Lecture 1: The Domino problem on groups, part I.

CANT 2016, CIRM (Marseille)

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Introduction

Objectives of this talk. . .

- Define the Domino problem (**DP**).
- Show the two main techniques to prove undecidability of **DP** on $\mathbb{Z}^2$

Outline of the talk.

Configurations and Subshifts (I)

- Let A be a finite alphabet, G be a finitely generated group.
- Colorings $x : G \to A$ are called **configurations**.
- Endowed with the prodiscrete topology A^G is a **compact** and **metrizable** set.
- **Cylinders** form a clopen basis

$$[a]_g = \left\{ x \in A^G \mid x_g = a \right\}.$$

- A **pattern** is a finite intersection of cylinders, or equivalently a finite configuration $p : S \to A$
- A **metric** for the cylinder topology is

$$d(x, y) = 2^{-\inf\{|g| \mid g \in G:\, x_g \neq y_g\}},$$

where $|g|$ is the length of the shortest path from 1_G to g in $\Gamma(G, S)$.

Configurations and Subshifts (II)

The **shift** action $\sigma : G \times A^G \to A^G$ is given by

$$(\sigma_g(x))_h = x_{g^{-1}h}.$$

The dynamical system (A^G, σ) is called the ***G*-fullshift over *A***.

Definition

A ***G*-subshift** is a closed and σ-invariant subset $X \subset A^G$.

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A **G-subshift** is a closed and σ-invariant subset $X \subset A^G$.

A pattern $p \in A^S$ **appears** in a configuration $x \in A^G$ if $(\sigma_g(x))_S = p$ for some $g \in G$.

Proposition

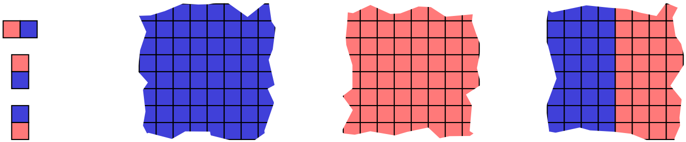
X is a G-subshift iff there exists a set $\mathcal{F}$ of forbidden patterns s.t.

$$X = X_{\mathcal{F}} := \left\{x \in A^G \mid \text{ no pattern of } \mathcal{F} \text{ appears in } x\right\}.$$

Subshifts of finite type

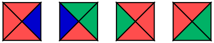
A G-subshift X is **of finite type** (G-SFT) if there exists a finite set of forbidden patterns $\mathcal{F}$ that defines it: $X = X_{\mathcal{F}}$.

Example:



SFTs and Wang tiles

Fix G a f.g. group and S a generating set for G. Wang tiles $\approx$ polygons with colored $2|S|$ edges.



Neighbourhood rule

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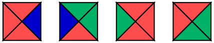
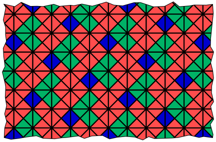


Neighbourhood rule

X_τ set of valid tilings by τ



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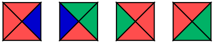

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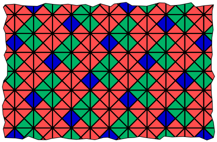
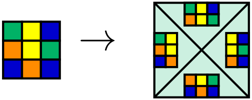


SFT $\approx X_\tau$



The Domino problem on groups

Fix G a f.g. group and S a generating set for G.

Domino problem on G

Input: A finite set of Wang tiles τ on S
Output: **Yes** if there exists a valid tiling by τ, **No** otherwise.

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Which f.g. groups have decidable Domino Problem ?

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$\rightarrow$ group property, quasi-isometry invariant.

Outline of the talk.

Sketch of the proof

Idea: encode **Turing machines** inside Wang tiles.

- Undecidability of the Halting problem of Turing machines.
- Reduction from the Halting problem of Turing machines.

Turing machines

$\delta(q,x)$		Symbol x			
		a	b	$\|$	$\sharp$
State q	q_0	$\bot$	$\bot$	$\bot$	$(q_{b^+}, a, \rightarrow)$
	q_{a^+}	$\bot$	$(q_{b^{++}}, a, \rightarrow)$	$\bot$	$\bot$
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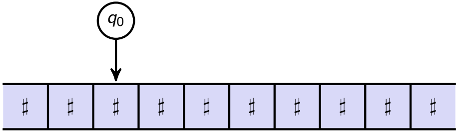

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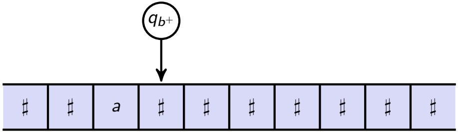

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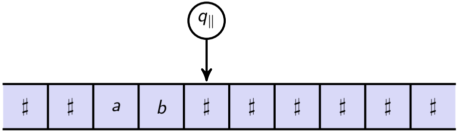

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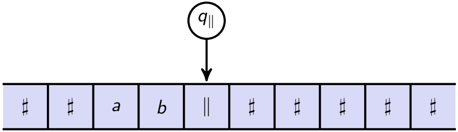

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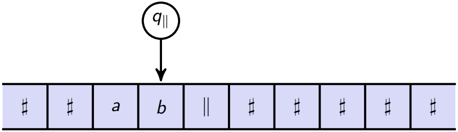

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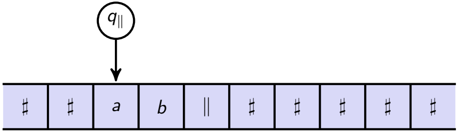

