Tiling problems on Baumslag-Solitar groups.

MCU 2013

Nathalie Aubrun[1] and Jarkko Kari[2]

[1] ENS de Lyon – CNRS
[2] University of Turku (Finland)

September 9, 2013

Outline

1. Tilings on groups
   - Definition
   - Finitely presented groups

2. Classical problems
   - Existence of aperiodic tile sets
   - Domino problem

3. Baumslag-Solitar groups
   - Why are they interesting ?
   - A weakly aperiodic tile set on $BS(2,3)$

Group presentations

- generators: a, b
- relations: $a^{-1}b^{-1}ab = \varepsilon$ (or $ab = ba$)

Group presentations

- generators: a, b
- relations: $a^{-1}b^{-1}ab = \varepsilon$ (or $ab = ba$)
- elements of the group: words on the alphabet $\{a, b, a^{-1}, b^{-1}\}$
- $aba = a^2b = ba^2 = b^{-1}a^2b^2 = \dots$

Group presentations

- generators: a, b
- relations: $a^{-1}b^{-1}ab = \varepsilon$ (or $ab = ba$)
- elements of the group: words on the alphabet $\{a, b, a^{-1}, b^{-1}\}$
- $aba = a^2b = ba^2 = b^{-1}a^2b^2 = \ldots$
- $< a, b \; | ab = ba >$

Group presentations

- generators: a, b
- relations: $a^{-1}b^{-1}ab = \varepsilon$ (or $ab = ba$)
- elements of the group: words on the alphabet $\{a, b, a^{-1}, b^{-1}\}$
- $aba = a^2b = ba^2 = b^{-1}a^2b^2 = \dots$
- $< a, b \mid ab = ba > \approx \mathbb{Z}^2$

Group presentations

- generators: a, b
- relations: $a^{-1}b^{-1}ab = \varepsilon$ (or $ab = ba$)
- elements of the group: words on the alphabet $\{a, b, a^{-1}, b^{-1}\}$
- $aba = a^2b = ba^2 = b^{-1}a^2b^2 = \dots$
- $< a, b \; | ab = ba > \approx \mathbb{Z}^2 \approx \; < a, b, c \; | ab = ba, ab = c, ac = ca, bc = cb >$

Cayley graph

Representation of a group with an undirected graph:

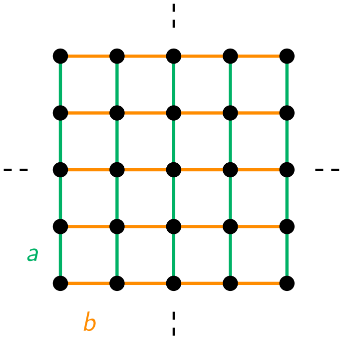
- vertices are elements of the group
- edges are labelled by the generators g_i
- an edge labelled by g_i between h and $h.g_i$

Cayley graph

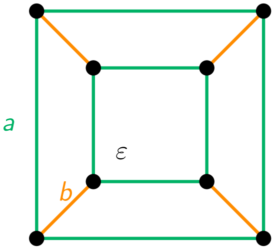
Representation of a group with an undirected graph:

- vertices are elements of the group
- edges are labelled by the generators g_i
- an edge labelled by g_i between h and $h.g_i$

$\mathbb{Z}^2 \approx < a, b / ab = ba >$



$< a, b / a^4 = b^2 = \varepsilon, b.a = a^3.b >$

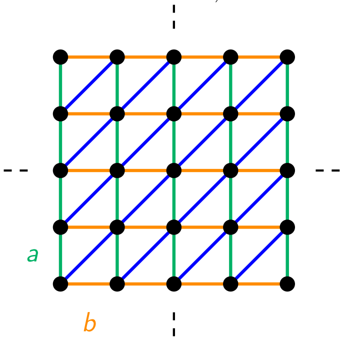


Cayley graph

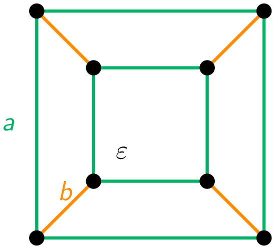
Representation of a group with an undirected graph:

