Sofic (and Effective) Subshifts on f.g. Groups

Lecture 4: Effective subshifts.

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Introduction

Mini-course divided into 4 lectures

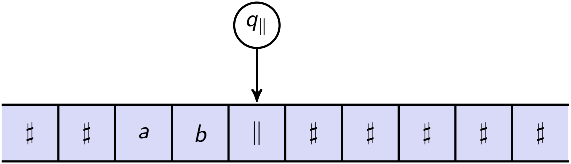
- Lecture 1: SD on f.g. groups: a computational approach.
- Lecture 2: Domino Problem, Part I: Wang tiles.
- Lecture 3: Domino Problem, Part II: f.g. groups.
- **Lecture 4: Effective subshifts.**

Lecture 4: Effective subshifts.

1. $\mathbb{Z}$-effective subshifts
   - Why effective subshifts ?
   - $\mathbb{Z}$-effectiveness
   - Limitations
2. A stronger notion of effectiveness
   - G-machines
   - G-effectiveness
   - Effectiveness and other classes of subshifts
3. G-Effective vs Sofic subshifts
   - Some groups with stricly G-effective subshifts
   - What about the hyperbolic plane ?

Reminder: Turing machines

A **Turing machine** is



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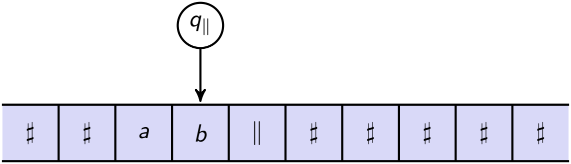

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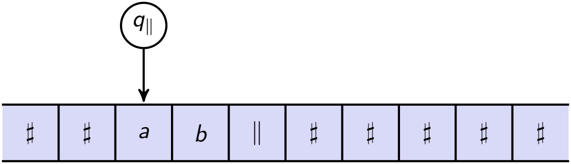

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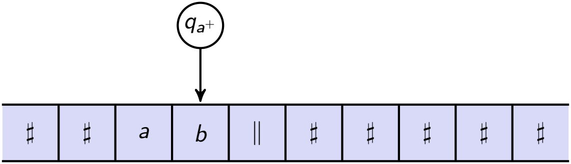

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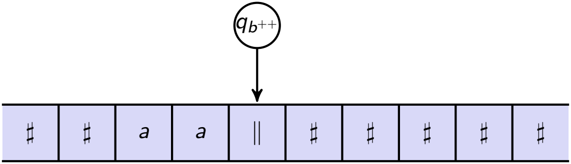

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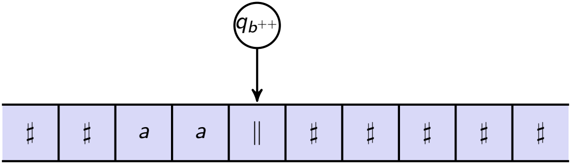

A finite word $w \in A^*$ is **accepted** (resp. **rejected**) by a Turing machine $\mathcal{M}$ if starting from the tape $\ldots \sharp \cdot w \sharp \ldots$, the machine $\mathcal{M}$ reaches an accepting (resp. rejecting) state in finite time.

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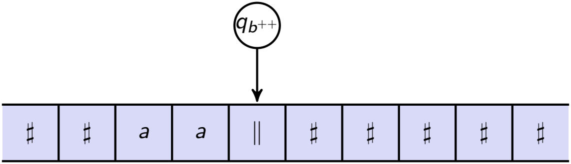

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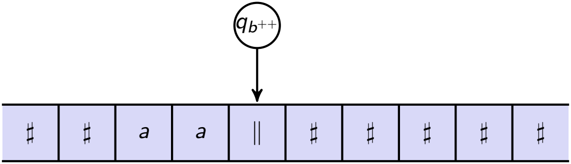

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A set of finite words $\mathcal{L} \subset A^*$ is **recognizable** if there exists a TM that accepts w iff $w \in \mathcal{L}$.

Effective subshifts on $\mathbb{Z}$

Definition

A $\mathbb{Z}$-subshift is **effective** if there exists a recognizable set of forbidden patterns that defines it.

Remark: The set of forbidden patterns F can be chosen to be maximal.

Examples:

- Sofic subshifts are effective.
- The sets of configurations on $\{a, b, c\}$ made of $a^n b^n$-blocks inside a sea of c's is an effective subshift.

