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# (Projective) Subdynamics of Multidimensional Subshifts, part I. SubTile 2013

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### Outline

### Symbolic dynamics

- Shift spaces and subshifts
- Classes of subshifts
- 2D vs 1D sofic subshifts

### Projective Subdynamics and Subactions

- Definitions
- Introductive examples
- Effective subshifts as projective subdynamics

Projective Subdynamics and Subactions

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### Full-shift, shift action and subshift

- $\mathcal{A}$  a finite alphabet and  $d \in \mathbb{N}$
- $x \in \mathcal{A}^{\mathbb{Z}^d}$  is a configuration
- $\mathcal{A}^{\mathbb{Z}^d}$  endowed with the prodiscrete topology is a compact metric space
- shift action  $\sigma : \mathbb{Z}^d \times \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ ,  $(\sigma_{(n_1,\ldots,n_d)}(x))_{(i_1,\ldots,i_d)} = x_{(i_1+n_1,\ldots,i_d+n_d)}$
- the dynamical system  $\left(\mathcal{A}^{\mathbb{Z}^d},\sigma\right)$  is the *d-dimensional full-shift on*  $\mathcal{A}$

Projective Subdynamics and Subactions

# Full-shift, shift action and subshift

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- the dynamical system  $\left(\mathcal{A}^{\mathbb{Z}^d},\sigma\right)$  is the *d-dimensional full-shift on*  $\mathcal{A}$

### Definition

A *subshift* is a closed and  $\sigma$ -invariant subset of  $\mathcal{A}^{\mathbb{Z}^d}$ .

$$\begin{cases} x \in \{0,1\}^{\mathbb{Z}^2} : x_{(i,j)} = 1 \Leftrightarrow i = j = 0 \end{cases} \text{ not } \sigma \text{-invariant } ! \\ \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \text{ only one 1 appears in } x \right\} \text{ not closed } ! \\ \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \text{ at most one 1 appears in } x \right\} \text{ is a subshift.} \end{cases}$$

# Combinatorial point of view

• A *pattern* is a local function  $p: S \to A$ , where  $S \subset \mathbb{Z}^d$  is finite.



• Given a pattern  $u \in \mathcal{A}^S$ , it generates the *cylinder* 

$$[u] = \left\{ x \in \mathcal{A}^{Z^d} : x|_S = u \right\}.$$

• If F is a set of patterns, the *subshift generated by F* is

$$X_F = \left\{ x \in \mathcal{A}^{Z^d} : \text{ no pattern of } F \text{ appears in } x 
ight\}.$$

• A subshift is thus the complement of a union of cylinders

$$X_{F} = \mathcal{A}^{\mathbb{Z}^{d}} \setminus \left( \bigcup_{\mathbf{i} \in \mathbb{Z}^{d}, u \in F} \sigma_{\mathbf{i}}([u]) \right).$$

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# Language of a subshift

#### Definition

The *language of size n* of a  $\mathbb{Z}^d$ -subshift X is

$$\mathcal{L}_n(X) := \left\{ p: [-n;n]^d 
ightarrow \mathcal{A}: \exists x \in X, p ext{ appears in } x 
ight\}.$$

The *language* of a  $\mathbb{Z}^d$ -subshift X is

$$\mathcal{L}(X) := \bigcup_{n\geq 0} \mathcal{L}_n(X).$$

The *complement of the language*  $\mathcal{L}(X)^c$  is the biggest set of forbidden patterns.

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#### Proposition

The topological and combinatorial definitions coincide.

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# Subshifts of finite type

The subshift  $X_{\{ \blacksquare, \ end \$ 



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# Subshifts of finite type



### Definition

A subshift is of finite type (SFT) if it can be defined by a finite set of forbidden patterns. It is of rank k if these finite patterns may be chosen of size k.

- simplest class for the combinatorial definition
- 2D-SFT  $\equiv$  tilings by Wang tiles
- closely related to cellular automata theory

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# Sofic subshifts

### Definition

A *sofic subshift* is the image of a SFT under a continuous and  $\sigma$ -commuting map.

continuous and  $\sigma$ -commuting map  $\Leftrightarrow$  Sliding block map (cellular automaton)  $\begin{matrix} [\mathsf{Hedlund, 1969}] \\ \Phi: \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d} \text{ given by the local function } \phi \end{matrix}$  $x \in \mathcal{A}^{\mathbb{Z}^2}$  $\Phi(x) \in \mathcal{B}^{\mathbb{Z}^2}$ 

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- SFT on which information can be erased.
- $\bullet\,$  On  $\mathbb Z,$  sofic subshifts are exactly those recognized by finite automata.

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### An example of purely sofic subshift

Let 
$$X_{\leq 1} = \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \text{ at most one } 1 \text{ appears in } x \right\}.$$

- Suppose that  $X_{\leq 1}$  is a rank k SFT.
- Then a configuration that contains two 1's at distance 2k + 1 cannot be rejected.

 $\Rightarrow X_{\leq 1}$  is not an SFT!

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# An example of non-sofic subshift

The *mirror subshift* is defined on alphabet  $\{ \Box, \blacksquare, \blacksquare \}$  by



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The *mirror subshift* is defined on alphabet  $\{ \Box, \blacksquare, \blacksquare \}$  by



Suppose  $X_{\text{mirror}}$  is sofic. Then  $\exists \Sigma \subset A^{\mathbb{Z}^2}$  a *k*-SFT and  $\Pi$  a block map of order *r*, such that

$$\Pi: \Sigma \to X_{\text{mirror}}$$
 is onto.



