we write $p \sqsubset x$.

We define a topology on $\mathcal{A}^{\mathbb{Z}^d}$ by endowing $\mathcal{A}$ with the discrete topology, and considering the product topology on $\mathcal{A}^{\mathbb{Z}^d}$. For this topology, $\mathcal{A}^{\mathbb{Z}^d}$ is a compact metric space on which $\mathbb{Z}^d$ acts by translation via $\sigma_{\mathcal{A}}$ – that will be denoted by σ if there is no ambiguity on the alphabet considered – defined for every $i \in \mathbb{Z}^d$ by

$$\sigma^i_{\mathcal{A}} : \begin{pmatrix} \mathcal{A}^{\mathbb{Z}^d} & \longrightarrow & \mathcal{A}^{\mathbb{Z}^d} & \\ x & \longmapsto & \sigma^i_{\mathcal{A}}(x) & \text{such that } \sigma^i_{\mathcal{A}}(x)_u = x_{i+u}\ \forall u \in \mathbb{Z}^d \end{pmatrix}.$$

The $\mathbb{Z}^d$-action $(\mathcal{A}^{\mathbb{Z}^d}, \sigma)$ is called the *fullshift.* If $\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}$ is a closed σ-invariant subset, the $\mathbb{Z}^d$-action $(\mathbf{T}, \sigma)$ is a *subshift.*

Let F be a set of finite patterns, we define the *subshift of forbidden patterns* F by

$$\mathbf{T}_F = \left\{ x \in \mathcal{A}^{Z^d} : \forall p \in F,\ p \text{ does not appear in } x \right\}.$$

It is well known that every subshift can be defined by this way [LM95]. Let $\mathbf{T}$ be a subshift. If there exists a finite set F of forbidden patterns such that $\mathbf{T} = \mathbf{T}_F$, then $\mathbf{T}$ is a *subshift of finite type.* If there exists a recursively enumerable set F of forbidden patterns – a set of patterns that can be enumerated by a Turing machine – such that $\mathbf{T} = \mathbf{T}_F$, then $\mathbf{T}$ is an *effective subshift.*

1.2. Factor and projective subaction

Let $(\mathbf{T} \subset \mathcal{A}^{\mathbb{Z}^d}, \sigma_{\mathcal{A}})$ and $(\mathbf{T}' \subset \mathcal{B}^{\mathbb{Z}^d}, \sigma_{\mathcal{B}})$ be two d-dimensional subshifts. A *factor map* is a continuous function $\pi : \mathbf{T} \to \mathbf{T}'$ such that $\pi \circ \sigma_{\mathcal{A}} = \sigma_{\mathcal{B}} \circ \pi$. If $\mathbf{T}$ is an SFT, then $\pi(\mathbf{T}) \subset \mathcal{B}^{\mathbb{Z}}$ is a subshift called a *sofic subshift.* In dimension 1, sofic subshifts are well understood, in particular because they possess a good representation with finite automata (see [LM95] for a complete survey).

Let $\mathbb{G}$ be a sub-group of $\mathbb{Z}^d$ finitely and freely generated by $\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_{d'}}$ ($d' \leq d$). Let $\mathbf{T} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a subshift, the *projective subdynamics* – or *projective subaction* – of $\mathbf{T}$ according to $\mathbb{G}$ is the subshift of dimension d' defined by

$$\mathbf{SA}_{\mathbb{G}}(\mathbf{T}) = \{y \in \mathcal{A}^{\mathbb{Z}^{d'}} : \exists x \in \mathbf{T} \text{ such that } \forall i_1, \dots, i_{d'} \in \mathbb{Z}^{d'}, \quad y_{i_1,\dots,i_{d'}} = x_{i_1\mathbf{u_1}+\dots+i_{d'}\mathbf{u_{d'}}}\}.$$

This notion was originally introduced in [JKM07]. In [PS] the authors use this definition of projective subaction, considering $\mathbb{Z}^2$ as a lattice and restricting subshifts to $\mathbb{Z}\mathbf{e}_1$ where $\mathbf{e}_1$ is the first canonical vector of $\mathbb{Z}^d$. They show that any 1-dimensional sofic subshift with positive entropy can be obtained as the projective subaction of a 2-dimensional SFT, and give some examples of subshifts that cannot be obtained that way. But the complete characterization of projective sub-actions of 2-dimensional SFT remains an open problem. Such a complete characterization was obtained by Hochman [Hoc09] if we allow factor maps in addition to projective subactions: the class of subshifts obtained by factor maps and projective subactions of SFT is exactly the class of effective subshifts. The original proof contains a construction that realizes any 1-dimensional effective subshift inside a 3-dimensional SFT. This construction has been simultaneously improved by two different techniques [AS13, DRS10] to get any 1-dimensional effective subshift inside a 2-dimensional SFT.

Theorem 1 ([Hoc09, AS13, DRS10])**.** *Any effective subshift of dimension d can be obtained with factor and projective subaction operations from a subshift of finite type of strictly higher dimension.*

2. Substitutive and S-adic subshifts

In this section we present substitutions, substitutive subshifts and S-adic subshifts.

2.1. Substitutions

Let $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ and $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$, we define $\mathbf{n} + \mathbf{k} = (n_1 + k_1, \dots, n_d + k_d) \in \mathbb{N}^d$, $\mathbf{n} \otimes \mathbf{k} = (n_1.k_1, \dots, n_d.k_d) \in \mathbb{N}^d$ and $\mathbf{n}^i = \mathbf{n} \otimes \dots \otimes \mathbf{n}$ with i factors. Given $\mathbf{k} = (k_1, \dots, k_d)$, we denote by $\mathbb{U}_{\mathbf{k}}$ the rectangle $[0; k_1] \times [0; k_2] \times \dots \times [0; k_d]$. We say that $\mathbf{i}$ is smaller (resp. strictly smaller) than $\mathbf{j}$ if for every $1 \leq l \leq d$, one has $i_l \leq j_l$ (resp. $i_l < j_l$). We denote it by $\mathbf{i} \leq \mathbf{j}$ (resp. $\mathbf{i} < \mathbf{j}$).

Let $\mathcal{A}$ be a finite alphabet, we define the *set of rectangular pattern* $\mathcal{P} = \bigcup_{\mathbf{k}\in\mathbb{N}^d} \mathcal{A}^{\mathbb{U}_\mathbf{k}}$. An $(\mathcal{A},d)$-*multidimensional substitution* of *size* $\mathbf{k}^{(s)} : \mathcal{A} \to \mathbb{N}^d$ is a function $s : \mathcal{A} \to \mathcal{P}$, such that for all $a \in \mathcal{A}$, we have $\mathrm{supp}(s(a)) = \mathbb{U}_{\mathbf{k}^{(s)}(a)}$ with $\mathbf{k}^{(s)}(a) = (k_1^s(a), \dots, k_d^s(a))$. An $(\mathcal{A},d)$-multidimensional substitution is non degenerate if $\mathbf{k}_l^{(s)}(a) \geq 1$ for every $l \in [1;d]$ and every $a \in \mathcal{A}$.

Let $p \in \mathcal{A}^{\mathbb{U}_\mathbf{k}}$ be a rectangular pattern with finite support $\mathbb{U}_\mathbf{k} \subset \mathbb{Z}^2$. We would like to apply a substitution s on this rectangular pattern p so that the result is also a rectangular pattern. Consider a bidimensional substitution, and suppose it has constant size, that is to say the function $\mathbf{k}^{(s)}$ is constant or equivalently the support of the images of a letter by the substitution does not depend on the letter. Take a partitioning of the support $\mathbb{U}_\mathbf{k}$ with unit squares, and apply the linear transformation that inflates each unit square by (λ_1, λ_2), you will obtain another partitioning of a bigger support $\mathbb{U}_{(\lambda_1\cdot k_1, \lambda_2\cdot k_2)}$ by rectangles of size (k_1, k_2), so that if two unit squares share an edge (resp. a vertex), then so do their inflated rectangles.

But if the substitution does not have constant size, the situation is more complicated since some overlaps or holes may appear. We would like to only consider substitutions applied on patterns that do not create such degenerate cases, which corresponds to the following notion of compatibility. We say that the substitution s is *compatible* with the pattern p (resp. with the configuration x) if for all $\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{U}_\mathbf{k}$ and $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{U}_\mathbf{k}$ (resp. $\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d$ and $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{Z}^d$) such that $i_l = j_l$ for one $l \in [1;d]$, one has $k_l^s(p_\mathbf{i}) = k_l^s(p_\mathbf{j})$.