Fact: There exist subshifts which are not effective (cardinality argument).

Proposition

A $\mathbb{Z}$-subshift is effective iff there exists a decidable set of forbidden patterns that defines it.

Why effective subshifts ?

- Chosmky hierarchy for formal languages.

- $\mathbb{Z}$-effective subshifts naturally appear as **projective subdynamics** of sofic $\mathbb{Z}^2$-subshifts.

Projective Subdynamics

Initially introduced by Johnson, Kass and Madden in 2007.

Idea: consider subsystems of lower dimension.

Definition

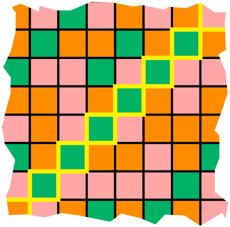
Let $X \subseteq A^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ subshift and $L \lneq \mathbb{Z}^d$ a k-dimensional sub-lattice $(1 \leq k < d)$. The L-projective subdynamics of X is

$$P_L(X) := \{x|_L : x \in X\} \subseteq A^L.$$

- $\left(P_L(X), \sigma_{L \times P_L(X)}\right)$ is a $\mathbb{Z}^k$-subshift.
- $P_L(X)$: globally admissible configurations of shape L in X.
- Loss of information about the original subshift.

Example of projective subdynamics

$L = \{(i,j) \in \mathbb{Z}^2 : i = j\}$

$x =$ 

$P_L(x) = \ldots$  $\ldots$

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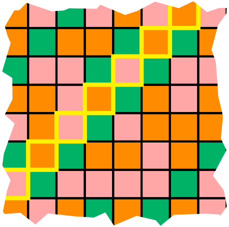
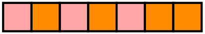
$x =$ 

$P_L\left(\sigma^{(1,1)}(x)\right) = \dots$  $\dots$

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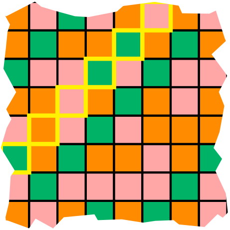
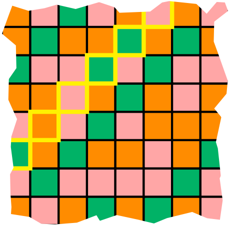
$x =$ 

$P_L\left(\sigma^{(2,2)}(x)\right) = \ldots$  $\ldots$

Example of projective subdynamics

$$L = \{(i,j) \in \mathbb{Z}^2 : i = j\} \qquad x = $$



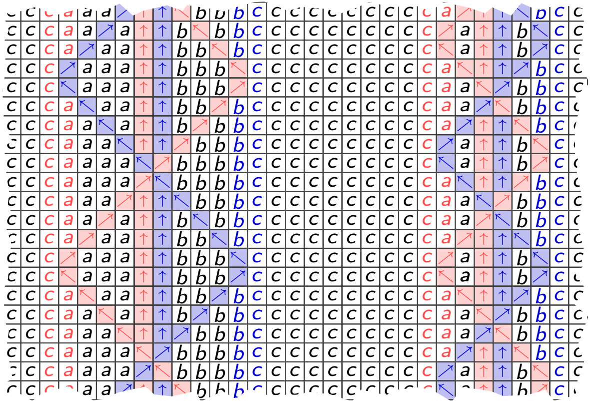
$$P_L\left(\sigma^{(2,2)}(x)\right) = \ldots$$

$\ldots$

In the sequel, we will concentrate on $P_{\vec{e}_1\mathbb{Z}}(X)$ (PS along the horizontal direction).

Stability under projective subdynamics

Fact: Sofic subshifts are not closed under PS.



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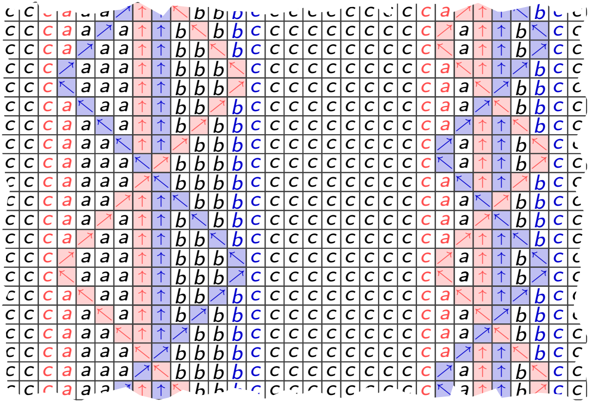

Proposition

The class of effective subshifts is stable under projective subdynamics.