- vertices are elements of the group
- edges are labelled by the generators g_i
- an edge labelled by g_i between h and $h.g_i$

$$\mathbb{Z}^2 \approx < a, b, c \; | ab = ba, ab = c, ac = ca, bc = cb >$$

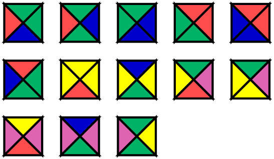


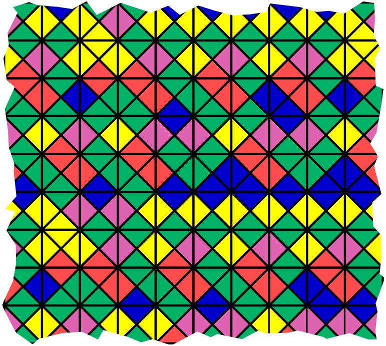
$$< a, b / a^4 = b^2 = \varepsilon, b.a = a^3.b >$$



Tilings on groups

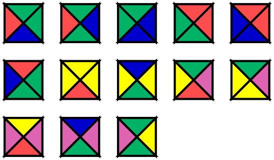
On $\mathbb{Z}^2$: Wang tiles

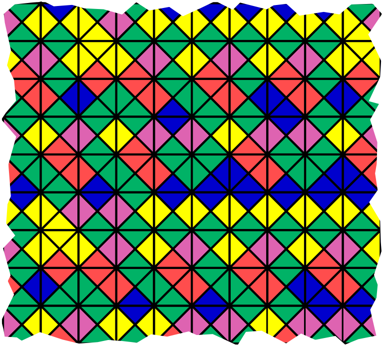




Tilings on groups

On $\mathbb{Z}^2$: Wang tiles





Generalization to a group G:

- a *tile* = pattern with one colour for each generator and each inverse ;
- finite tile set τ;
- a *configuration* (or *tiling*) $\in \tau^G$ = colouring of the Cayley graph that respects the neighbourhood rule.

Finitely presented groups

A group is finitely presented if it possesses a presentation having

- a finite number of generators ;
- a finite number of relations.

Finitely presented groups

A group is finitely presented if it possesses a presentation having

- a finite number of generators ;
- a finite number of relations.

Interest:

- structure with a finite representation. . .
- which may nevertheless be complex:

Theorem (Novikov, 1955 & Boone, 1957)

There are finitely presented groups with an undecidable word problem.

Outline

Classical problems: aperiodic tile sets

A tiling $x \in A^G$ is *m-periodic* with $m \in G$ non-trivial if

$$\forall g \in G, x_g = x_{m.g}.$$

The set of periods of a tiling x, denoted by $Per(x)$, is thus a sub-group of G.

Classical problems: aperiodic tile sets

A tiling $x \in A^G$ is *m-periodic* with $m \in G$ non-trivial if

$$\forall g \in G, x_g = x_{m.g}.$$

The set of periods of a tiling x, denoted by $Per(x)$, is thus a sub-group of G.

- x is *weakly periodic* if $Per(x)$ contains an infinite cyclic subgroup
  x is *strongly non-periodic* if it is not weakly periodic
  τ is *strongly aperiodic* if a valid tiling exists and if it admits only strongly non-periodic tilings.
- x is *strongly periodic* if $Per(x)$ is a finite index subgroup of G
  x is *weakly non-periodic* if it is not strongly periodic
  τ is *weakly aperiodic* if a valid tiling exists and if it admits only weakly non-periodic tilings.

Classical problems: aperiodic tile sets

Remarks:

- Strong aperiodicity implies weak aperiodicity.
- On $\mathbb{Z}^2$ the two notions coincide (but not on $\mathbb{Z}^3 \ldots$).

Classical problems: aperiodic tile sets

Remarks:

- Strong aperiodicity implies weak aperiodicity.
- On $\mathbb{Z}^2$ the two notions coincide (but not on $\mathbb{Z}^3 \dots$).

Question: Given a group G, is it possible to build a weakly/strongly aperiodic tile set ?

Classical problems: aperiodic tile sets

Remarks:

- Strong aperiodicity implies weak aperiodicity.
- On $\mathbb{Z}^2$ the two notions coincide (but not on $\mathbb{Z}^3 \ldots$).

Question: Given a group G, is it possible to build a weakly/strongly aperiodic tile set ?

Aperiodicity

- On free groups, every tile set has a strongly periodic configuration (compactness argument).
- There exist strongly aperiodic tile sets on $\mathbb{Z}^2$ [Ber66, Rob71].

Classical problems: domino problem

Question: Let G b a group generated by $\mathcal{G}$. Is it possible to find an algorithm that takes as input a finite set of Wang tiles τ on $\mathcal{G}$, and outputs **Yes** if and only if there exists a valid tiling by τ ?