Example 1. Let $\mathcal{A}$ be the two elements alphabet $\mathcal{A} = \{\circ, \bullet\}$ and s be the two-dimensional substitution whose rules are

$$\circ \mapsto \begin{array}{cc}\circ & \circ\\ \circ & \circ\end{array} \quad \text{and} \quad \bullet \mapsto \begin{array}{cc}\circ & \circ\\ \bullet & \circ\end{array}.$$

For instance, the substitution s acts on the pattern p described below

$$s: \quad p = \begin{array}{ccc}\circ & \bullet & \bullet\\ \bullet & \circ & \circ\end{array} \quad \mapsto \quad s(p) = \begin{array}{cc|cc|cc}\circ & \circ & \circ & \circ & \circ & \circ\\ \circ & \circ & \bullet & \circ & \bullet & \circ\\ \hline \circ & \circ & \circ & \circ & \circ & \circ\\ \bullet & \circ & \circ & \circ & \circ & \circ\end{array}.$$

Consider now the substitution s' whose rules are

$$\circ \mapsto \begin{array}{cc}\circ & \circ\\ \circ & \circ\end{array} \quad \text{and} \quad \bullet \mapsto \begin{array}{ccc}\circ & \circ & \bullet\\ \bullet & \circ & \circ\end{array}.$$

The substitution s' is not compatible with the pattern p since the pattern $s'(p)$ is not a rectangular pattern – it contains holes.

$$s'(p) = \begin{array}{ccc|ccc|ccc} \circ & \circ & ? & \circ & \circ & \bullet & \circ & \circ & \bullet \\ \circ & \circ & ? & \bullet & \circ & \circ & \bullet & \circ & \circ \\ \hline \circ & \circ & \bullet & \circ & \circ & ? & \circ & \circ & ? \\ \bullet & \circ & \circ & \circ & \circ & ? & \circ & \circ & ? \end{array} .$$

However, the substitution s' acts on the pattern p' described below

$$s' : \quad p' = \begin{array}{cc} \bullet & \circ \\ \bullet & \circ \end{array} \quad \mapsto \quad s(p) = \begin{array}{ccc|cc} \circ & \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ & \circ \\ \hline \circ & \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ & \circ \end{array} .$$

The reader may have noticed that the notion of compatibility defined above imposes actually more constraints on the partition by rectangles than just avoiding holes and overlaps. We say that a partition of $\mathbb{Z}^d$ – or a finite region of $\mathbb{Z}^d$ – with rectangles is a *rigid partition* if rectangles are edge-to-edge, that is to say the vertex of a rectangle can only intersect another rectangle at a vertex. By definition, if you have a rigid partition of $\mathbb{Z}^d$ by rectangles, it suffices to know rectangles along a diagonal to deduce the whole partition (see Figure 1).

Suppose now that the substitution s is compatible with the configuration x. Assume that the pattern $s(x_{(0,\dots,0)})$ appears in position $(0,\dots,0)$ in $s(x)$, is it possible to deduce the positions of the patterns $s(x_{\mathbf{i}})$ in $s(x)$ for every $\mathbf{i} \in \mathbb{Z}^d$? For a given $\mathbf{i} = (i_1,\dots,i_d) \in \mathbb{Z}^d$, this position depends on the sequence of positions of all the patterns $s(x_{(j_1,j_2,\dots,j_d)})$, for j_ℓ between 0 and k_ℓ, hence we can define it recursively. This is the goal of following function φ, and which uses the fact that a rigid partition is entirely determined by one of its diagonals.

Let $(\mathbf{k}^{(n)})_{n\in\mathbb{Z}}$ be a sequence of d-dimensional vectors with entries in the naturals. We define the function

$$\varphi^{(\mathbf{k}^{(n)})_{n\in\mathbb{Z}}} : \begin{pmatrix} \mathbb{Z}^d & \to & \mathbb{Z}^d \\ \mathbf{i} & \mapsto & (\varphi_1(i_1), \varphi_2(i_2), \dots, \varphi_d(i_d)) \end{pmatrix}$$

where $\varphi_l(0) = 0$, $\varphi_l(r) = \sum_{j=0}^{r-1} (\mathbf{k}^{(j)})_l$ if $r \geq 0$ and $\varphi_l(r) = \sum_{j=r}^{-1} (\mathbf{k}^{(j)})_l$ if $r < 0$ for every $l \in [1; d]$. This function $\varphi^{(\mathbf{k}^{(n)})_{n\in\mathbb{Z}}}$ provides a way to distort the grid $\mathbb{Z}^d$ in order to obtain a rigid partition of $\mathbb{Z}^d$ with rectangles (see Figure 1).

Given a substitution s compatible with a configuration $x \in \mathbb{Z}^d$, we can now describe the new configuration $s(x)$ thanks to the auxiliary function φ (see Figure 2). Define

$$\phi^{(x,s)} = \varphi^{(\mathbf{k}^{(s)}(x_{(n,\dots,n)}))_{n\in\mathbb{Z}}}.$$

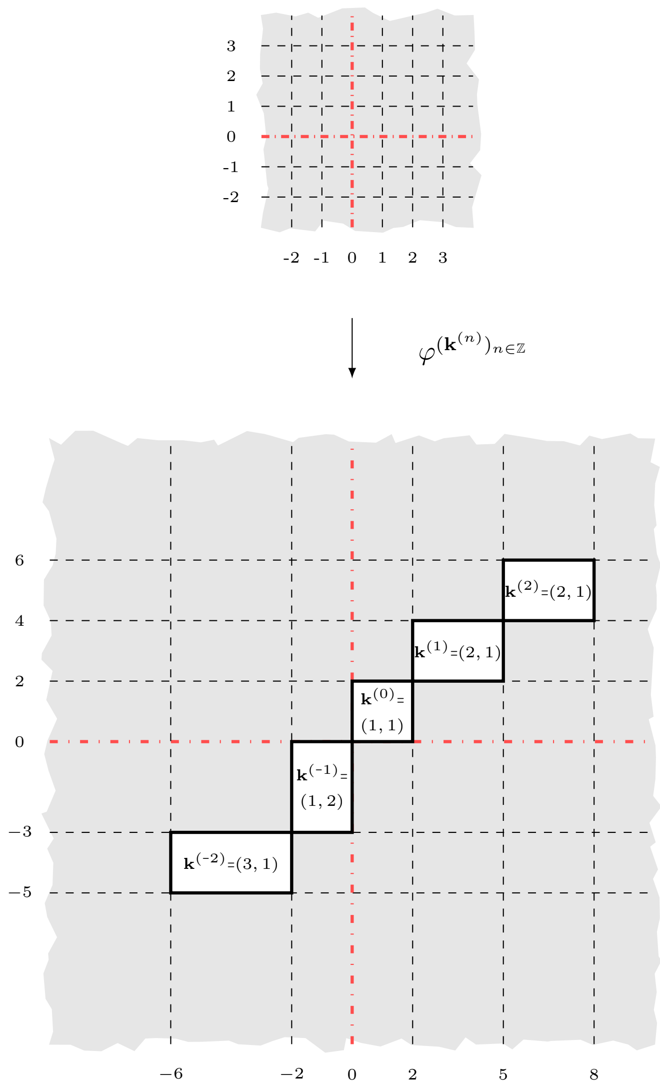

FIGURE 1. An example of sequence $(\mathbf{k}^{(n)})_{n\in\mathbb{Z}}$ of 2-dimensional vectors with entries in the naturals and the function $\varphi^{(\mathbf{k}^{(n)})_{n\in\mathbb{Z}}}$ that define a rigid partition of $\mathbb{Z}^2$. The sequence $(\mathbf{k}^{(n)})_{n\in\mathbb{Z}}$ represents the sizes of the rectangles of the rigid partition.

If the substitution s is compatible with the configuration x and if p is a pattern of x, the substitution s acts on p and we obtain a pattern $s(p)$ whose support is

$$\mathrm{supp}(s(p)) = \bigcup_{\mathbf{i}\in\mathrm{supp}(p)} \mathbb{U}_{\mathbf{k}^{(s)}(p_{\mathbf{i}})} + \phi^{(x,s)}(\mathbf{i})$$

and such that

$$\forall \mathbf{i} \in \mathrm{supp}(p), \forall \mathbf{j} \in \mathrm{supp}(s(p_{\mathbf{i}})), s(p)_{\phi^{(x,s)}(\mathbf{i})+\mathbf{j}} = s(p_{\mathbf{i}})_{\mathbf{j}}.$$

So the substitution s can easily be extended to a function on configurations $s_\infty : \begin{pmatrix} \mathcal{A}^{\mathbb{Z}^d} & \to & \mathcal{A}^{\mathbb{Z}^d} \\ x & \mapsto & s(x) \end{pmatrix}$ such that if the substitution s is compatible with the configuration $x \in \mathcal{A}^{\mathbb{Z}^d}$, then the configuration $s_\infty(x)$ is defined by $s(p)_{\phi^{(x,s)}(\mathbf{i})+\mathbf{j}} = s(p_{\mathbf{i}})_{\mathbf{j}}$ for all $\mathbf{i} \in \mathbb{Z}^d$ and $\mathbf{j} \in \mathrm{supp}(s(x_{\mathbf{i}}))$.

Auxiliary function ϕ can also be seen as a way to express how substitutions commute in a certain sense with the shift σ. This is expressed by Proposition 2 and illustrated in Figure 2.