Motivation: Hochman's result

Theorem (Hochman 2008)

Any effective $\mathbb{Z}$-subshift may be obtained as the projective subdynamics of a sofic $\mathbb{Z}^3$-subshift.

The proof is based on

- the use of ***Turing machines as SFT***,
- ***substitutive subshifts*** to construct computation zones in 3D.

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Theorem (A.& Sablik 2013, Durand, Romaschenko & Shen 2012)

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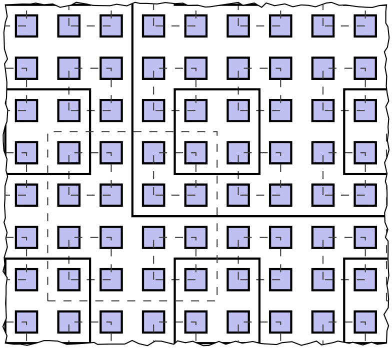
Hochman's result: idea of the proof

What about Robinson tiling ?



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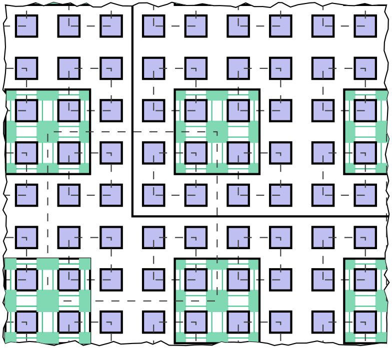

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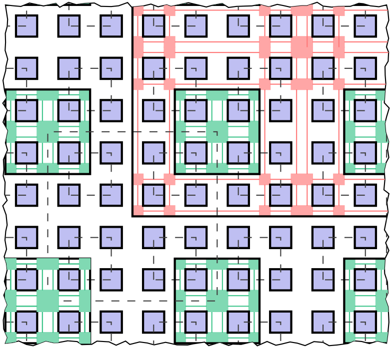

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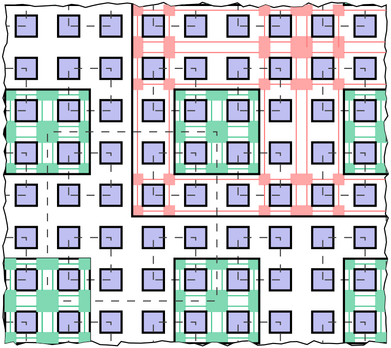

But words produced will be disconnected !

- Go to dimension 3 to define rectangular computations zones.
- Use a hierarchy of Turing machines to compare disconnected words with the content of the tape.

How to go further ?

Question 1

Does there exist a f.g. group such that ***Effective*** = ***Sofic*** ?

- Candidate: group containing a structure similar to $\mathbb{H}^2$?

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Question 2

Which groups admit a ***Hochman like result*** ?

- Which G s.t. effective G-subshifts can be embedded inside sofic $G \times \mathbb{Z}$-subshifts ?
- Which G s.t. effective G-subshifts can be embedded inside sofic $G \times \mathbb{Z}^2$-subshifts ? (easier ?)

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$\Rightarrow$ **Define a notion of effectiveness for G-subshifts.**

$\mathbb{Z}$-effectiveness

Let G be a f.g. group. How to define effectiveness for G-subshifts ?

$\mathbb{Z}$-effectiveness

Let G be a f.g. group. How to define effectiveness for G-subshifts ?

First idea: Use Turing machines.

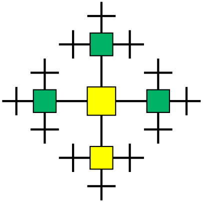
Problem: Turing machines take ***words*** as input and not ***patterns*** on G.

Tentative: Encode patterns inside words.

Pattern codings

How to encode a ***pattern*** $p \in A^*_G$ on G inside a ***word*** $w_p \in A^*$?

