Computational Aspects of $\mathbb{Z}^d$ Symbolic Dynamics

Mathematical Congress of the Americas 2013

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In this talk. . .

- Multidimensional SFT and effective subshifts
- Turing machines, Computability obstruction
- Projective subdynamics

Outline

Multidimensional subshifts

- $\mathcal{A}$ a finite alphabet and $d \in \mathbb{N}$
- $\mathcal{A}^{\mathbb{Z}^d}$, the *set of configurations*, is a compact metric space (for the prodiscrete topology)
- *shift action* $\sigma : \mathbb{Z}^d \times \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$, $\left(\sigma_{(n_1,\dots,n_d)}(x)\right)_{(i_1,\dots,i_d)} = x_{(i_1+n_1,\dots,i_d+n_d)}$
- the dynamical system $\left(\mathcal{A}^{\mathbb{Z}^d}, \sigma\right)$ is the *d-dimensional full-shift on $\mathcal{A}$*

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Definitions

- *(topological)* A *subshift* is a closed and σ-invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$.
- *(combinatorial)* If F is a set of patterns, the *subshift generated by F* is

$$X_F = \left\{x \in \mathcal{A}^{\mathbb{Z}^d} : \text{ no pattern of } F \text{ appears in } x\right\}.$$

Effective subshifts

SFT $\subsetneq$ Sofic susbhifts $\subsetneq$ *Effectively closed*

Definition

A subshift is *effectively closed* (or *effective*) if its complement is a computable union of cylinders.

Property

X is effectively closed if and only one of the followings holds

(i) $X = X_{\mathcal{F}}$ for some recursively enumerable set $\mathcal{F}$ of forbbiden patterns

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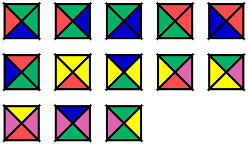
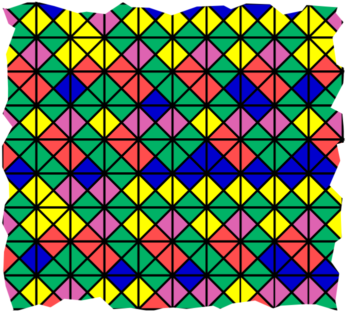
Wang tiles and SFT

We consider tilings of $\mathbb{R}^2$ by unit squares with one color on each edge, such that two adjacent squares wear the same color on their common edge.





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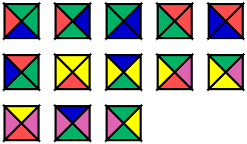
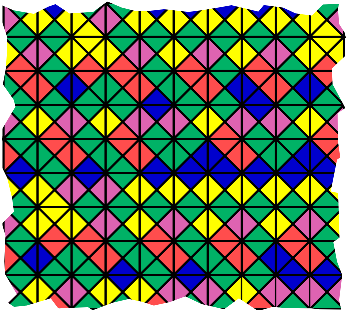
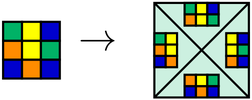




Every SFT is equivalent (up to conjugacy) to a finite set of Wang tiles



Turing machines and Wang tiles (I)

A *Turing machine* is a tuple $\mathcal{M} = (Q, \Gamma, \sharp, q_0, \delta, Q_F)$ where:

- Q is a finite set of states, $q_0 \in Q$ is the initial state;
- Γ is a finite alphabet;
- $\sharp \notin \Gamma$ blank symbol
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\leftarrow, \downarrow, \rightarrow\}$ transition function;
- $F \subset Q_F$ finite set of final states.

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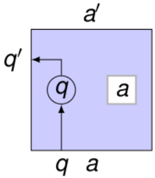
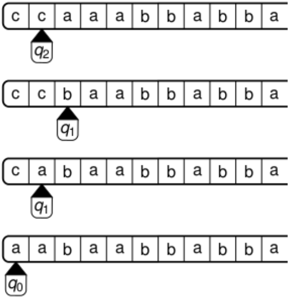
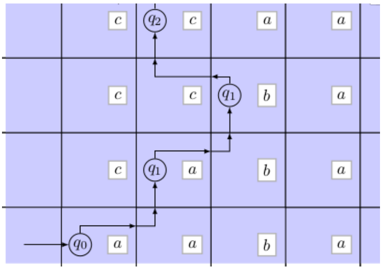
The rule $\delta(q, a) = (q', a', \leftarrow)$ will be encoded by the Wang tile



Turing machine $\mathcal{M} \rightsquigarrow$ finite set of Wang tiles $\tau_{\mathcal{M}}$

Turing machines and Wang tiles (II)





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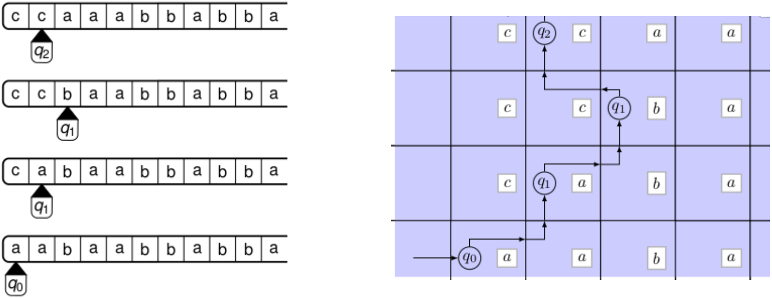

But the set of tilings by $\tau_{\mathcal{M}}$ may contain more than valid computations by $\mathcal{M}$...