Proposition 2. *One has $s \circ \sigma^{\mathbf{i}}(x) = \sigma^{\phi^{(x,s)}(\mathbf{i})} \circ s(x)$ for all $\mathbf{i} \in \mathbb{Z}^d$.*

Proof. Let $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$. By definition of ϕ, one has the two following properties

- there exist $\mathbf{j}', \mathbf{j}'' \in \mathbb{Z}^d$ such that $\mathbf{j} = \phi^{(x,s)}(\mathbf{j}') + \mathbf{j}''$ with $\mathbf{j}''_\ell \in [0, \phi^{(x,s)}(\mathbf{j}' + \mathbf{e}_\ell)_\ell - \phi^{(x,s)}(\mathbf{j}')_\ell - 1]$, where $\mathbf{e}_\ell$ is the ℓth canonical vector of $\mathbb{Z}^d$, and $\|\mathbf{j}'\| \leq \|\mathbf{j}\|$;
- $\phi^{(x,s)}(\mathbf{i} + \mathbf{j}) = \phi^{(x,s)}(\mathbf{i}) + \phi^{(\sigma^{\mathbf{i}}(x),s)}(\mathbf{j})$.

Then one has (see Figure 2 for an example)

$$\left[s \circ \sigma^{\mathbf{i}}(x)\right]_{\mathbf{j}} = \left[s(\sigma^{\mathbf{i}}(x))\right]_{\phi^{(\sigma^{\mathbf{i}}(x),s)}(\mathbf{j}')+\mathbf{j}''} = \left[s(x_{\mathbf{i}+\mathbf{j}'})\right]_{\mathbf{j}''}$$

with $\mathbf{j} = \phi^{(\sigma^{\mathbf{i}}(x),s)}(\mathbf{j}') + \mathbf{j}''$ and $\mathbf{j}''_\ell \in [0, \phi^{(\sigma^{\mathbf{i}}(x),s)}(\mathbf{j}' + \mathbf{e}_\ell)_\ell - \phi^{(\sigma^{\mathbf{i}}(x),s)}(\mathbf{j}')_\ell - 1]$ for all $\ell \in [1, d]$. Thus

$$\begin{aligned}\left[s \circ \sigma^{\mathbf{i}}(x)\right]_{\mathbf{j}} &= [s(x)]_{\phi^{(x,s)}(\mathbf{i}+\mathbf{j}')+\mathbf{j}''} = [s(x)]_{\phi^{(x,s)}(\mathbf{i})+\phi^{(\sigma^{\mathbf{i}}(x),s)}(\mathbf{j}')+\mathbf{j}''} \\ &= \left[\sigma^{\phi^{(x,s)}(\mathbf{i})} \circ s(x)\right]_{\mathbf{j}}.\end{aligned}$$

□

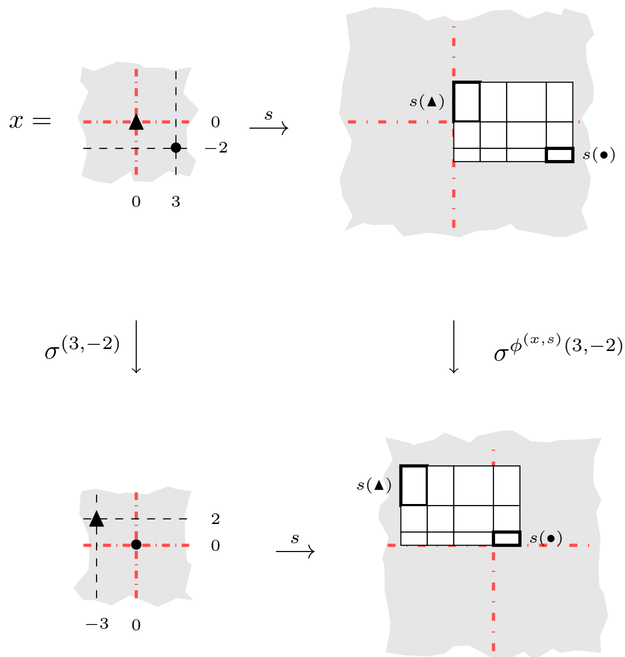

FIGURE 2. If the configuration x is compatible with the substitution s, then one can define the function $\phi^{(x,s)} = \varphi^{(\mathbf{k}^{(s)}(x_{(n,\dots,n)}))_{n\in\mathbb{Z}}}$ which verifies $s \circ \sigma^{\mathbf{i}}(x) = \sigma^{\phi^{(x,s)}(\mathbf{i})} \circ s(x)$.

2.2. Composition of substitutions

Consider now that one wants to apply not only one but a finite set of substitutions on a (finite or infinite) pattern p. We first define how to compute the composition of two or more substitutions. Let s, s' be two substitutions. We say that s' is *compatible* with s if for any letter a, the pattern $s(a)$ is compatible with s'. If s' is compatible with s, we can thus define the *composition* $s' \circ s$ such that the image of a letter a by $s' \circ s$ is the pattern $s'(s(a))$. For a sequence of substitutions $S_{[k;n]} = (s_k, \dots, s_n)$, one defines by induction the substitution $\widehat{\mathrm{S}}_{[k;n]}$:

- $\widehat{\mathrm{S}}_{[n;n]} = s_n$;
- $\widehat{\mathrm{S}}_{[k;n]} = s_k \circ \widehat{\mathrm{S}}_{[k+1;n]}$ if $k < n$ and s_k is compatible with $\widehat{\mathrm{S}}_{[k+1;n]}(a)$ for all $a \in \mathcal{A}$.

Note that with this definition, substitutions are applied by decreasing index – substitution s_0 is actually the last to be applied in $\widehat{\mathrm{S}}_{[0;n]}$. This reverse order could seem a bit surprising at first sight, but doing this we ensure that the sequence of finite patterns $\widehat{\mathrm{S}}_{[0;n]}(a)$ will converge for every letter $a \in \mathcal{A}$.

Let $\mathscr{S}$ be a finite set of $(\mathcal{A}, d)$-multidimensional substitutions. We present the two classical points of view to make $\mathscr{S}$ act on the set of configurations $\mathcal{A}^{\mathbb{Z}^d}$. In the first one, the set $\mathscr{S}$ acts on a configuration x via a sequence of substitutions $\mathrm{S} = (s_i)_{i\in\mathbb{N}} \in \mathscr{S}^{\mathbb{N}}$, and at iteration i the substitution s_i is applied to every letter in x (see Section 2.3). In the second one, the set $\mathscr{S}$ acts on a configuration x in a non uniform way, that is to say at each iteration the applied substitution depends on the position in x (see Section 2.4).

2.3. S-adic subshifts

Let $\mathscr{S}$ be a finite set of $(\mathcal{A}, d)$-multidimensional substitutions and let $\mathrm{S} \in \mathscr{S}^{\mathbb{N}}$ be a sequence of substitutions. We want to define how this sequence acts on a letter $a \in \mathcal{A}$. The principle is that at in the i-th iteration, the substitution s_0 is applied to the whole pattern $s_1 \circ \cdots \circ s_i(a)$. We define the following two S-adic subshifts based on this action of S on letters of $\mathcal{A}$

$$\begin{aligned}
\mathbf{T}_{\mathrm{S}} &= \left\{x \in \mathcal{A}^{\mathbb{Z}^d} : \forall p \sqsubset x,\ \exists a \in \mathcal{A}, \exists n \in \mathbb{N}, \quad p \sqsubset \widehat{\mathrm{S}}_{[0;n]}(a)\right\}\\
\mathbf{T}'_{\mathrm{S}} &= \left\{x \in \mathcal{A}^{\mathbb{Z}^d} : \forall n \in \mathbb{N},\ \exists y \in \mathcal{A}^{\mathbb{Z}^d}, \exists \mathbf{i} \in \mathbb{Z}^d, \quad \widehat{\mathrm{S}}_{[0;n]}(y) = \sigma^{\mathbf{i}}(x)\right\}.
\end{aligned}$$

The first subshift $\mathbf{T}_{\mathrm{S}}$ will be called the *sub-pattern* S*-adic subshift*. The set $\mathbf{T}_{\mathrm{S}}$ corresponds to a symbolic dynamics approach, since it is defined in terms of allowed patterns. The sequence of substitutions S produces patterns, that are the $\widehat{\mathrm{S}}_{[0;n]}(a)$ for any letter $a \in \mathcal{A}$, and these patterns are seen as the allowed patterns of the subshift $\mathbf{T}_{\mathrm{S}}$ – the fact that $\mathbf{T}_{\mathrm{S}}$ is a subshift is obvious. The second subshift $\mathbf{T}'_{\mathrm{S}}$ will be called the *limit* S*-adic subshift*, and this time the idea is to consider only configurations $x \in \mathcal{A}^{\mathbb{Z}^2}$ for which it is possible to find a pre-image of any order under the sequence of substitutions S. The study of $\mathbf{T}'_{\mathrm{S}}$ can be justified under a dynamical point of view: it can be seen as the shift closure of the limit set

$$\bigcap_{n\in\mathbb{N}} \widehat{\mathrm{S}}_{[0;n]}\left(\mathcal{A}^{\mathbb{Z}^d}\right).$$

Note that writing the set $\mathbf{T}'_{\mathrm{S}}$ as below gives a direct proof that it is a subshift – closed and shift-invariant.