Remark: The problem does not depend on the set of generators chosen for G.

Classical problems: domino problem

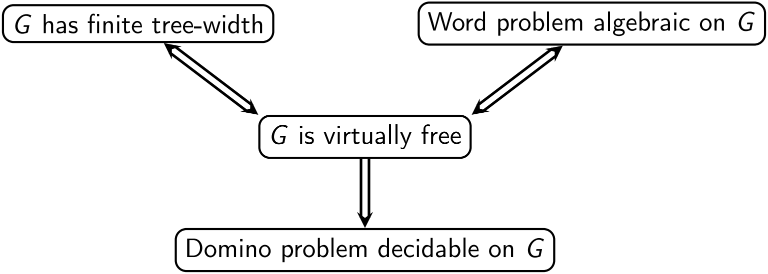
Question: Let G b a group generated by $\mathcal{G}$. Is it possible to find an algorithm that takes as input a finite set of Wang tiles τ on $\mathcal{G}$, and outputs **Yes** if and only if there exists a valid tiling by τ ?

Remark: The problem does not depend on the set of generators chosen for G.

Domino problem

- Decidable on free groups.
- Undecidable on $\mathbb{Z}^2$ [Ber66, Rob71]
- Undecidable on the hyperbolic plane [Kar07, Mar08].
- Decidable when G is virtually free [MS85] (= has a free sub-group of finite index).

Domino problem on a group



Outline

Why Baumslag-Solitar groups ?

Aim: Necessary condition on G to make the domino problem decidable ?

Why Baumslag-Solitar groups ?

Aim: Necessary condition on G to make the domino problem decidable ?

Examples of groups:

- non virtually free

Why Baumslag-Solitar groups ?

Aim: Necessary condition on G to make the domino problem decidable ?

Examples of groups:

- non virtually free (otherwise DP is decidable)

Why Baumslag-Solitar groups ?

Aim: Necessary condition on G to make the domino problem decidable ?

Examples of groups:

- non virtually free (otherwise DP is decidable)
- with decidable word problem

Why Baumslag-Solitar groups ?

Aim: Necessary condition on G to make the domino problem decidable ?

Examples of groups:

- non virtually free (otherwise DP is decidable)
- with decidable word problem (otherwise... ☹)

Definition

Baumslag-Solitar group: $\mathrm{BS}(m,n) =< a, b | a^m b = b a^n >$

Definition

Baumslag-Solitar group: $BS(m,n) =< a, b | a^m b = ba^n >$

Theorem (Magnus, 1932)

Every finitely presented group defined by a single relation has a decidable word problem.

Definition

Baumslag-Solitar group: $BS(m, n) =< a, b | a^m b = ba^n >$

Theorem (Magnus, 1932)

Every finitely presented group defined by a single relation has a decidable word problem.

Theorem (Baumslag-Solitar, 1962)

The groups $BS(m, n)$ are not virtually free.

Definition

Baumslag-Solitar group: $BS(m, n) =< a, b | a^m b = ba^n >$

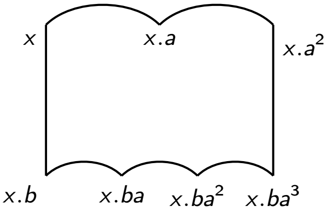
Theorem (Magnus, 1932)

Every finitely presented group defined by a single relation has a decidable word problem.

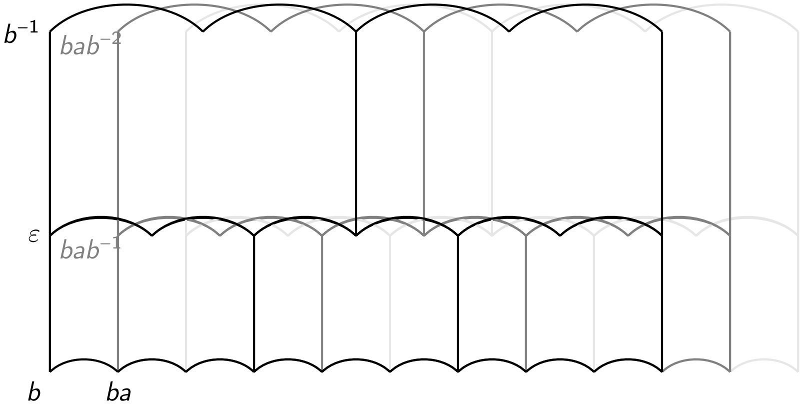
Theorem (Baumslag-Solitar, 1962)

The groups $BS(m, n)$ are not virtually free.