Consequences for 2D SFT

From the previous encoding, one can prove that there exists no algorithm to decide whether

- a SFT is empty [Berger, 1964]
- a pattern is globally admissible [Robinson, 1971]
- an SFT has periodic configurations [Gurevich & Koryakov, 1972]

2D vs 1D sofic subshifts

1D sofic subshifts

- $X_F = \emptyset$? is decidable
- entropy is computable (nonnegative rational multiples of log of Perron numbers)
- representation by finite automata/matrix
- every SFT has a periodic configuration
- soficness $\Leftrightarrow$ finite number of followers set

2D sofic subshifts

- $X_F = \emptyset$? is undecidable
- entropy is not computable (right recursively enumerable numbers)
- representation by Wang tiles, textile systems
- $\exists$ aperiodic SFT

Outline

Projective subdynamics

Initially introduced by Johnson, Kass and Madden in 2007.

Definition

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ subshift and $L \lneq \mathbb{Z}^d$ a k-dimensional sublattice $(1 \leq k < d)$. The *L-projective subdynamics of X* is

$$P_L(X) := \{x|_L : x \in L\} \subseteq \mathcal{A}^L.$$

- $(P_L(X), \sigma_{L \times P_L(X)})$ is a $\mathbb{Z}^k$-subshift.
- $P_L(X)$: globally admissible configurations of shape L in X.
- Loss of information about the original subshift.

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In the sequel, we will concentrate on $P_{\vec{e}_1\mathbb{Z}}(X)$ (PS along the horizontal direction).

Projective subdynamics of sofic subshifts

Proposition

Projective subdynamics of SFT (sofic subshifts) are effective subshifts.

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Theorem (Hochman 2008)

Any effective $\mathbb{Z}^d$ subshift may be obtained as the projective subdynamics of a $\mathbb{Z}^{d+2}$ sofic subshift.

The proof is based on

- the use of *Turing machines as SFT*,
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Theorem (Durand, Romaschenko & Shen 2011, A.& Sablik 2013)

Any effective $\mathbb{Z}^d$-subshift may be obtained as the projective subdynamics of a $\mathbb{Z}^{d+1}$ sofic subshift.

A four layers construction

How to realize an effective 1D-subshift $\Sigma \subset {\mathcal{A}_\Sigma}^{\mathbb{Z}}$ as PS of a 2D sofic subshift ?

- SFT made of four layers
  - first layer: configuration $x \in \mathcal{A}_\Sigma^{\mathbb{Z}}$ that will be checked
  - second layer: hierarchical structure: computation zones for TM
  - third layer: TM $\mathcal{M}_{\mathtt{F}}$ that enumerates forbidden patterns of Σ and checks if $x \in \Sigma$
  - fourth layer: TM $\mathcal{M}_{\mathtt{Search}}$ that helps the TM $\mathcal{M}_{\mathtt{F}}$ to scan entirely x
- all layers but the first are finally erased with a letter-to-letter block map

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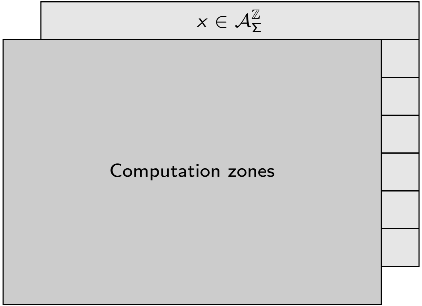

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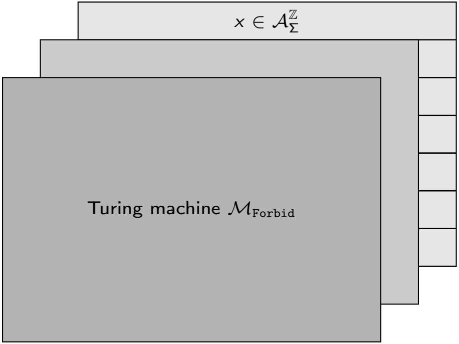

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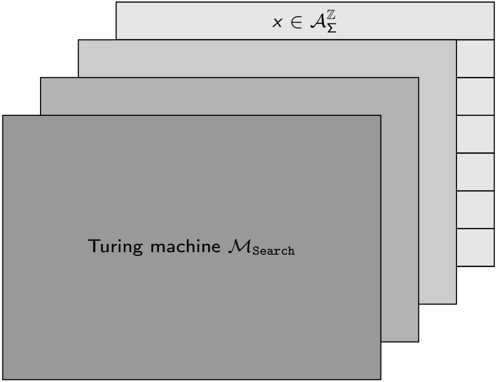

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What are PSA of SFT ?

Previous theorems completely characterize PSA of sofic subshifts (the computability obstruction is the only one).

Concerning PSA of SFT, there are only partial answers

- Complete classification of 1D sofic subshifts as PSA of 2D SFT. [Pavlov & Schraudner, preprint]
- Every $\mathbb{Z}$-effective subshift that contains a sofic subshift of positive entropy is the PSA of some $\mathbb{Z}^2$-SFT. [Guillon, 2011]
- A certain class of $\mathbb{Z}$-effective subshifts that contains a subshift of positive entropy is the PSA of some $\mathbb{Z}^2$-SFT. [Sablik & Schraudner, preprint]

Possible improvements

The construction is optimal in terms of dimension, but can we go further ?

- Is it possible to determinize the construction (deterministic SFT) ?
  $\rightsquigarrow$ It should be. . . [Guillon & Zinoviadis, in progress]

- The construction is highly constrained, in the sense that the sofic subshift is constant along the vertical direction ($\Rightarrow$ zero entropy).
  $\rightsquigarrow$ What are PS of mixing sofic subshifts/SFT ?

Conclusion

- Challenging question: characterize soficness in higher dimension.
- PSA: decrease dimension to better understand 2D subshifts.
- Example of result where computability obstruction is the only one.
- Another approach: impose that lines are in some subshift X_H, what subshift X_V can you get on the columns ?

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Thank you for your attention !