These two subshifts $\mathbf{T}_{\mathrm{S}}$ and $\mathbf{T}'_{\mathrm{S}}$ are actually almost the same. First notice that the inclusion $\mathbf{T}_{\mathrm{S}} \subset \mathbf{T}'_{\mathrm{S}}$ always holds. Take some configuration x in $\mathbf{T}_{\mathrm{S}}$. Then by compactness, for every integer $n \in \mathbb{N}$, one can construct some configuration y such that $x = \widehat{\mathrm{S}}_{[0;n]}(y)$.

Moreover it can be proven, but we will not do it here since it is not the goal of this article, that if the substitutions are all *primitive* – every letter will eventually appear in the pattern created by iteration of any substitution on any letter – the two subshifts are equal. If one substitution in $\mathscr{S}$ is not primitive, one can easily construct example in which the reciprocal inclusion does not hold (see Example 2). And even in the general case, the two subshifts do not differ that much: the set $\mathbf{T}'_{\mathrm{S}} \setminus \mathbf{T}_{\mathrm{S}}$ is always countable – again we do not give the proof here.

Example 2. Let s be the first substitution of Example 1. Then if we choose $\mathscr{S} = \{s\}$ and so $\mathrm{S} = s^{\mathbb{N}}$, the two S-adic subshifts defined above are in this case

$$\mathbf{T}_{\mathrm{S}} = \left\{\circ^{\mathbb{Z}^2}\right\} \quad \text{and} \quad \mathbf{T}'_{\mathrm{S}} = \left\{\circ^{\mathbb{Z}^2}\right\} \cup \left\{\sigma^i(x_\bullet), i \in \mathbb{Z}^2\right\}$$

where the configuration $x_\bullet$ is such that $x_{(i,j)} = \bullet$ if and only if $(i,j) = (0,0)$. Indeed, $x_\bullet$ is in the subshift $\mathbf{T}'_{\mathrm{S}}$ – $x_\bullet$ is a fixed-point of s – but not in the subshift $\mathbf{T}_{\mathrm{S}}$, since the central pattern $x_{[-1;0]\times[-1;0]}$ appears neither in a $\widehat{\mathrm{S}}_{[0;n]}(\bullet)$ nor in a $\widehat{\mathrm{S}}_{[0;n]}(\circ)$.

2.4. Non-deterministic substitutions

Let $\mathscr{S}$ be a finite set of d-dimensional substitutions on alphabet $\mathcal{A}$, and $x \in \mathcal{A}^{\mathbb{Z}^d}$ be a configuration. In the previous section we have seen that one substitution $s \in \mathscr{S}$ can be applied on x – provided s is compatible with x – to get a new configuration $s(x) \in \mathcal{A}^{\mathbb{Z}^d}$. With this formalism, the same substitution s is applied to any letter a that appears in x, so that we could roughly speaking say that the set of substitutions $\mathscr{S}$ acts in a uniform way on configurations. But we could also imagine that different substitutions are applied to letters, depending on the position of the letters. In other words, some substitution $s_i \in \mathscr{S}$ is applied on the letter x_i, which means that we do not apply one substitution on x, but a configuration of substitutions living in $\mathscr{S}^{\mathbb{Z}^d}$.

For a finite set $\mathbb{U} \subset \mathbb{Z}^d$, we consider the pattern of substitutions $\mathbf{s} \in \mathscr{S}^{\mathbb{U}}$. We say that the pattern of substitutions $\mathbf{s} \in \mathscr{S}^{\mathbb{U}}$ is *compatible* with a pattern $p \in \mathcal{A}^{\mathbb{U}}$ if for all $i = (i_1, \dots, i_d) \in \mathbb{U}$ and $j = (j_1, \dots, j_d) \in \mathbb{U}$ such that $i_l = j_l$ for one $l \in [1; d]$, one has $k_l^{\mathbf{s}_i}(p_i) = k_l^{\mathbf{s}_j}(p_j)$. Compatibility thus means that the pattern of substitutions $\mathbf{s}$ transforms p into a rigid partition of some finite rectangular region of $\mathbb{Z}^d$ with rectangles $s_i(p_i)$ for $i \in \mathbb{U}$.

If the pattern of substitutions $\mathbf{s} \in \mathscr{S}^{\mathbb{U}_\mathbf{k}}$ is compatible with a pattern $p \in \mathcal{A}^{\mathbb{U}_\mathbf{k}}$ that appears in a configuration x, it acts on p and we obtain the pattern $\mathbf{s}(p)$

- whose support is $\mathrm{supp}(\mathbf{s}(p)) = \bigcup_{i \in \mathrm{supp}(p)} \mathbb{U}_{\mathbf{k}^{\mathbf{s}_i(p_\mathbf{i})}} + \phi^{(x,s)}(\mathbf{i})$, since each letter p_i generates a patterns with support $\mathbb{U}_{\mathbf{k}^{\mathbf{s}_i(p_\mathbf{i})}}$ shifted by $\phi^{(x,s)}(\mathbf{i})$;
- and such that $\forall \mathbf{i} \in \mathrm{supp}(p), \forall \mathbf{j} \in \mathrm{supp}(\mathbf{s}_i(p_\mathbf{i})), \mathbf{s}(p)_{\phi^{(x,s)}(\mathbf{i})+\mathbf{j}} = \mathbf{s}_i(p_\mathbf{i})_\mathbf{j}$ (see Figure 3).

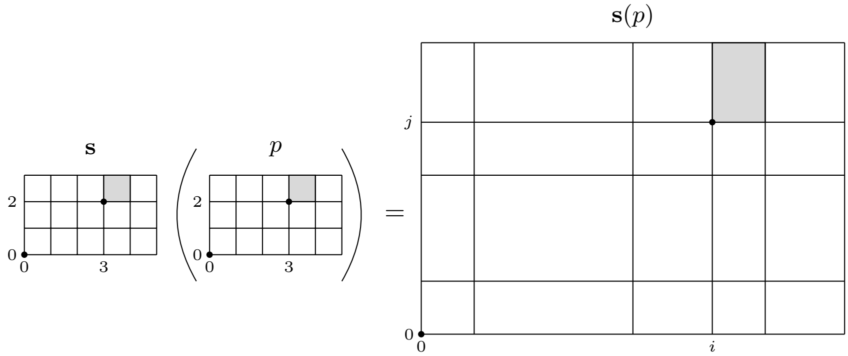

FIGURE 3. A pattern of substitutions $\mathbf{s}$ compatible with a pattern p that appears in position $(0,0)$ in some configuration x. On the picture, coordinates i and j are defined by $\phi^{(x,s)((3,2))} = (i,j)$.

EXAMPLE 3. Let $\mathscr{S} = \{s_a, s_b, s_c, s_d\}$ be a set of two-dimensional substitutions on the alphabet $\mathcal{A} = \{\circ, \bullet\}$ defined by the following rules

$$s_a : \circ \mapsto \begin{array}{cc} \circ & \circ \\ \circ & \circ \end{array} \qquad s_b : \circ \mapsto \begin{array}{ccc} \circ & \bullet & \circ \\ \circ & \bullet & \circ \end{array}$$

$$\bullet \mapsto \begin{array}{cc} \circ & \circ \\ \bullet & \circ \end{array} \qquad \bullet \mapsto \begin{array}{ccc} \circ & \circ & \circ \\ \bullet & \circ & \circ \end{array}$$

$$s_c : \circ \mapsto \begin{array}{cc} \circ & \circ \\ \bullet & \circ \\ \circ & \bullet \end{array} \qquad s_d : \circ \mapsto \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \circ & \circ & \circ \end{array}$$

$$\bullet \mapsto \begin{array}{cc} \circ & \circ \\ \circ & \bullet \\ \bullet & \bullet \end{array} \qquad \bullet \mapsto \begin{array}{ccc} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \bullet & \bullet & \bullet \end{array}.$$

Then given the pattern p pictured below, the patterns of substitutions $\mathbf{s}$ and $\mathbf{s}'$ are compatible with p and we can define the patterns $\mathbf{s}(p)$ and $\mathbf{s}'(p)$, while the pattern of substitutions $\mathbf{s}''$ is not – on the bottom right $s_d(\circ)$ is of height 3 while $s_a(\circ)$ is of height 2.

$$p = \begin{array}{cccc} \circ & \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \end{array},$$

$$\mathbf{s} = \begin{array}{cccc} s_a & s_a & s_b & s_a \\ s_c & s_c & s_d & s_c \end{array}, \ \mathbf{s}' = \begin{array}{cccc} s_c & s_d & s_d & s_c \\ s_a & s_b & s_b & s_a \end{array}, \ \mathbf{s}'' = \begin{array}{cccc} s_a & s_a & s_b & s_a \\ s_c & s_c & s_d & s_a \end{array},$$

$$\mathbf{s}(p) = \begin{array}{cc|cc|ccc|cc} \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \bullet & \circ & \bullet & \circ & \circ & \bullet & \circ \\ \hline \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet \end{array}$$

$$\mathbf{s}'(p) = \begin{array}{cc|ccc|ccc|cc} \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \bullet & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \bullet \\ \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \circ & \circ & \circ & \circ & \circ & \circ & \bullet & \circ & \circ & \circ \\ \bullet & \circ & \bullet & \circ & \circ & \circ & \bullet & \circ & \circ & \circ \end{array}.$$