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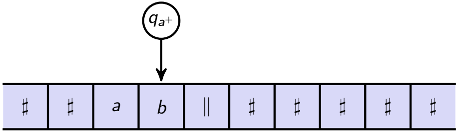

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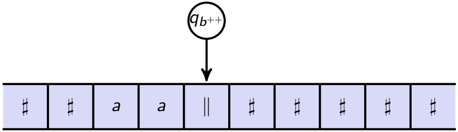

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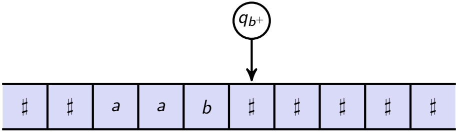

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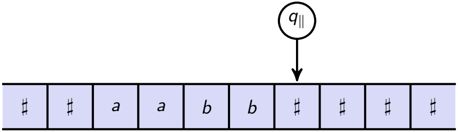

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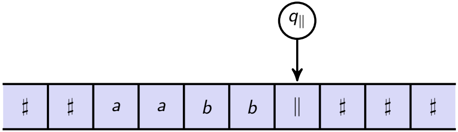

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Theorem (Turing, 1936)

The Halting problem (to know whether a Turing machine $\mathcal{M}$ halts on input w or not) is undecidable.

Theorem

The Blank tape Halting problem (to know whether a Turing machine $\mathcal{M}$ halts on the empty input) is undecidable.

Turing machines and Wang tiles

Encode Turing machine computations inside Wang tiles:

- no computation head
- initial configuration $({}^{\infty}\sharp^{\infty}, q_0)$
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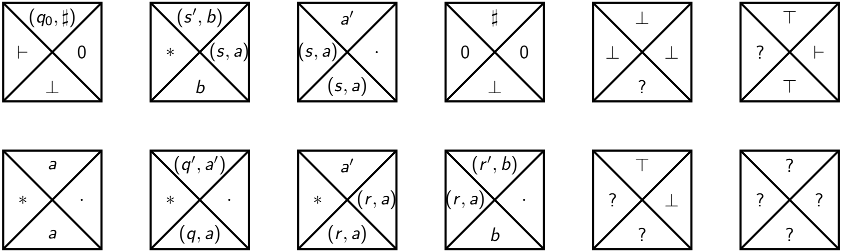

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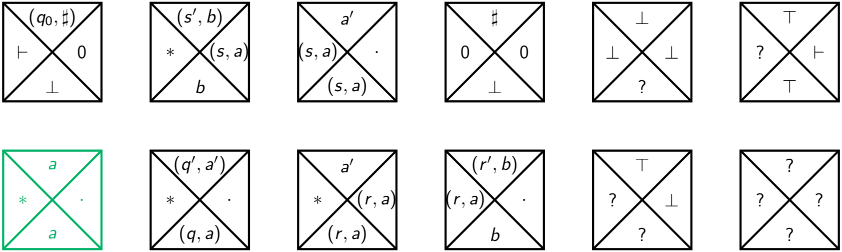

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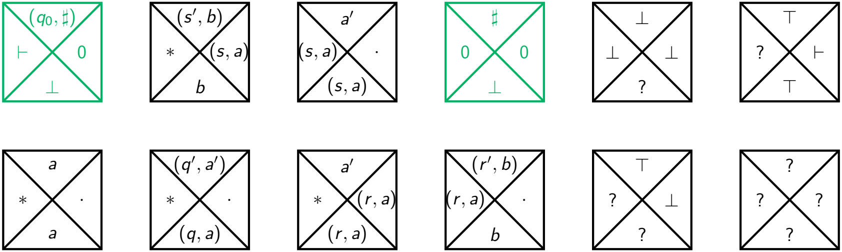

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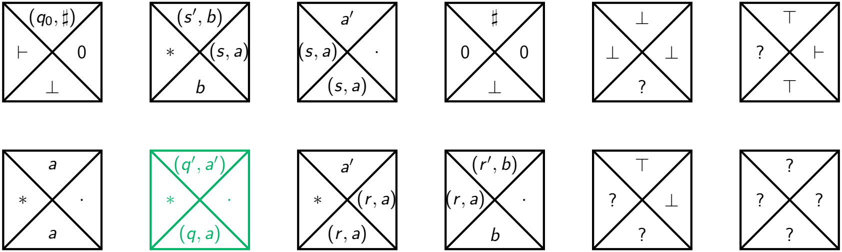

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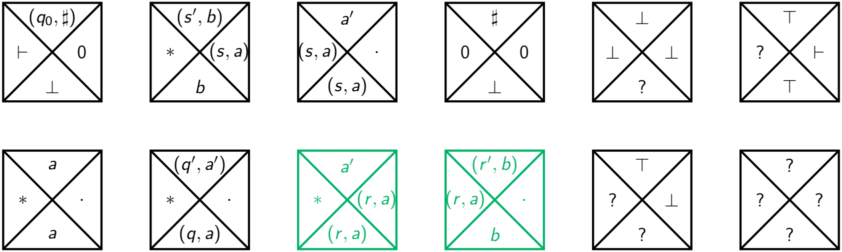

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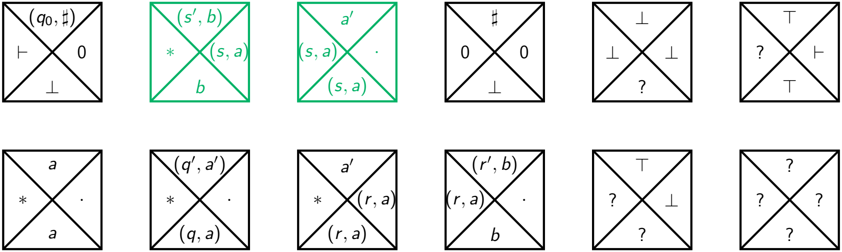