Case of $\mathbb{F}_2$



→ $(1_{\mathbb{F}_2},$ 🟨$), (a,$ 🟩$), (b,$ 🟩$), (a^{-1},$ 🟩$), (b^{-1},$ 🟨$)$

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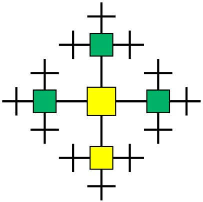

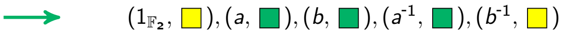

Definition

Let $S \subset G$ be a finite generator. A **pattern coding** c is a finite set of tuples $c = (w_i, a_i)_{1 \leq i \leq n}$ where $w_i \in (S \cup S^{-1})^*$ and $a_i \in A$.

Consistent pattern codings

Suppose now you are given a word on $A \times \left(S \cup S^{-1}\right)^*$, does it code a pattern on G ?

Examples: On $BS(1,2) = \langle a, b \mid ab = ba^2 \rangle$,

- $\begin{matrix} (\epsilon, 0) \\ (ab, 0) \end{matrix} \quad \begin{matrix} (b, 1) \\ (ba^2, 0) \end{matrix} \quad \begin{matrix} (a, 1) \\ (ba, 1) \end{matrix}$ corresponds to a pattern,
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Definition

A pattern coding c is **consistent** if for every words w_i, w_j that represent the same element in G one has $a_i = a_j$.

If c is a consistent pattern coding, we define the pattern $\Pi(c) \in A_G^*$ such that $supp(\Pi(c)) = \bigcup_{i \in I} w_i$ and $\Pi(c)_{w_i} = a_i$.

$\mathbb{Z}$-effective susbhifts

Let G be a f.g. group and $S \subset G$ a finite generating set.

Definition

A G-subshift $X \subset A^G$ is **$\mathbb{Z}$-effective** if there exists $F \subset A^*_G$ such that $X = X_F$ and a Turing machine $\mathcal{M}$ that accepts a pattern coding c if and only if it is either inconsistent or $\Pi(c) \in F$.

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Question: Is it always possible to recognize if a pattern coding is inconsistent?

Limitations of $\mathbb{Z}$-effectiveness: recursively presented groups.

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Theorem

Let $|A| \geq 2$ then the following are equivalent:

- G is recursively presented.
- The $WP(G)$ is recognizable.
- The set of inconsistent patterns codings is recognizable.

Limitations of $\mathbb{Z}$-effectiveness: decidable Word Problem.

Remark: Even if G is finitely presented, there may be simple G-subshifts which are not $\mathbb{Z}$-effective !

Limitations of $\mathbb{Z}$-effectiveness: decidable Word Problem.

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Theorem

The one-or-less subshift

$$X_{\leq 1} := \{x \in \{0,1\}^G \mid |\{g \in G : x_g = 1\}| \leq 1\}$$

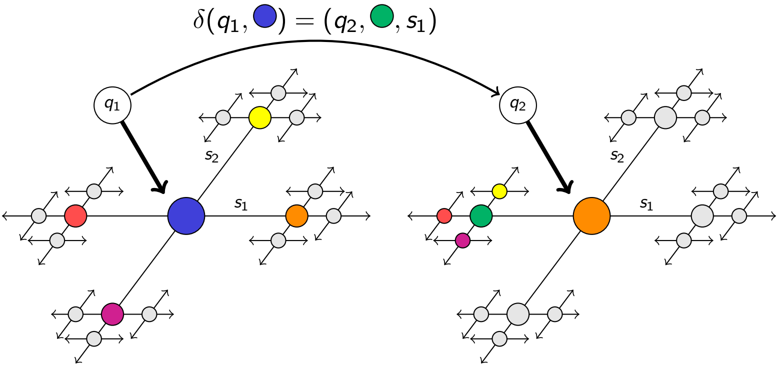
is not $\mathbb{Z}$-effective if G has undecidable WP.

Lecture 4: Effective subshifts.

G-machines

Definition

A ***G*-machine** is a Turing machine whose tape has been replaced by the group G. The transition function is $\delta : Q \times \Sigma \to Q \times \Sigma \times (S \cup S^{-1} \cup \{1_G\})$ where S is a finite set of generators of G.



G-machines as computational model

Similarly to TM, we define notions of G-**decidable** and G-**recognizable** languages of patterns $\mathcal{L} \subset A_G^*$.

Proposition

Let $\mathcal{L}$ be a language that can be decided by a multiple head G-machine. Then $\mathcal{L}$ can be decided by a G-machine.

G-effectiveness

Definition

A G-subshift $X \subset A^G$ is **G-effective** if there exists a set of forbidden patterns $F \subset A^*_G$ such that $X = X_F$ and F is G-recognizable.