We define the set of *$\mathscr{S}$-patterns* by induction. An $\mathscr{S}$-pattern of level 0 is an element of $\mathcal{A}$, and p is an $\mathscr{S}$-pattern of level $n+1$ if there exists an $\mathscr{S}$-pattern $p' \in \mathcal{A}^{\mathbb{U}}$ of level n and a pattern of substitutions $\mathbf{s} \in \mathscr{S}^{\mathbb{U}}$ compatible with p' such that $\mathbf{s}(p') = p$. Of course the support of each $\mathscr{S}$-pattern is rectangular. The $\mathscr{S}$-patterns lead us to define $\mathbf{T}_{\mathscr{S}}$, the *sub-pattern subshift generated by the set of substitutions $\mathscr{S}$*

$$\mathbf{T}_{\mathscr{S}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall p \sqsubset x,\ p \text{ is a sub-pattern of an } \mathscr{S}\text{-pattern} \right\}.$$

Suppose that $\mathbf{s} \in \mathscr{S}^{\mathbb{Z}^d}$ is an infinite pattern of substitutions and $x \in \mathcal{A}^{\mathbb{Z}^d}$ is a configuration. We denote by $\mathbf{s}(x)$ the configuration in $\mathcal{A}^{\mathbb{Z}^d}$ obtained if one applies $s_{\mathbf{i}}$ on $x_{\mathbf{i}}$ for every $\mathbf{i} \in \mathbb{Z}^d$. We thus define $\mathbf{T}'_{\mathscr{S}}$ the *limit subshift generated by the set of substitutions $\mathscr{S}$* as follows

$$\mathbf{T}'_{\mathscr{S}} = \Big\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall n \in \mathbb{N}, \exists y \in \mathcal{A}^{\mathbb{Z}^d}, \exists (\mathbf{s}_0, \dots, \mathbf{s}_{n-1}) \in \left(\mathscr{S}^{\mathbb{Z}^d}\right)^n, \\ \exists \mathbf{i} \in \mathbb{Z}^d, \mathbf{s}_0 \circ \cdots \circ \mathbf{s}_{n-1}(y) = \sigma^{\mathbf{i}}(x) \Big\}.$$

This subset of configurations is obviously shift-invariant, and writing it as a limit set as we did for $\mathbf{T}'_{\mathrm{S}}$ provides a simple proof that it is also closed.

Remark. A compactness argument leads to the inclusion $\mathbf{T}_{\mathscr{S}} \subseteq \mathbf{T}'_{\mathscr{S}}$, one also has $\mathbf{T}_{\mathrm{S}} \subseteq \mathbf{T}_{\mathscr{S}}$ and $\mathbf{T}'_{\mathrm{S}} \subseteq \mathbf{T}'_{\mathscr{S}}$ for any sequence S – it corresponds to the particular case where patterns of substitutions $\mathbf{s}_i$ contain a single substitution s_i.

2.5. Subshift generated by a set of S-adic sequences

Let $\boldsymbol{\mathfrak{S}} \subset \mathscr{S}^{\mathbb{N}}$ be a subset of S-adic sequences. We are interested in the sets

$$\mathbf{T}_{\boldsymbol{\mathfrak{S}}} = \bigcup_{\mathrm{S} \in \boldsymbol{\mathfrak{S}}} \mathbf{T}_{\mathrm{S}} \text{ and } \mathbf{T}'_{\boldsymbol{\mathfrak{S}}} = \bigcup_{\mathrm{S} \in \boldsymbol{\mathfrak{S}}} \mathbf{T}'_{\mathrm{S}}.$$

These sets are shift-invariant. We can consider the product topology on $\mathscr{S}^{\mathbb{N}}$, for this topology $\mathscr{S}^{\mathbb{N}}$ is compact. In the following proposition, we show that if $\boldsymbol{\mathfrak{S}} \subset \mathscr{S}^{\mathbb{N}}$ is closed then $\mathbf{T}_{\boldsymbol{\mathfrak{S}}}$ and $\mathbf{T}'_{\boldsymbol{\mathfrak{S}}}$ are closed. In this case $\mathbf{T}_{\boldsymbol{\mathfrak{S}}}$ and $\mathbf{T}'_{\boldsymbol{\mathfrak{S}}}$ are called respectively the *sub-pattern* $\boldsymbol{\mathfrak{S}}$*-adic subshift* and the *limit* $\boldsymbol{\mathfrak{S}}$*-adic subshift*.

Proposition 3. *Let $\boldsymbol{\mathfrak{S}} \subset \mathscr{S}^{\mathbb{N}}$ be a closed set. Then $\mathbf{T}_{\boldsymbol{\mathfrak{S}}}$ and $\mathbf{T}'_{\boldsymbol{\mathfrak{S}}}$ are subshifts.*

Proof. The proof is only realized for $\mathbf{T}_{\boldsymbol{\mathfrak{S}}}$, the same arguments hold for $\mathbf{T}'_{\boldsymbol{\mathfrak{S}}}$. Let $(x^i)_{i\in\mathbb{N}}$ be a sequence of element of $\mathbf{T}_{\boldsymbol{\mathfrak{S}}}$ which converges toward $x \in \mathcal{A}^{\mathbb{Z}^d}$, it is possible to assume that $x^i_{[-i,i]^d} = x_{[-i,i]^d}$ for all $i \in \mathbb{N}$. We want to show that $x \in \mathbf{T}_{\boldsymbol{\mathfrak{S}}}$. By definition, for all $i \in \mathbb{N}$, $x^i \in \mathbf{T}_{\mathrm{S}_i}$ where $\mathrm{S}_i = s^i_0 s^i_1 \cdots s^i_n \cdots \in \boldsymbol{\mathfrak{S}}$.

By compactness of $\boldsymbol{\mathfrak{S}}$, the sequence $(\mathrm{S}_i)_{i\in\mathbb{N}}$ admits an adherence value $\mathrm{S} = s_0 s_1 \cdots s_n \cdots \in \boldsymbol{\mathfrak{S}}$. Leaving to take a subsequence, it is possible to assume that $s^i_0 s^i_1 \cdots s^i_i = s_0 s_1 \cdots s_i = \widehat{\mathrm{S}}_{[0;i]}$ for all $i \in \mathbb{N}$. Thus for $i \in \mathbb{N}$, every sub-pattern $p \sqsubset x_{[-i,i]^d} = x^i_{[-i,i]^d}$ appears as sub-pattern of $\widehat{\mathrm{S}}_{[0;i]}(a)$ for $a \in \mathcal{A}$. We deduce that $x \in \mathbf{T}_{\mathrm{S}} \subset \mathbf{T}_{\boldsymbol{\mathfrak{S}}}$. $\square$

A set $\boldsymbol{\mathfrak{S}} \subset \mathscr{S}^{\mathbb{N}}$ is *effectively closed* if there is a sequence of words $(w_n)_{n\in\mathbb{N}}$ on the alphabet $\mathscr{S}$ enumerated by a Turing machine such that $\mathrm{S} \in \boldsymbol{\mathfrak{S}}$ if and only if $\widehat{\mathrm{S}}_{[0,|w_n|-1]} \neq w_n$ for all $n \in \mathbb{N}$. This means that there exists an algorithm with the following behavior: it loops forever on a sequence that belongs to the effectively closed set, but ends in finite time on other sequences. This definition is standard in recursive analysis [Wei00]. Note that an effectively closed set of sequences of substitutions my contain some non computable sequences.

If $\boldsymbol{\mathfrak{S}} = \{\mathrm{S}\}$ is a singleton, then $\boldsymbol{\mathfrak{S}}$ is effectively closed if and only if S is an effective sequence.

3. Realization by sofic subshifts

We prove that multidimensional S-adic subshifts given by an effective sequence of substitutions are sofic.

3.1. Mozes' theorem and property A

In [Moz89] Mozes studied non deterministic multidimensional substitutions, and proved that provided a non deterministic substitution satisfies a good property – called property A, defined below – then the subshift generated is sofic.

All substitutions we consider here are deterministic since the substitutions rules are given by a function. Nevertheless this formalism provides a way to study non deterministic substitutions. Given s a non deterministic substitution, if a letter $a \in \mathcal{A}$ has two patterns p_1, p_2 as images, one replaces s by s_1 and s_2, where s_1 has the same substitution rules as s without the rule $a \to p_2$, and s_2 has the same substitutions rules as s without the rule $a \to p_1$. By iterating this process, we can transform a non deterministic substitution into a set finite $\mathscr{S}$ of deterministic substitutions, so that the subshift called $(\Omega, \mathbb{Z}^2)$ by Mozes is exactly the subshift $\mathbf{T}_{\mathscr{S}}$.