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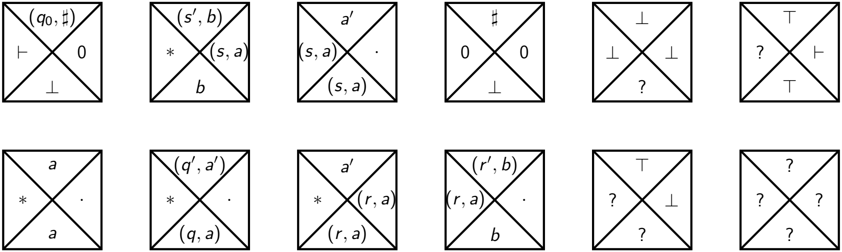

We want: τ **admits a tiling iff $\mathcal{M}$ does not halt on the empty input.**

Which tilings ?

We **forbid** tiles with an halting state q_f.

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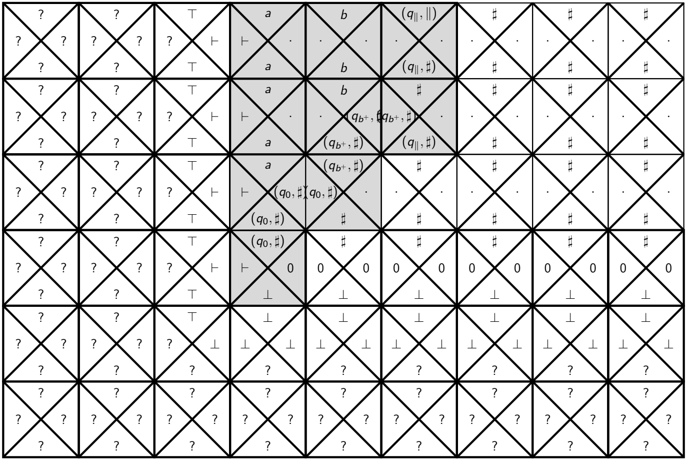
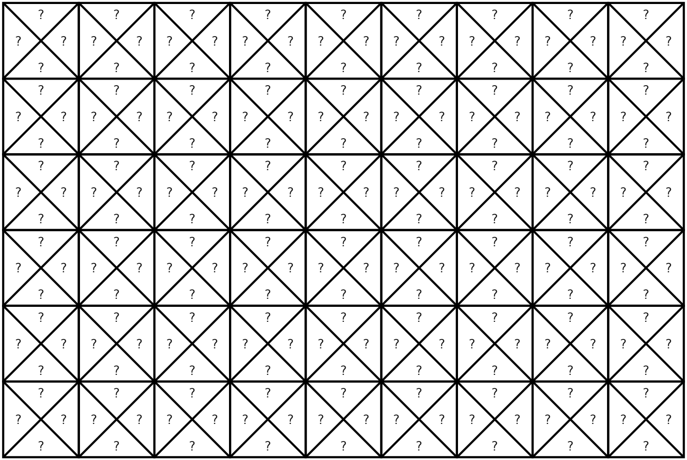


Which tilings ?

We **forbid** tiles with an halting state q_f.

If $\mathcal{M}$ does not halt on the empty input, we have a tiling. But...



The Origin Constrained Domino problem

What we have not proven:

Not-Yet-Theorem

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What we have proven:

Theorem (Kahr, Moore & Wang 1962, Büchi 1962)

The Origin Constrained Domino problem is undecidable on $\mathbb{Z}^2$.

where

Origin Constrained Domino problem

Input: A finite set of Wang tiles τ, a tile $t \in \tau$
Output: **Yes** if there exists a valid tiling by τ with t at the origin, **No** otherwise.

How to initialize computations ?

Build one infinite in time and space computation zone?

- **Compactness** $\Rightarrow$ we cannot force one given tile to appear exactly once in every valid tiling

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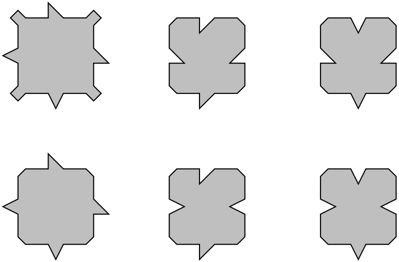
One solution: hierarchy of computation zones (thus arbritrarily big zones) that intersect a lot.

Robinson tileset

The Robinson tileset, where tiles can be rotated and reflected.



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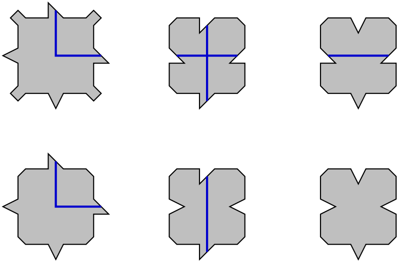

Existence of a valid tiling

Proposition

Robinson's tileset admits at least one valid tiling.

Existence of a valid tiling

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Robinson's tileset admits at least one valid tiling.

Proof:

- We can build arbitrarily large patterns (called macro-tiles) with the same structure.
- We thus conclude by compactness.