Example: The one-or-less subshift is G-effective.

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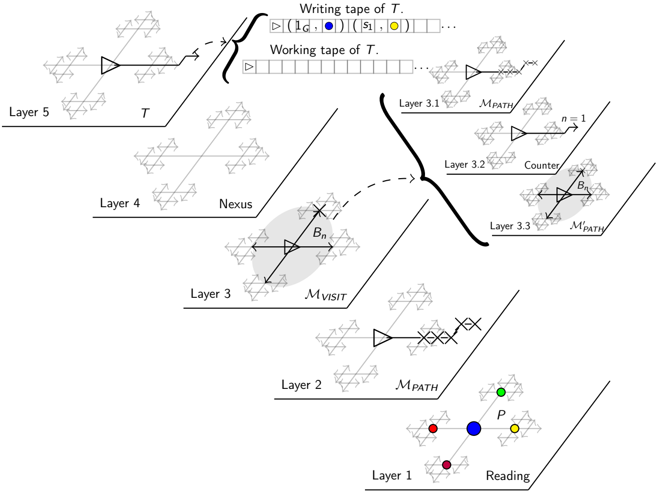
Example: The one-or-less subshift is G-effective.

Theorem

Let G be an infinite, finitely generated group, then every $\mathbb{Z}$-effective subshift is G-effective.

- Initiate a backtracking over G in order to mark a one-sided infinite path.
- Use the path to simulate one-sided Turing machines.

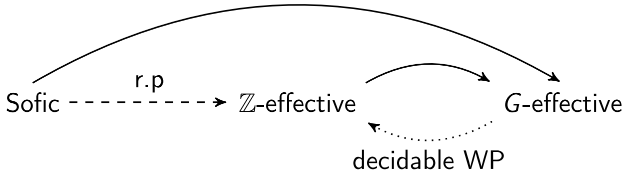
$\mathbb{Z}$-effective subshifts are G-effective



Effectiveness and other classes of subshifts

Theorem

- If G has decidable WP then every G-effective subshift is $\mathbb{Z}$-effective.
- The class of G-effective subshifts is closed under factors.
- Every G-SFT is G-effective.
- Every Sofic G-subshift is G-effective.



Lecture 4: Effective subshifts.

Groups with stricly effective subshifts

Proposition

If G is a recursively presented group with undecidable WP there exists G-effective subshifts which are not sofic.

Proof: $X_{\leq 1}$.

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If G is a recursively presented group with undecidable WP there exists G-effective subshifts which are not sofic.

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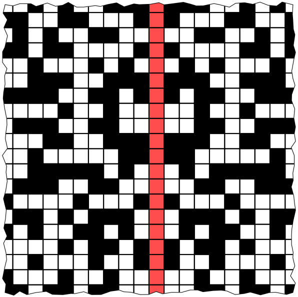
Question: Is it possible to construct G-effective subshifts which are not sofic in big classes of groups?

Mirror subshift in $\mathbb{Z}^2$

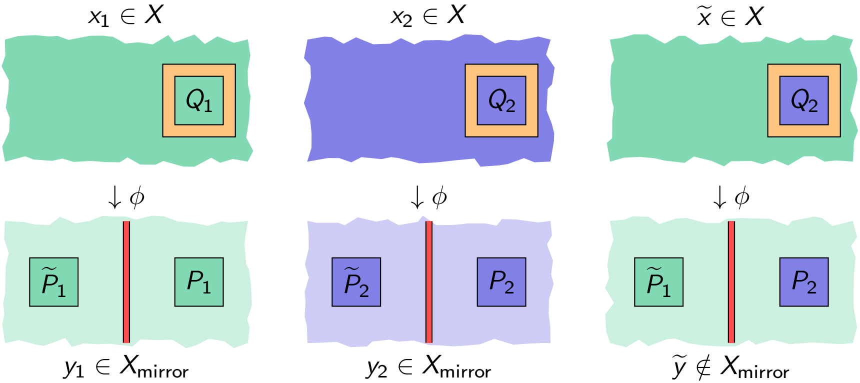
Let $A = \{ \square , \blacksquare , {\color{red}\blacksquare} \}$ and $X_{\text{mirror}} = X_{F_{\text{mirror}}} \subset A^{\mathbb{Z}^2}$ where