We say that a set of substitutions $\mathscr{S}$ is *of type A*, or *has property A*, if it satisfies the following condition. Let $p = \mathbf{u}_0 \circ \cdots \circ \mathbf{u}_k(a)$ be an $\mathscr{S}$-pattern, where pattern substitutions $\mathbf{u}_i$ are chosen among $\mathscr{S}$, and l a $2 \times \cdots \times 2$ pattern that appears in p. Suppose there exists a sequence of patterns of substitutions $\mathbf{s}_0, \ldots, \mathbf{s}_n$ compatible with the $2 \times \cdots \times 2$ pattern l that produce a sequence of patterns $l_0 = l, l_1 = \mathbf{s}_0(l_0), \ldots, l_n = \mathbf{s}_n(l_{n-1})$. Then it is possible to find a sequence of patterns of substitutions $\mathbf{s}'_0, \ldots, \mathbf{s}'_n$ compatible with the pattern p such that the blocks that derive from l in $p_0 = p, p_1 = \mathbf{s}'_0(p_0), \ldots, p_n = \mathbf{s}'_n(p_{n-1})$ are exactly $l_0, l_1, \ldots, l_n$ (see Figure 4). It is possible that the composition of substitution rules chosen for l is not compatible with the pattern p, and in this case it has to be possible to find another sequence of substitution rules compatible with p and such that the the blocks that derive from l are exactly the $l_0 = l, l_1, \ldots, l_n$.

Remark. This property A for sets of substitutions is actually not very restrictive. For instance any set of substitutions $\mathscr{S}$ such that for every substitution $s \in \mathscr{S}$, the support of $s(a)$ is the same for any $a \in \mathcal{A}$, has the property A. Moreover, if the set $\mathscr{S}$ is reduced to a single deterministic substitution s, then $\mathscr{S}$ is of type A.

Theorem 4 ([Moz89]). *Let $\mathscr{S}$ be a set of non degenerate deterministic multidimensional substitutions – all letters are mapped to a pattern of size at least 2 in all directions – that possesses property A. Then the subshift $\mathbf{T}_{\mathscr{S}}$ is a sofic.*

In the sequel results are proven for the subshift $\mathbf{T}_{\mathscr{S}}$ only, but remain the same for the subshift $\mathbf{T}'_{\mathscr{S}}$ if we admit that Mozes' result generalizes to $\mathbf{T}'_{\mathscr{S}}$. Proof of Theorem 4 can actually be adapted, without new ideas, to get almost the same

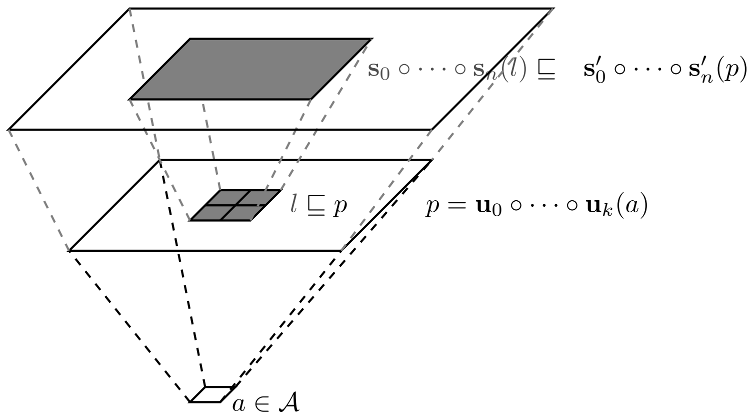

FIGURE 4. A set of substitutions $\mathscr{S}$ having property A.

result for $\mathbf{T}'_{\mathscr{S}}$ – the only difference is that the set of substitutions $\mathscr{S}$ is no longer required to have property A.

ADDENDUM (to Theorem 4)**.** *Let $\mathscr{S}$ be a set of deterministic multidimensional substitutions. Then the subshift $\mathbf{T}'_{\mathscr{S}}$ is sofic.*

We give here some elements to adapt Mozes' result. This sketch of a proof is addressed to readers already familiar with Mozes' proof, others may skip this part.

Proof. We first give some ideas of the proof of Theorem 4. Then we will explain how it can be adapted to get a proof of its addendum.

Let $\mathscr{S}$ be a set of substitutions of type A. Mozes constructs a sofic subshift Σ such that $\mathbf{T}_{\mathscr{S}}$ is a factor of Σ. The subshift Σ contains a grid that ensures that a configuration x is in the sofic subshift Σ if and only if one can find, for any $n \in \mathbb{N}$, a sequence of infinite patterns of substitutions $\mathbf{s}_0, \dots \mathbf{s}_{n-1} \in \mathscr{S}^{\mathbb{Z}^d}$, a configuration y_n and $\mathbf{i} \in \mathbb{Z}^d$ such that $\mathbf{s}_0 \circ \cdots \circ \mathbf{s}_{n-1}(y_n) = \sigma^{\mathbf{i}}(x)$. In Σ all the y_n are coded in a hierarchical structure. Let Q be the set of $2 \times \cdots \times 2$ patterns that appear in an $\mathscr{S}$-pattern. There is an additional condition, that we call condition Q: any $2 \times \cdots \times 2$ pattern that appears in any configuration y_n has to be in Q. So both type A condition and Q condition are made to ensure that any pattern that appears in a configuration x also appears in an $\mathscr{S}$-pattern.

This construction works: given a configuration $x \in \mathbf{T}_{\mathscr{S}}$ it is easy to construct a configuration $y \in \Sigma$ that encodes x and all its pre-images. Reciprocally given a pattern p that appears in a configuration $x \in \Sigma$, one can find a sequence of finite patterns of substitutions $(\mathbf{s}_0, \dots, \mathbf{s}_{n-1})$ such that p appears in $\mathbf{s}_0 \circ \cdots \circ \mathbf{s}_{n-1}(p')$, where p' is either a letter or appears in a $2 \times \cdots \times 2$ pattern. If p' is a letter then p appears in an $\mathscr{S}$-pattern. Otherwise, p' appears in a $2 \times \cdots \times 2$ pattern that appears itself in an $\mathscr{S}$-pattern – thanks to condition Q –, hence property A ensures that p also appears in an $\mathscr{S}$-pattern, that is to say generated by one letter a (see Figure 4). So any pattern appearing in x appears in an $\mathscr{S}$-pattern.

The difference between the subshifts $\mathbf{T}_{\mathscr{S}}$ and $\mathbf{T}'_{\mathscr{S}}$ is that we remove the condition that forces a pattern appearing in a configuration x to occur in an $\mathscr{S}$-pattern – and of course we still require that x has a pre-image of any order by $\mathscr{S}$. Hence property A is no longer needed, and if we adapt Mozes' construction by replacing the set Q by the set of all the $2 \times \cdots \times 2$ patterns, then $\mathbf{T}'_{\mathscr{S}}$ is a factor of the sofic subshift obtained. This proves the corollary.

□

3.2. Effective S-adic subshifts are sofic

Let $\mathrm{S} \in \mathscr{S}^{\mathbb{N}}$, of course one has $\mathbf{T}_{\mathrm{S}} \subset \mathbf{T}_{\mathscr{S}}$, but there is no immediate reason for $\mathbf{T}_{\mathrm{S}}$ to be also sofic. Moreover there are only countably many sofic subshifts but as stated in the Introduction there are uncountably many different non-conjugate S-adic subshifts, thus there exist non-sofic S-adic subshifts.

Theorem 5. *Let $\mathscr{S}$ be a finite set of non degenerate multidimensional substitutions and $\mathfrak{S} \subset \mathscr{S}^{\mathbb{N}}$ be effective closed. Then $\mathbf{T}'_{\mathfrak{S}}$ is sofic. If $\mathscr{S}$ has property A, then $\mathbf{T}_{\mathfrak{S}}$ is sofic.*

Remark. We only present the proof that $\mathbf{T}_{\mathfrak{S}}$ is sofic. The proof is similar for $\mathbf{T}'_{\mathfrak{S}}$, one just needs to replace $\mathbf{T}_{\mathscr{S}}$ by $\mathbf{T}'_{\mathscr{S}}$ in the proof.

P r o o f. We now assume that $d = 2$, the proof is similar for $d \geq 3$. Let $\mathscr{S}$ be a finite set of non degenerate $(\mathcal{A}, 2)$-substitutions, we define $\mathcal{A}' = \mathcal{A} \times \mathscr{S}^2$. To every $s \in \mathscr{S}$ we associate a $(\mathcal{A}', 2)$-substitution $\widetilde{s}$ with same support as represented in Figure 5.

All these substitutions $\widetilde{s}$ form a set $\widetilde{\mathscr{S}} = \{\widetilde{s} : s \in \mathscr{S}\}$. Let $\mathrm{S} = (s_i)_{i \in \mathbb{N}} \in \mathscr{S}^{\mathbb{N}}$ be an effective sequence, we can thus consider the effective sequence $\widetilde{\mathrm{S}} = (\widetilde{s}_i)_{i \in \mathbb{N}} \in \widetilde{\mathscr{S}}^{\mathbb{N}}$. The aim of substitutions $\widetilde{s}$ is to keep a record of the sequence of substitutions that have previously been applied, to ensure that the same substitution is used everywhere on a given level. An $\widetilde{s}$-pattern can be divided into nine zones (see Figure 5). The border zones are used to transfer information containing both the last substitution applied, but also the sequence of all substitutions previously applied. Note that in this construction, it is crucial to consider only non degenerate substitutions.