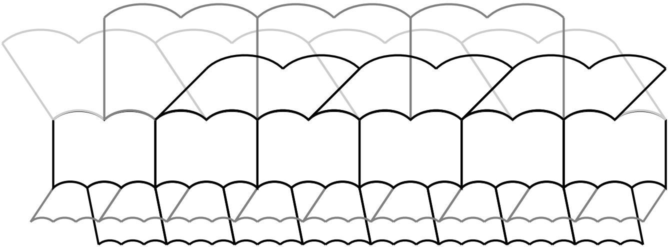
In the sequel: $BS(2, 3) =< a, b | a^2 b = ba^3 >$



Structure



Structure



Techniques known

How to build aperiodic tile sets ?

- give ad-hoc *local rules*
  $\rightsquigarrow$ strongly aperiodic tile set on $\mathbb{Z}^2$ [Rob71], $\mathbb{H}^2$ [GS10]
- use *substitutions* [Oll08] or *fixpoint theorem* [DRS09]
  $\rightsquigarrow$ gives self-similar tilings, hence strongly aperiodic tile set, but only for $\mathbb{Z}^d$ (or amenable groups)
- simulate an *aperiodic dynamical system*
  $\rightsquigarrow$ strongly aperiodic tile set on $\mathbb{Z}^2$ [Kar96], and $\mathbb{H}^2$ [Kar07]
- ...

Techniques known

How to build aperiodic tile sets ?

- give ad-hoc *local rules*
  $\rightsquigarrow$ strongly aperiodic tile set on $\mathbb{Z}^2$ [Rob71], $\mathbb{H}^2$ [GS10]
- use *substitutions* [Oll08] or *fixpoint theorem* [DRS09]
  $\rightsquigarrow$ gives self-similar tilings, hence strongly aperiodic tile set, but only for $\mathbb{Z}^d$ (or amenable groups)
- simulate an *aperiodic dynamical system*
  $\rightsquigarrow$ strongly aperiodic tile set on $\mathbb{Z}^2$ [Kar96], and $\mathbb{H}^2$ [Kar07]
- ...

How to prove the undecidability of the domino problem ?

- reduction from the Halting problem
- reduction from the immortality problem for piecewise affine maps
- ...

Techniques known

How to build aperiodic tile sets ?

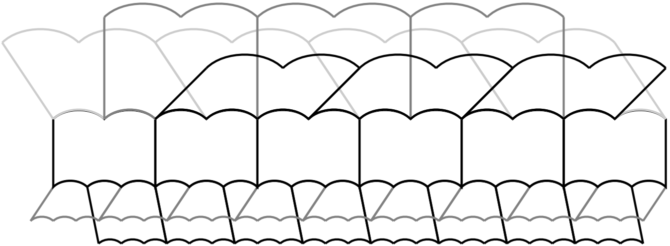
- give ad-hoc *local rules*
  $\rightsquigarrow$ strongly aperiodic tile set on $\mathbb{Z}^2$ [Rob71], $\mathbb{H}^2$ [GS10]
- use *substitutions* [Oll08] or *fixpoint theorem* [DRS09]
  $\rightsquigarrow$ gives self-similar tilings, hence strongly aperiodic tile set, but only for $\mathbb{Z}^d$ (or amenable groups)
- simulate an *aperiodic dynamical system*
  $\rightsquigarrow$ strongly aperiodic tile set on $\mathbb{Z}^2$ [Kar96], and $\mathbb{H}^2$ [Kar07]
- ...

How to prove the undecidability of the domino problem ?

- reduction from the Halting problem
- reduction from the immortality problem for piecewise affine maps
- ...

Remark: On $\mathbb{Z}^d$ the undecidability of DP implies the existence of a strongly aperiodic tile set !

Which technique on BS groups ?



- Infinitely many layers that merge infinitely often.
- Each layer is isomorphic to a tesselation of $\mathbb{H}^2$.
- But we cannot directly use the tileset of [Kar07] $\rightsquigarrow$ synchronization problems !!