$$F_{\text{mirror}} = \left\{ \begin{smallmatrix}\square\\ {\color{red}\blacksquare}\end{smallmatrix}, \begin{smallmatrix}\blacksquare\\ {\color{red}\blacksquare}\end{smallmatrix}, \begin{smallmatrix}{\color{red}\blacksquare}\\ \square\end{smallmatrix}, \begin{smallmatrix}{\color{red}\blacksquare}\\ \blacksquare\end{smallmatrix} \right\} \cup \bigcup_{w \in A^*} \{ {\color{red}\blacksquare}\, w \,{\color{red}\blacksquare} , \blacksquare\, w \,{\color{red}\blacksquare}\, \tilde{w} \,\square , \square\, w \,{\color{red}\blacksquare}\, \tilde{w} \,\blacksquare \}$$

where $\tilde{w}$ denotes the mirror image of the word w.



The mirror subshift is not sofic



Amenable groups

Key ingredients in the previous proof

- A $\mathbb{Z}^2$-effective subshift X with highly non-local conditions.
- The existence of an increasing sequence of finite sets whose border grows slower than the sets themselves.

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Theorem

If G is an amenable f.g. group, then there exist G-effective subshifts which are not sofic.

Proof: Ball mimic subshift **(S. Barbieri's poster)**

Groups with more than two ends

Theorem

If G is a f.g. group where $e(G) \geq 2$, then there exist G-effective subshifts which are not sofic.

Proof: Mimic subshift **(S. Barbieri's poster)**

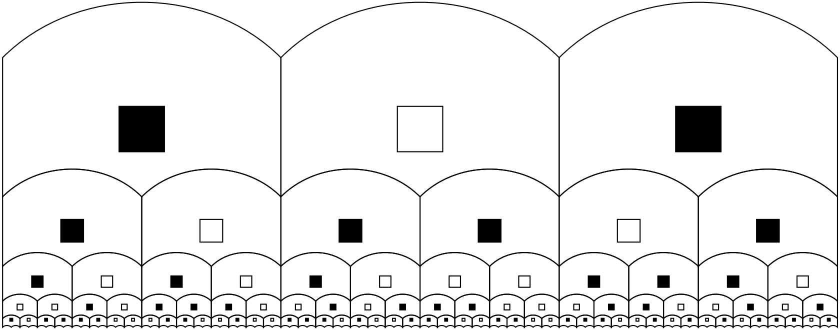
What about the hyperbolic plane ?

Question

Can we construct a f.g. group s.t. all G-effective subshifts are sofic?

Information Compression in $\mathbb{H}^2$ (I)

Start with a configuration in $\{ \blacksquare , \square \}^{\mathbb{H}^2}$



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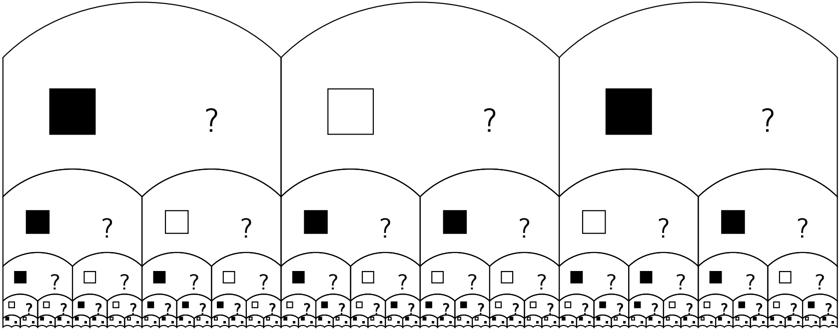

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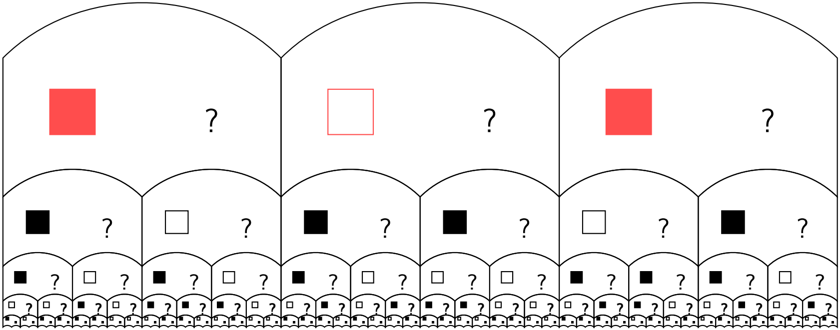