Example 4. Let $\mathscr{S}$ be the set of 2-dimensional substitutions on the alphabet $\mathcal{A} = \{\circ, \bullet\}$ defined in Example 3. In Figure 5 where $\widetilde{\mathrm{S}} = (\widetilde{s}_d, \widetilde{s}_a, \widetilde{s}_a, \dots)$ applied on the letter $\bullet$, one can find on the second coordinate of the bottom line – or third coordinate of the rightmost column – of the patterns $s_2(\bullet) = \widetilde{s}_a(\bullet), s_1 \circ s_2(\bullet) = \widetilde{s}_a \circ \widetilde{s}_a(\bullet), s_0 \circ s_1 \circ s_2(\bullet) = \widetilde{s}_d\widetilde{s}_a\widetilde{s}_a(\bullet), \dots$ the sequence of substitutions already applied appears.

One considers $\pi : \mathcal{A}'^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ the letter-to-letter block map which keeps the letter of $\mathcal{A}$ and $\pi_V : \mathcal{A}'^{\mathbb{Z}^d} \to \mathscr{S}^{\mathbb{Z}^d}$ (resp. $\pi_H : \mathcal{A}'^{\mathbb{Z}^d} \to \mathscr{S}^{\mathbb{Z}^d}$) the letter-to-letter block map which keeps the substitution $s_V \in \mathscr{S}$ (resp. $s_H \in \mathscr{S}$) of an element $(a, s_V, s_H) \in \mathcal{A}'$.

Claim 1: $\mathbf{T}_{\boldsymbol{\mathfrak{S}}} = \pi\left(\mathbf{T}_{\widetilde{\boldsymbol{\mathfrak{S}}}}\right)$ *where* $\widetilde{\boldsymbol{\mathfrak{S}}} = \{\widetilde{s} : s \in \boldsymbol{\mathfrak{S}}\}$.

Proof: This is straightforward, since the alphabet $\mathcal{A}'$ contains alphabet $\mathcal{A}$, and substitution $\widetilde{s}$ restricted to alphabet $\mathcal{A}$ is exactly substitution s. ◇ Claim 1

Consequently, it is sufficient to prove that $\mathbf{T}_{\widetilde{\boldsymbol{\mathfrak{S}}}}$ is sofic.

Claim 2: *The subshift* $\Sigma = \mathbf{SA}_{\mathbb{Z}\times\{0\}}\left(\pi_V(\mathbf{T}_{\widetilde{\boldsymbol{\mathfrak{S}}}})\right)$ *is effective.*

Proof: The class of effective subshifts is closed under factor, but also under projective subaction. This follows from the fact that projective subactions are special cases of factors of subactions. Indeed, Theorem 3.1 and Proposition 3.3 of [Hoc09] establish that symbolic factors and subactions preserve effectiveness. Thus it is sufficient to prove that $\mathbf{T}_{\widetilde{\boldsymbol{\mathfrak{S}}}}$ is an effective subshift.

Let $(w_n)_{n\in\mathbb{N}}$ be the effective sequence of word such that $\boldsymbol{\mathfrak{S}} = \mathscr{S}^{\mathbb{N}} \setminus \cup_n [w_n]$ where $[w_n] = \{\mathrm{S} \in \mathscr{S}^{\mathbb{N}} : \widehat{\mathrm{S}}_{[0,|w_n|-1]} = w_n\}$. One has

$$\mathbf{T}_{\boldsymbol{\mathfrak{S}}} = \bigcap_{n\in\mathbb{N}} \left\{x \in \mathcal{A}^{\mathbb{Z}^2} : \widehat{\mathrm{S}}_{[0,|w_n|-1]}(a) \text{ with } a|! \in \mathcal{A} \text{ and } \widehat{\mathrm{S}}_{[0,|w_m|-1]} \neq w_m,\ \forall m \leq n\right\}.$$

Thus $\mathbf{T}_{\boldsymbol{\mathfrak{S}}}$ is defined as an intersection of subshift of finite type where the forbidden patterns are given recursively. One deduces that it is an effective subshift. ◇ Claim 2

By Theorem 1 there exists a 2-dimensional subshift of finite type $\mathbf{T}_\Sigma$ on an alphabet $\mathcal{B}$ and a factor $\pi_\Sigma : \mathcal{B}^{\mathbb{Z}^d} \to \mathscr{S}^{\mathbb{Z}^d}$ such that

$$\Sigma = \mathbf{SA}_{\mathbb{Z}\times\{0\}}\left(\pi_\Sigma\left(\mathbf{T}_\Sigma\right)\right).$$

Note that the fact that $d \geq 2$ is crucial here, since the previous statement is not true for $d = 1$.

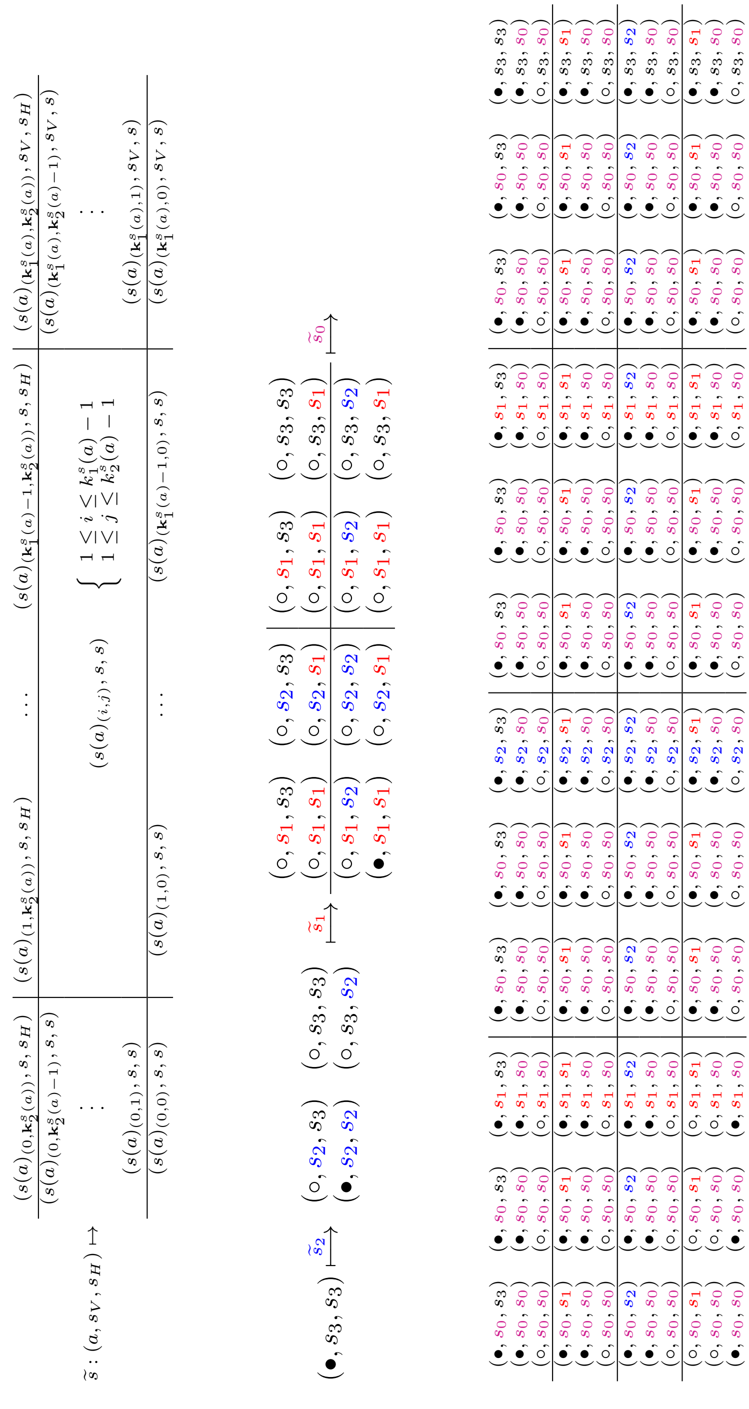

FIGURE 5. On the left the definition of new substitutions $\widetilde{s}$. On the right an example.