Macro-tiles of level 1

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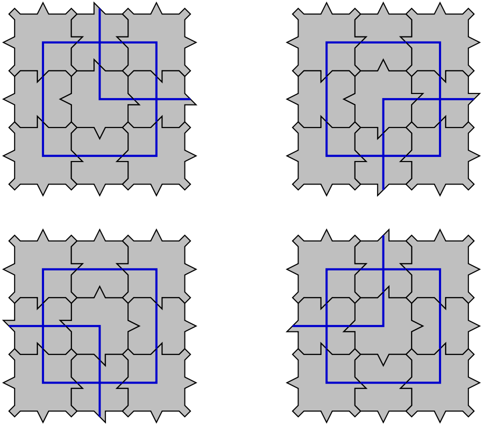

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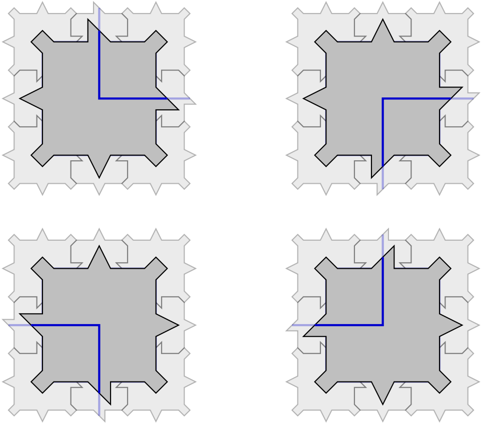
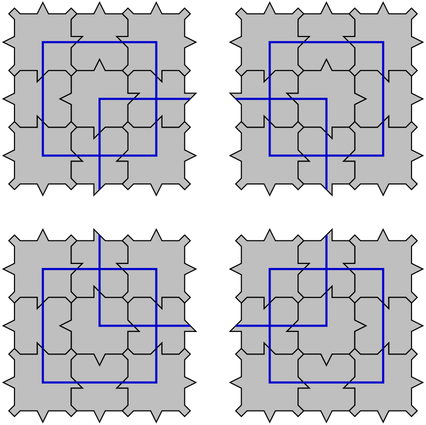


They behave like large .

From macro-tiles of level 1 to macro-tiles of level 2



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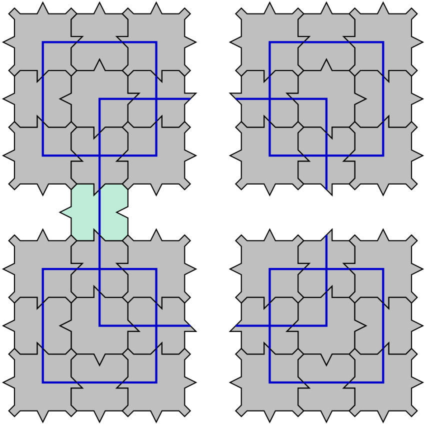

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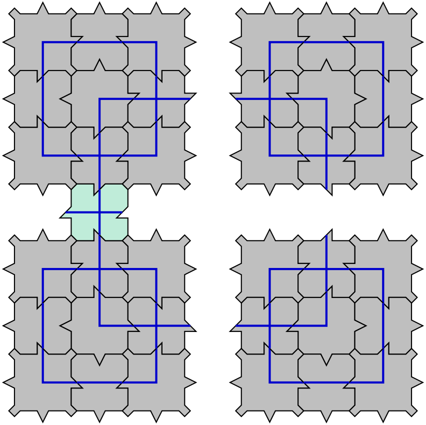

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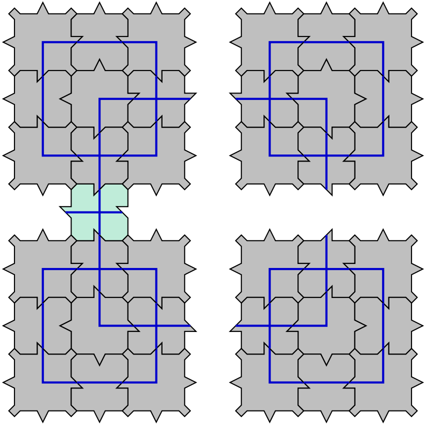

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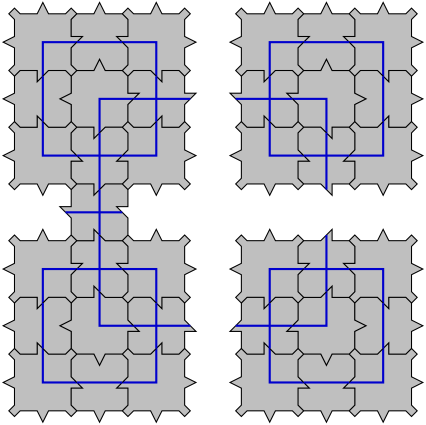

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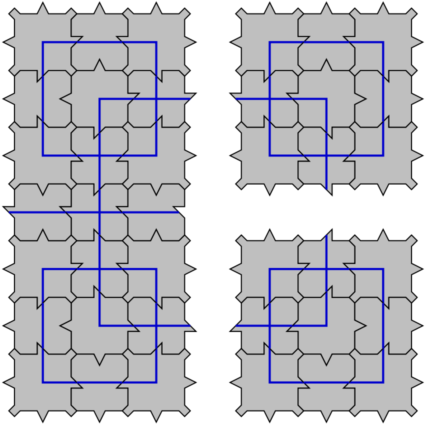

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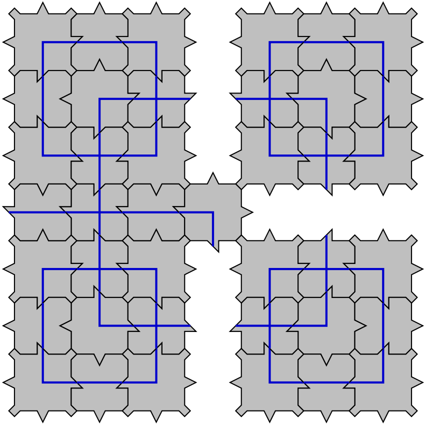