An aperiodic tile set: sketch of the construction

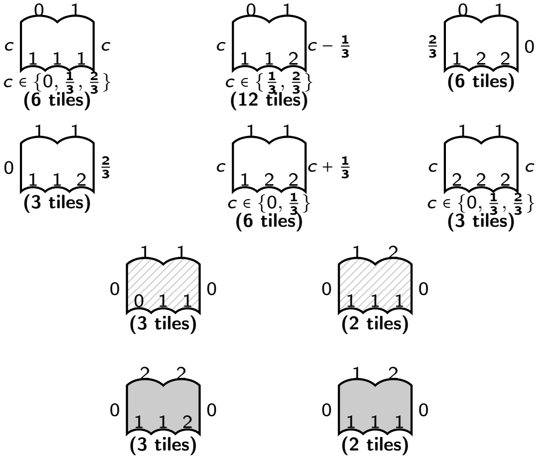
Let $T : [\frac{2}{3}; 2] \to [\frac{2}{3}; 2]$ be the piecewise linear map defined by

$$T : x \mapsto \begin{cases} 2x \text{ if } x \in [\frac{2}{3}; 1] \\ \frac{2}{3}x \text{ if } x \in]1; 2] \end{cases}$$

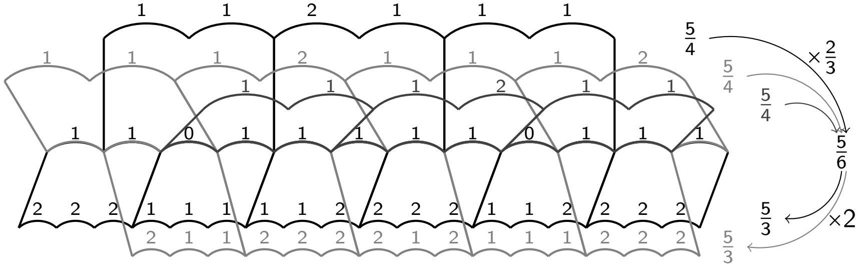
Properties

- The dynamical system T is aperiodic.
- Following [Kar07], we construct a finite tile set τ.
- There does not exist a strongly periodic valid tiling by τ.
- There exists a weakly periodic valid tiling by τ (period $\omega = bab^{-1}a^2ba^{-1}b^{-1}a^{-2}$).

The tile set τ



Example of tiling by τ



Tiling by τ corresponding to the orbit $\left(\dots, \frac{5}{4}, \frac{5}{6}, \frac{5}{3}, \dots\right)$ in T.

Tiling problems on BS groups

Theorem (A.& Kari)

There exist weakly aperiodic tile sets on $BS(m, n)$ for every $m, n > 0$.

Theorem (A.& Kari)

The domino problem is undecidable on $BS(m, n)$.

Proof: Reduction from the undecidability of the mortality problem for piecewise affine maps.

Conclusion

- $\Rightarrow$ A class of groups with undecidable domino problem...

Conclusion

- $\Rightarrow$ A class of groups with undecidable domino problem...
- but no progress about the reciprocal statement of [MS85].

Conclusion

- $\Rightarrow$ A class of groups with undecidable domino problem. . .
- but no progress about the reciprocal statement of [MS85].
- More interesting: what happens on $< a, b | ab^m = ba^n >$?

Conclusion

- $\Rightarrow$ A class of groups with undecidable domino problem...
- but no progress about the reciprocal statement of [MS85].
- More interesting: what happens on $< a, b | ab^m = ba^n >$?
- Use different characterizations of virtually free groups.

Conclusion

- $\Rightarrow$ A class of groups with undecidable domino problem...
- but no progress about the reciprocal statement of [MS85].
- More interesting: what happens on $< a, b | ab^m = ba^n >$?
- Use different characterizations of virtually free groups.

Thank you for your attention !

References

Robert Berger.
The Undecidability of the Domino Problem.
American Mathematical Society, 1966.

B. Durand, A. Romashchenko, and A. Shen.
Fixed point theorem and aperiodic tilings.
Bull. Eur. Assoc. Theor. Comput. Sci. EATCS, (97):126–136, 2009.

Chaim Goodman-Strauss.
A hierarchical strongly aperiodic set of tiles in the hyperbolic plane.
Theoretical Computer Science, 411(7-9):1085–1093, 2010.

Jarkko Kari.
A small aperiodic set of wang tiles.
Discrete Mathematics, 160(1-3):259–264, 1996.

Jarkko Kari.
The tiling problem revisited.
In *MCU*, pages 72–79, 2007.

Maurice Margenstern.
The domino problem of the hyperbolic plane is undecidable.
Theoretical Computer Science, 407(1-3):29–84, 2008.

David E. Muller and Paul E. Schupp.
The theory of ends, pushdown automata, and second-order logic.
Theoretical Computer Science, 37:51–75, 1985.