If we consider a configuration of the subshift $\mathbf{T}_{\widetilde{\mathscr{S}}}$ defined in Section 2.4, any substitution that appears in the set $\mathscr{S}$ may be chosen, provided it is still compatible with the configuration. But on each level, different substitutions may appear, which does not fit the definition of an S-adic subshift. To solve this problem we synchronize substitutions so that the same substitution is everywhere consistently used on a given level. To do that we need to ensure that for any configuration $x \in \mathbf{T}_{\widetilde{\mathscr{S}}}$, the same substitution appears in $\pi_V(x)$ on each row (resp. in $\pi_H(x)$ on each column). This can be enforced by local rules, and we thus define the subshift $\widetilde{\mathbf{T}}_{\widetilde{\mathscr{S}}}$ in the following way :

$$\begin{aligned}\widetilde{\mathbf{T}}_{\widetilde{\mathscr{S}}} \;=\; \Big\{ x \in \mathbf{T}_{\widetilde{\mathscr{S}}} : \forall (i,j) \in \mathbb{Z}^2, \pi_H(x)_{(i,j)} = \pi_H(x)_{(i,j+1)} \\ \text{and } \pi_V(x)_{(i,j)} = \pi_V(x)_{(i+1,j)} \Big\}.\end{aligned}$$

Obviously $\widetilde{\mathbf{T}}_{\widetilde{\mathscr{S}}} \subset \mathbf{T}_{\widetilde{\mathscr{S}}}$, and local rules added in $\widetilde{\mathbf{T}}_{\widetilde{\mathscr{S}}}$ ensure that substitutions applied on a given level are synchronized. Moreover these local rules do not impose more constraints on the substitutions: every sequence of substitutions $\widetilde{\mathrm{S}} \in \widetilde{\mathscr{S}^{\mathbb{N}}}$ can be obtained. We deduce that

$$\widetilde{\mathbf{T}}_{\widetilde{\mathscr{S}}} = \bigcup_{\widetilde{\mathrm{S}} \in \widetilde{\mathscr{S}^{\mathbb{N}}}} \mathbf{T}_{\widetilde{\mathrm{S}}} \subset \mathbf{T}_{\widetilde{\mathscr{S}}}.$$

Finally we consider

$$\mathbf{T}_{\mathtt{Final}} = \left\{ (x,s) \in \widetilde{\mathbf{T}}_{\widetilde{\mathscr{S}}} \times \mathbf{T}_{\Sigma} : \forall (i,j) \in \mathbb{Z}^2,\ \pi_V(x)_{(i,j)} = \pi_{\Sigma}(s)_{(i,j)} \right\}.$$

Thanks to Theorem 4, we know that the subshift $\mathbf{T}_{\widetilde{\mathscr{S}}}$ is sofic, hence so is $\widetilde{\mathbf{T}}_{\widetilde{\mathscr{S}}}$ since it is built by putting in more local rules into sofic subshift. Hence by construction, $\mathbf{T}_{\mathtt{Final}}$ is a sofic subshift. Consider the letter-to-letter factor map $\pi_{\mathtt{Final}} : \mathbf{T}_{\mathtt{Final}} \to \mathcal{A}'^{\mathbb{Z}^2}$ which keeps the letters of $\mathcal{A}'$.

Claim 3: $\pi_{\mathtt{Final}}(\mathbf{T}_{\mathtt{Final}}) = \mathbf{T}_{\widetilde{\mathfrak{S}}}$.

Proof: Given a configuration $x \in \mathbf{T}_{\widetilde{\mathfrak{S}}}$, it is easy to construct a corresponding element in $\mathbf{T}_{\mathtt{Final}}$.

Reciprocally, suppose you are given a configuration $x_{\mathtt{Final}} \in \mathbf{T}_{\mathtt{Final}}$. Replacing substitutions in $\mathscr{S}$ by composition of two substitutions of $\mathscr{S}$ if necessary, we assume that for all $s \in \mathscr{S}$ and all $a \in \mathcal{A}$, $\mathbf{k}_1^{(s)}(a), \mathbf{k}_2^{(s)}(a) \geq 2$. First the $\widetilde{\mathbf{T}}_{\widetilde{\mathscr{S}}}$ part of $x_{\mathtt{Final}}$ ensures that $\pi_{\mathtt{Final}}(x_{\mathtt{Final}})$ is an element of one $\mathbf{T}_{\widetilde{\mathrm{S}}'}$ for some $\widetilde{\mathrm{S}}' \in \widetilde{\mathscr{S}^{\mathbb{N}}}$. Secondly the condition that links the $\widetilde{\mathbf{T}}_{\widetilde{\mathscr{S}}}$ part with the $\mathbf{T}_{\Sigma}$ part certifies that $\mathrm{S}' \in \mathfrak{S}$: all substitutions are periodically repeated, but substitution s_0 is the only one which is repeated at least twice systematically – since $\mathbf{k}_1^{(s)}, \mathbf{k}_2^{(s)} \geq 2$. If we apply the same reasoning to a pre-image of $x_{\mathtt{Final}}$ by s_0, we can find s_1

and so on. At steep n, the words $s_0s_1\dots s_n$ verifies $[s_0s_1\dots s_n]\cap\mathbf{S}\neq\emptyset$ and we can find a pre-image of each patterns of $x_{\mathtt{Final}}$ by $s_0s_1\dots s_n$. $\diamond$ `Claim 3`

$\square$

4. On the sequences of substitutions defining S-adic subshifts that are effective

In Section 3 we proved that effective S-adic subshifts are sofic. A natural question would be to wonder what conditions are imposed on the set of S-adic sequences of substitutions that defines an S-adic subshift known to be effective. We present here a reciprocal statement to Theorem 5.

Theorem 6. *Let $\mathscr{S}$ be a finite set of substitutions and let $\mathbf{S}\subseteq\mathscr{S}^{\mathbb{N}}$ be a closed subset of S-adic sequences. If the* S-*adic subshift $\mathbf{T}_{\mathbf{S}}$ is non-empty and effective (and in particular if $\mathbf{T}_{\mathbf{S}}$ is sofic) then $\mathbf{S}$ is effectively closed.*

Proof. We describe an effective procedure that computes a sequence of words $(w_n)_{n\in\mathbb{N}}$ on the alphabet $\mathscr{S}$ such that $\mathrm{S}\in\mathbf{S}$ if and only if $\mathrm{S}_{[0,|w_n|-1]}\neq w_n$ for all $n\in\mathbb{N}$. This procedure runs forever and produces successively some words $w_n\in\mathscr{S}^*$. The procedure is divided into different steps. For every integer n, the n^{th} step consists in rejecting some words $w\in\mathscr{S}^*$ of length $|w|=n$ such that no sequence in $\mathbf{S}$ starts with w. The principle is to obtain the subshift $\mathbf{T}_{\mathbf{S}}$ by approaching it with a decreasing sequence of SFT that contains it.

For every integer n, the n^{th} step of the algorithm computes $\mathcal{F}$ the n first forbidden patterns produced by the Turing machine $\mathcal{M}$ which defines the effective subshift in view to produce the set $\mathcal{P}$ of all patterns of size $[-k^n,k^n]^d$, where k is the maximal size of the pattern of the substitution, where no pattern of $\mathcal{F}$ appears. Then for all $w\in\mathscr{S}^m$ with $m\leq n$ and $a\in\mathcal{A}$ the algorithm check whether $\widehat{w}(a)$ appears in the center of an element of $\mathcal{P}$. If not, the word w is returned by the procedure. Algorithm 1 describes this procedure.

Clearly if a word w is rejected by Algorithm 1 then it is not in the begging of an element $\mathrm{S}\in\mathbf{S}$ since no pattern of $T_{\mathbf{S}}$ contains the pattern $\widehat{w}(a)=\widehat{S}_{[0,|w|-1]}(a)$ for $a\in\mathcal{A}$. The next Claim proof the reciprocal.

Claim 1: *Let $w=s_0\dots s_m\in\mathscr{S}^{m+1}$. If $\mathrm{S}_{[0;k]}\neq w$ for $\mathrm{S}\in\mathbf{T}_{\mathbf{S}}$ then Algorithm 1 rejects w in finite time.*

Proof: Suppose that the Algorithm 1 does not reject $w=s_0\dots s_k$. Then for every integer n, there exists a pattern $p\in\mathcal{P}$ with support $[-k^n,k^n]$ and a letter $a\in\mathcal{A}$ such that the pattern $s_0\circ\dots\circ s_k(a)$ appears in the center of p. By a compactness

Algorithm 1: Compute forbidden beginnings for sequences in $\mathbf{S}$

```
n ← 0, F ← ∅, N ← 1, P ← ∅;
k ← max_{s∈𝒮, a∈A, 1≤l≤d} k_l^(s)(a);
while n ≥ 0 do
    F ← n first forbidden patterns of T_S produced by M;
    N ← k^n;
    P ← all patterns with support [−N, N]^d that do not contain any f ∈ F;
    for each s_0 ... s_m ∈ 𝒮^(m+1) with m + 1 ≤ n do
        for each a ∈ 𝒜 do
            if ∀p ∈ P, s_0 ∘ ··· ∘ s_m(a) does not appear in the center of p then
                Reject the word s_0 ... s_m;
    n ← n + 1;
```

argument we get a configuration $x \in \mathbf{T}_{\mathbf{S}}$ such that $x = s_0 \circ \cdots \circ s_k(y)$ for some other configuration y. Since $\mathbf{S}$ is effectively closed, it imposes the existence of a sequence S in $\mathbf{S}$ such that $\mathrm{S}_{[0;k]} = w$ which is a contradiction. ◇ `Claim 1`

Claim 1 suffices to conclude the proof, thus Theorem 6 holds.

□

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