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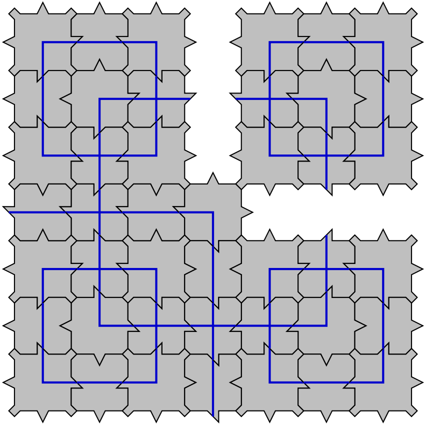

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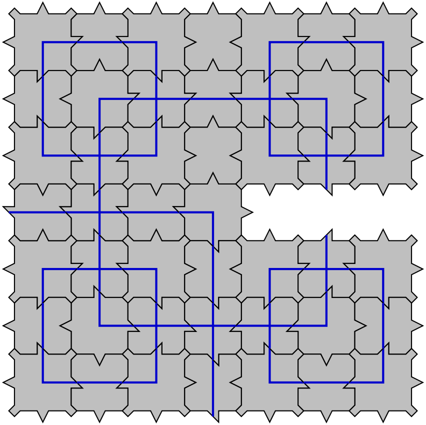

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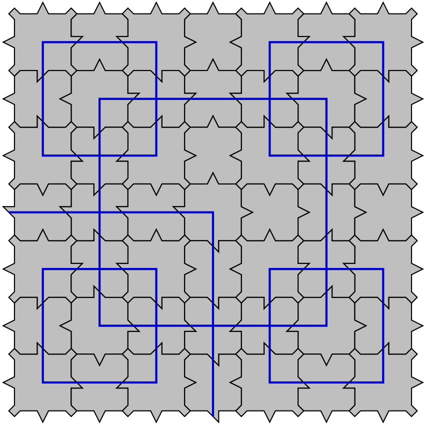

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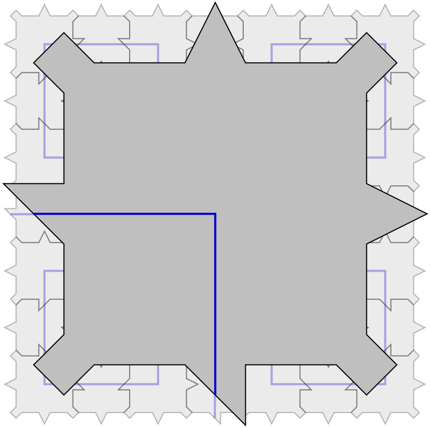
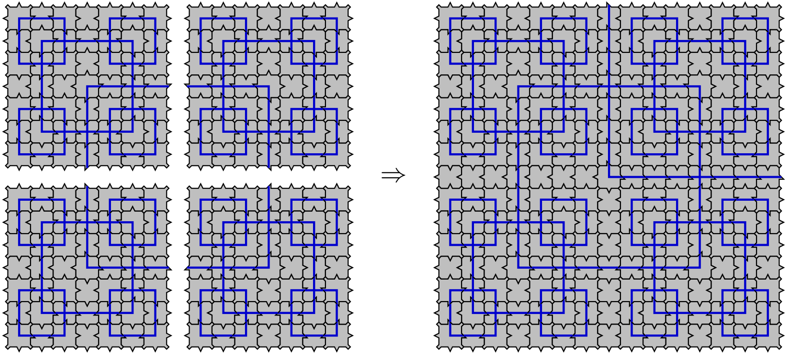
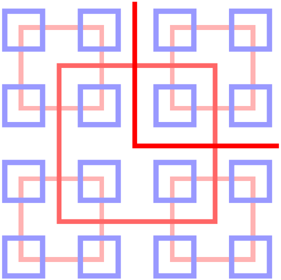


From macro-tiles of level n to macro-tiles of level $n+1$



About Robinson's tiling structure

Hierarchy of squares: squares of level n are gathered by 4 to form a square of level $n+1$



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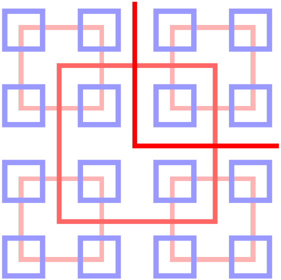
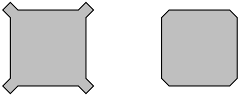


Proposition

The only valid tilings by the Robinson tileset form a hierarchy of squares.

Valid tilings (I)

The two forms in Robinson tileset, cross (bumpy corners) and arms (dented corners).



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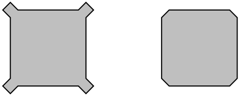
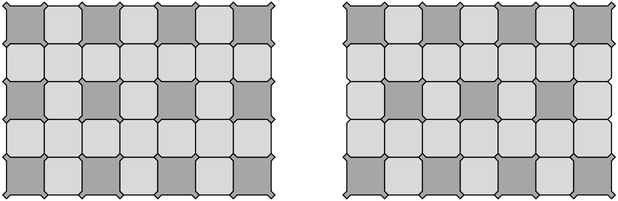
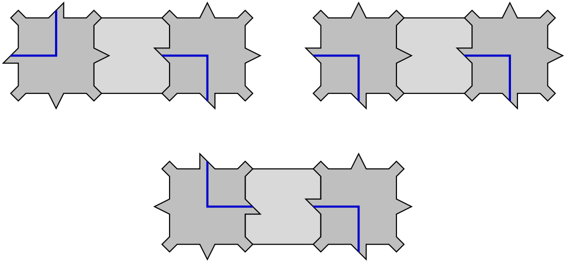


Obviously, two crosses cannot be in contact (neither through an edge nor a vertex) thus a cross must be surrounded by eight arms.