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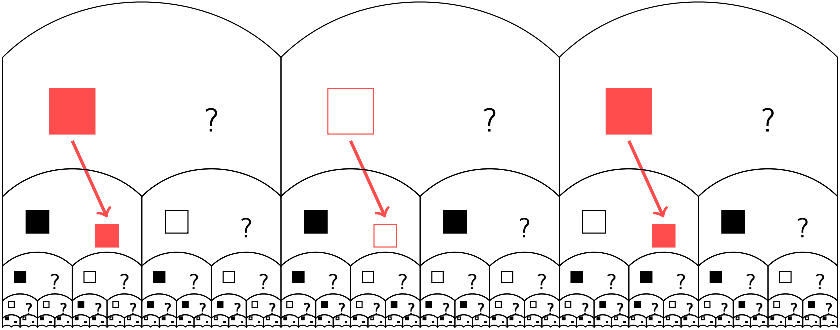
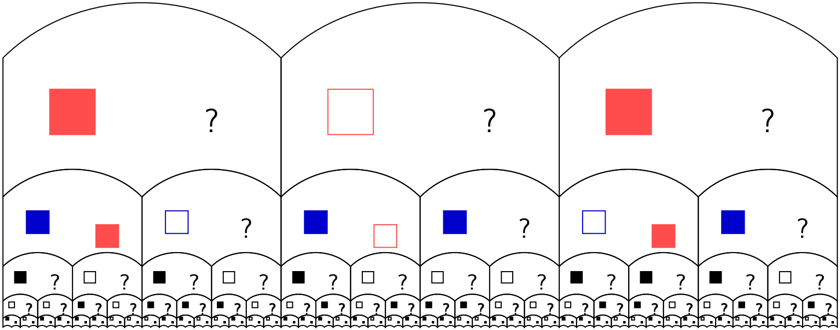


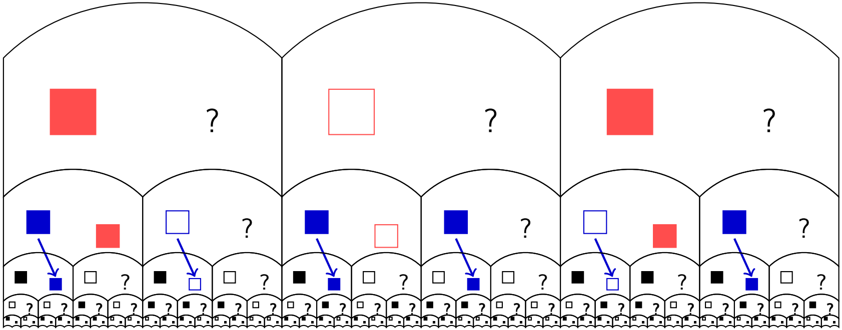
Information Compression in $\mathbb{H}^2$ (I)

Start with a configuration in $\{ ■ , □ \}^{\mathbb{H}^2}$



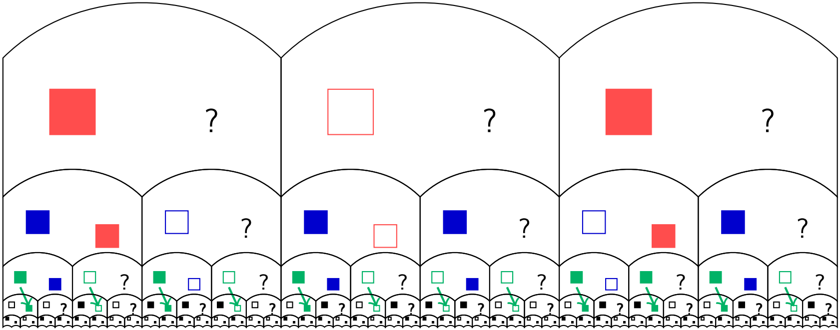
Information Compression in $\mathbb{H}^2$ (I)

Start with a configuration in $\{ \blacksquare , \square \}^{\mathbb{H}^2}$



Information Compression in $\mathbb{H}^2$ (I)