Valid tilings (II)

You cannot have things like



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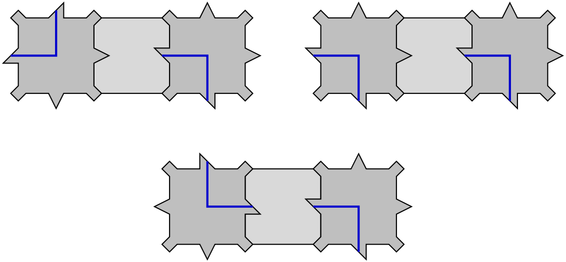

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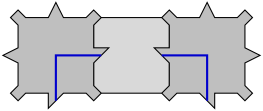

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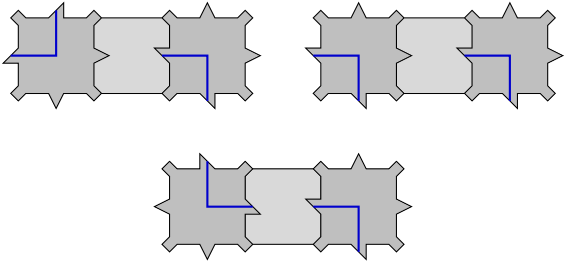

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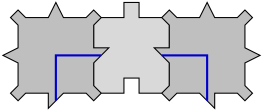
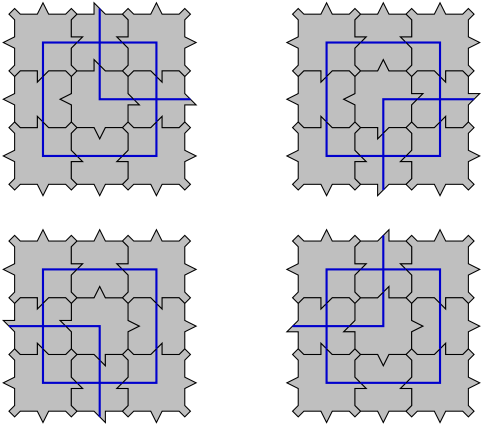


Valid tilings (III)

So each  is part of a macro tile of level 1

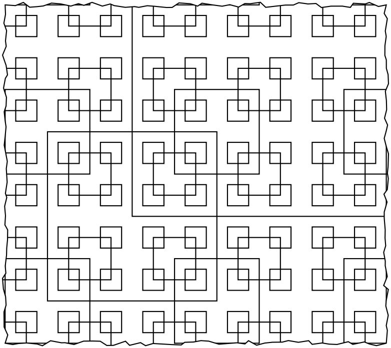


that behaves like a big , and so on. . .

Undecidability of the Domino Problem (II)

Solution

Embed Turing machine computations inside the hierarchy of squares given by Robinson's tiling.



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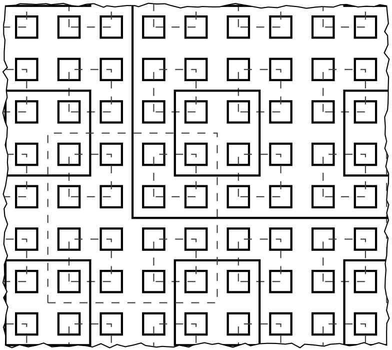

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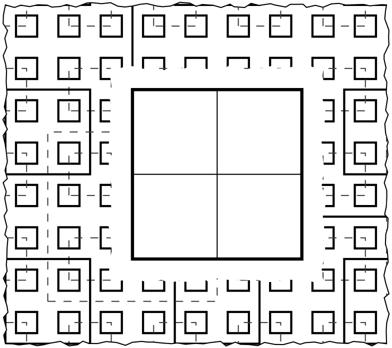

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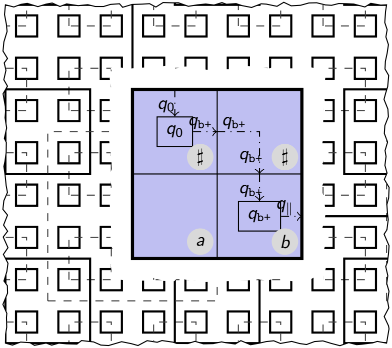

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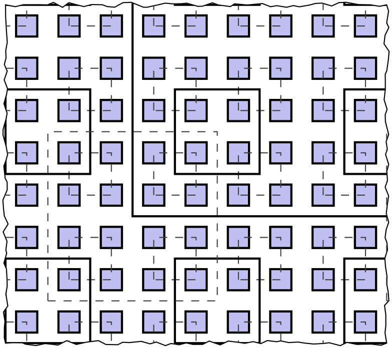

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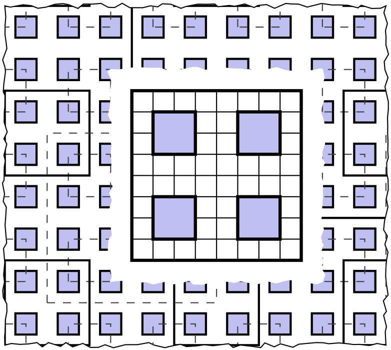