Start with a configuration in $\{ \blacksquare , \square \}^{\mathbb{H}^2}$



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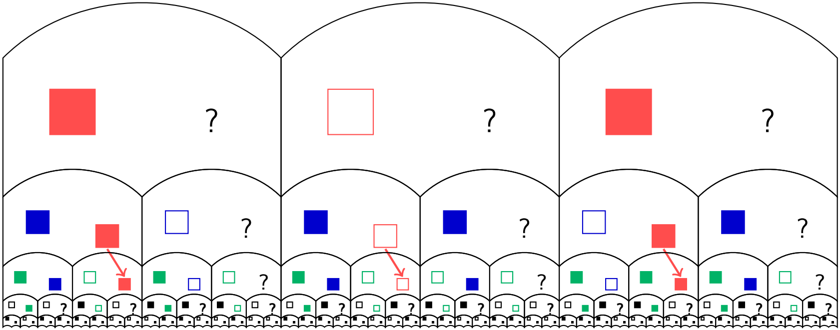
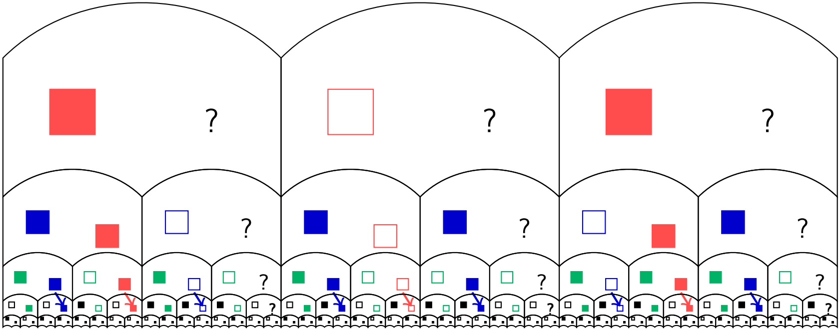


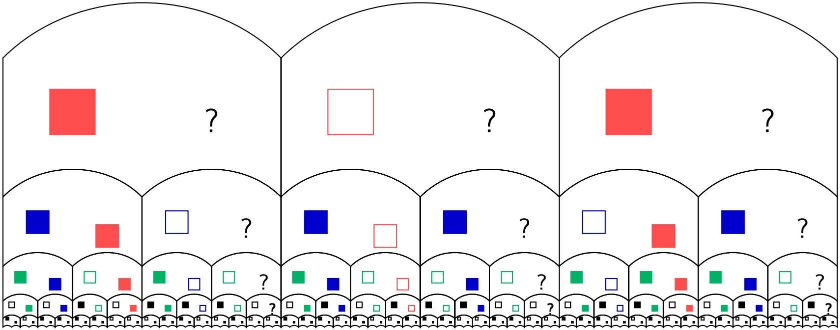
Information Compression in $\mathbb{H}^2$ (I)

Start with a configuration in $\{ \blacksquare , \square \}^{\mathbb{H}^2}$



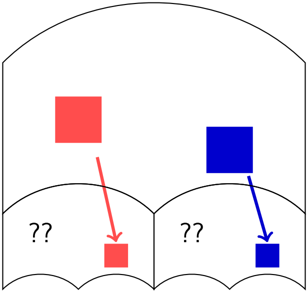
Information Compression in $\mathbb{H}^2$ (I)

Start with a configuration in $\{ \blacksquare , \square \}^{\mathbb{H}^2}$



Information Compression in $\mathbb{H}^2$ (II)

This coding can be imposed by local rules of the form



We thus add **finite type** constraints which ensure that

- for every row, the k^{th} row above is coded every 2^k ;
- every row codes its upper half-plane.

Proposition

If $\mathbf{X}$ is an SFT (resp. sofic subshift, effective subshift), then $\Phi(\mathbf{X})$ is an SFT (resp. sofic subshift, effective subshift).

Groups with a dyadic encoding ?

With dyadic encoding, patterns can be *replaced* by words.

Can this encoding be used to get rid of the *extra dimension(s)* needed in results for effective $\mathbb{Z}$-subshifts ?

Examples of groups with dyadic encoding ?

- Natural candidate: Baumslag-Solitar groups $\mathrm{BS}(1, n)$...
- ...but (un)fortunately they are amenable.

Conclusion

- Two notions of effectiveness for G-subshifts, that coincide iff G has decidable WP
- Are these two notions always weaker than soficness ?
- Find groups that admit a *Hochman like theorem* ?

Thank you for your attention !!