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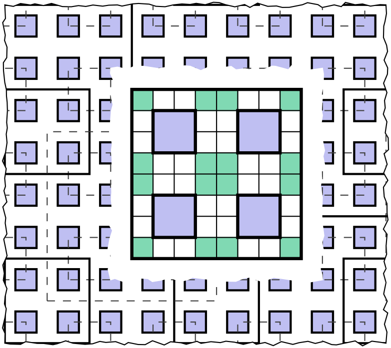

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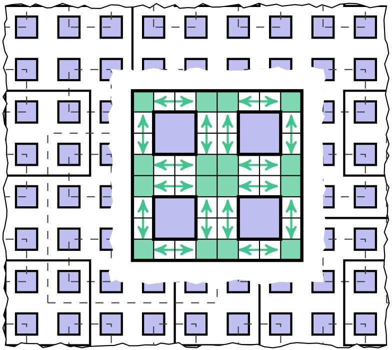

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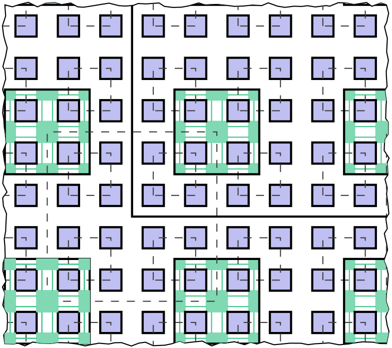

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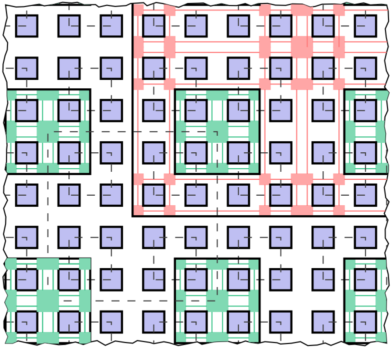

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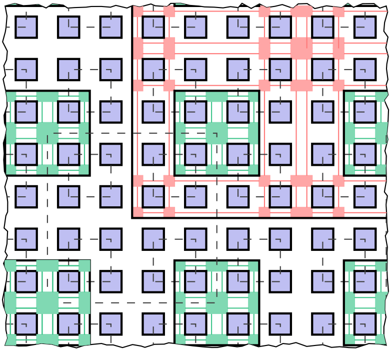

Theorem (Berger 1966, Robinson 1971)

The Domino Problem is undecidable on $\mathbb{Z}^2$.

Outline of the talk.

Sketch of the proof

Idea: encode **piecewise affine maps** inside Wang tiles.

- Undecidability of the Mortality problem of Turing machines.
- Undecidability of the Mortality problem of piecewise affine maps.
- Reduction from the Mortality problem of piecewise affine maps.

Mortality problem of Turing machines

Take $\mathcal{M}$ a deterministic Turing machine with an halting state q_f.

!! configurations of $\mathcal{M}$ do not have finite support !!

A configuration (x, q) is a **non-halting configuration** if it never evolves into the halting state.

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Mortality problem of Turing machines

Input: a deterministic Turing machine $\mathcal{M}$ with an halting state.
Output: **Yes** if $\mathcal{M}$ has a non-halting configuration, **No** otherwise.

Theorem (Hooper, 1966)

The Mortality problem of Turing machines is undecidable.

Proof: very technical, uses Minsky 2-counters machines.

Rational piecewise affine maps in $\mathbb{R}^2$

Take $f_i : U_i \to \mathbb{R}^2$ for $i \in [1; n]$ some rational affine maps, with $U_1, U_2, \ldots, U_n$ disjoint unit squares with integer corners.

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ with domain $U = \cup_{i=1}^{n} U_i$ by

$$\overrightarrow{x} \mapsto f_i(\overrightarrow{x}) \text{ if } \overrightarrow{x} \in U_i.$$

A point $\overrightarrow{x} \in \mathbb{R}^2$ is an **immortal starting point** for $(f_i)_{i=1\ldots n}$ if for every $n \in \mathbb{N}$, the point $f^n(\overrightarrow{x})$ lies inside the domain U.

Mortality problem of piecewise affine maps

Input: a system of rational affine maps $f_1, f_2, \ldots, f_n$ with disjoint unit squares $U_1, U_2, \ldots, U_n$ with integer corners.
Output: **Yes** the system has an immortal starting point, **No** otherwise.

Rational piecewise affine maps and Turing machines (I)

We use the **moving tape** Turing machines model.

Assume that $\mathcal{M}$ has alphabet $A = \{0, 1, \ldots, a-1\}$ and states $Q = \{0, 1, \ldots, b-1\}$.

Given $\mathcal{M}$ a Turing machine, we construct a system $f_1, f_2, \ldots, f_n$ of piecewise affine maps s.t.

- A configuration of $\mathcal{M}$ is coded by two real numbers.
- A transition of $\mathcal{M}$ is coded by one f_i.
- $f_1, f_2, \ldots, f_n$ has an immortal starting point if and only if $\mathcal{M}$ has an immortal configuration.

Rational piecewise affine maps and Turing machines (II)

Configuration (x, q) is coded by $(\ell, r) \in \mathbb{R}^2$ where

$$\ell = \sum_{i=-1}^{-\infty} M^i x_i$$

and

$$r = Mq + \sum_{i=0}^{\infty} M^{-i} x_i,$$

where M is an integer s.t. $M > a$ and $M > b$.

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The transition $\delta(q, a) = (q', a', \rightarrow)$ is coded by the affine transformation

$$\begin{pmatrix} \ell \\ r \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{M} & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \ell \\ r \end{pmatrix} + \begin{pmatrix} a' \\ M(q' - a - Mq) \end{pmatrix}$$

with domain $[0, 1] \times [Mq, Mq + 1]$.

Rational piecewise affine maps and Turing machines (II)

- A Turing machine $\mathcal{M}$ is transformed into a system $f_1, \ldots, f_n$ of rational piecewise affine maps.

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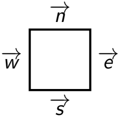
- $\mathcal{M}$ has an immortal starting point iff $f_1, \ldots, f_n$ has.

Theorem

The Mortality problem of piecewise affine maps is undecidable.

Rational affine maps inside Wang tiles (I)

Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ a rational affine map as before. The tile



is said to **compute** the function f if

$$f(\vec{n}) + \vec{w} = \vec{s} + \vec{e}.$$

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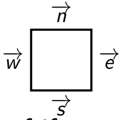

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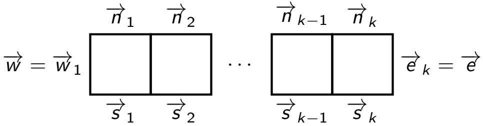
And on a row:



$$f\left(\frac{\vec{n}_1 + \cdots + \vec{n}_k}{k}\right) + \frac{1}{k}\vec{w} = \frac{\vec{s}_1 + \cdots + \vec{s}_k}{k} + \frac{1}{k}\vec{e}$$

Rational affine maps inside Wang tiles (II)

For $x \in \mathbb{R}$, a **representation of x** is a sequence of integers $(x_k)_{k\in\mathbb{Z}}$ s.t.

- $\forall k \in \mathbb{Z}, x_k \in \{\lfloor x \rfloor, \lfloor x \rfloor + 1\}$;
- $\forall k \in \mathbb{Z}$,
$$\lim_{n\to\infty} \frac{x_{k-n} + \cdots + x_{k+n}}{2n+1} = x.$$

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Define $B_k(x) = \lfloor kx \rfloor - \lfloor (k-1)x \rfloor$ for every $k \in \mathbb{Z}$. Then

$$B(x) = (B_k(x))_{k\in\mathbb{Z}}$$

is the **balanced representation of x**.

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For $\overrightarrow{x} \in \mathbb{R}^2$ and $k \in \mathbb{Z}$, define $B_k(\overrightarrow{x})$ coordinate by coordinate.

If $\overrightarrow{x}$ is in $U_i = [n, n+1] \times [m, m+1]$, then
$B_k(\overrightarrow{x}) \in \{(n,m), (n, m+1), (n+1, m), (n+1, m+1)\}$ for every $k \in \mathbb{Z}$.

Rational affine maps inside Wang tiles (III)

The tile set corresponding to $f_i(\overrightarrow{x}) = M\overrightarrow{x} + \overrightarrow{b}$ consists of tiles

$$\begin{array}{ccc} & B_k(\overrightarrow{x}) & \\ \begin{array}{r} f_i(A_{k-1}(\overrightarrow{x})) - A_{k-1}(f_i(\overrightarrow{x})) \\ +(k-1)\overrightarrow{b} \end{array} & \square & \begin{array}{l} f_i(A_k(\overrightarrow{x})) - A_k(f_i(\overrightarrow{x})) \\ +k\overrightarrow{b} \end{array} \\ & B_k(f_i(\overrightarrow{x})) & \end{array}$$

for every $k \in \mathbb{Z}$ and $\overrightarrow{x} \in U_i$.

Rational affine maps inside Wang tiles (III)

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Top: $B_k(\overrightarrow{x})$

Left: $f_i(A_{k-1}(\overrightarrow{x})) - A_{k-1}(f_i(\overrightarrow{x})) + (k-1)\overrightarrow{b}$

Right: $f_i(A_k(\overrightarrow{x})) - A_k(f_i(\overrightarrow{x})) + k\overrightarrow{b}$

Bottom: $B_k(f_i(\overrightarrow{x}))$

for every $k \in \mathbb{Z}$ and $\overrightarrow{x} \in U_i$.

Since U_i is bounded and f_i rational, there are **finitely many** tiles !

Rational affine maps inside Wang tiles (IV)

- A system of rational affine maps $f_1, f_2, \ldots, f_n$ defined on $U_1, U_2, \ldots, U_n$ with integer corners.
- Each $f_i \rightsquigarrow$ a finite set of tiles T_i
- Set of tiles $T = \cup T_i$ with additional markings (every row tiled by a single T_i)
- T admits a tiling of the plane iff $f_1, f_2, \ldots, f_n$ has an immortal point.

Rational affine maps inside Wang tiles (IV)

- A system of rational affine maps $f_1, f_2, \ldots, f_n$ defined on $U_1, U_2, \ldots, U_n$ with integer corners.
- Each $f_i \rightsquigarrow$ a finite set of tiles T_i
- Set of tiles $T = \cup T_i$ with additional markings (every row tiled by a single T_i)
- T admits a tiling of the plane iff $f_1, f_2, \ldots, f_n$ has an immortal point.

Theorem (Kari, 2007)

The Domino problem is undecidable on $\mathbb{Z}^2$.

Conclusion

- Two proofs of the undecidability of **DP** on $\mathbb{Z}^2$.
- Encode a small computational model inside Wang tiles.
- What about f.g. groups ?

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Thank you for your attention !!