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 is onto.

 $|A|^{4nr+8nk+4r^2} < 2^{n^2}$ 



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# Effectively closed subshifts

 $\mathsf{SFT} \subsetneq \mathsf{Sofic \ susbhifts} \subsetneq \textit{Effectively \ closed}$ 

#### Definition

A subshift is *effectively closed* (or *effective*) if its complement is a computable union of cylinders.

#### Property

X is effectively closed if and only one of the followings holds (i)  $X = X_F$  for some recursively enumerable set F of forbbiden patterns (ii)  $X = X_F$  for some recursive set F of forbbiden patterns

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**Remark:** There exist non effectively closed subshifts (countability argument).

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# Turing machines and SFT (I)

- A *Turing machine* is a tuple  $\mathcal{M} = (Q, \Gamma, \sharp, q_0, \delta, Q_F)$  where:
  - Q is a finite set of states,  $q_0 \in Q$  is the initial state;
  - Γ is a finite alphabet;
  - # ∉ Γ blank symbol
  - $\delta: Q \times \Gamma \to Q \times \Gamma \times \{\leftarrow, \downarrow, \rightarrow\}$  transition function;
  - $F \subset Q_F$  finite set of final states.

The rule  $\delta(q_1, x) = (q_2, y, \leftarrow)$  will be encoded by the pattern

$z \leftarrow q_2$	у	z′
Z	$x \leftarrow q_1$	z′

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# Turing machines and SFT (II)

 $\mathcal{M}$  Turing machine  $\rightsquigarrow$  finite set of patterns  $F_{\mathcal{M}} \rightsquigarrow$  SFT  $X_{F_{\mathcal{M}}}$ 



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# 2D vs 1D sofic subshifts

### 1D sofic subshifts

- ►  $X_F = \emptyset$ ? is decidable
- entropy is computable (nonnegative rational multiples of log of Perron numbers)
- representation by finite automata/matrix
- every SFT has a periodic configuration
- ► soficness ⇔ finite number of followers set

### 2D sofic subshifts

- ►  $X_F = \emptyset$ ? is undecidable
- entropy is not computable (right recursively enumerable numbers)
- representation by Wang tiles, textile systems

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► ∃ aperiodic SFT

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### Necessary conditions for soficness in 2D

• If X is a minimal subshift with positive entropy, then X is not sofic. [Desai, 2006]

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### Necessary conditions for soficness in 2D

• If X is a minimal subshift with positive entropy, then X is not sofic. [Desai, 2006]

• If X is effective and if the Kolmogorov complexity of every  $p \in \mathcal{L}_n(X)$  is greater than  $\mathcal{O}(n)$ , then X is not sofic. [Durand, Romaschenko & Shen, 2008]

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• Too many extender sets implies non-soficness. [Kass & Madden 2013] and [Pavlov, 2013]

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### Outline

### Symbolic dynamics

- Shift spaces and subshifts
- Classes of subshifts
- 2D vs 1D sofic subshifts

### Projective Subdynamics and Subactions

- Definitions
- Introductive examples
- Effective subshifts as projective subdynamics

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# Projective Subdynamics

Initially introduced by Johnson, Kass and Madden in 2007.

#### Definition

Let  $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$  be a  $\mathbb{Z}^d$  subshift and  $L \lneq \mathbb{Z}^d$  a k-dimensional sublattice  $(1 \leq k < d)$ . The *L-projective subdynamics of* X is

$$P_L(X) := \{x|_L : x \in L\} \subseteq \mathcal{A}^L.$$

- $(P_L(X), \sigma_{L \times P_L(X)})$  is a  $\mathbb{Z}^k$ -subshift.
- $P_L(X)$ : globally admissible configurations of shape L in X.
- Loss of information about the original subshift.

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In the sequel, we will concentrate on  $P_{\vec{e}_1\mathbb{Z}}(X)$  (PS along the horizontal direction).

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# Entropy and PS

Proposition (Johnson, Kass & Madden, 2007)

$$h_{top}(P_{\vec{e}_1\mathbb{Z}}(X)) \geq h_{top}(X).$$

**Proof:** 

$$h_{top}(X) = \lim_{n \to \infty} \frac{1}{n^2} \log \left( |\mathcal{L}_n(X)| \right)$$
$$= \lim_{n \to \infty} \frac{1}{n^2} \log \left( |\mathcal{L}_n(P_{\vec{e}_1 \mathbb{Z}}(X))|^n \right)$$
$$\leq \lim_{n \to \infty} \frac{1}{n} \log \left( |\mathcal{L}_n(P_{\vec{e}_1 \mathbb{Z}}(X))| \right)$$
$$= h_{top}(P_{\vec{e}_1 \mathbb{Z}}(X))$$

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### Subdynamics

### Definition

Let  $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$  be a  $\mathbb{Z}^d$  subshift and  $Y \subseteq B^{\mathbb{Z}^k}$  a  $\mathbb{Z}^k$ -subshift  $(1 \le k < d)$ . Then Y is a *subaction of* X if the dynamical systems  $(X, \sigma|_{\mathbb{Z}^k})$  and  $(Y, \sigma|_{\mathbb{Z}^k})$  are isomorphic.

- Much stronger than projective subdynamics
- The subshift Y is defined on a possibly non-finite alphabet
- No loss of information

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- What are projective subdynamics of 2D sofic subshifts
- What are projective subdynamics of 2D SFT ?
- What are subactions of sofic subshifts ?
- What are subactions of 2D SFT ?

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### Questions

- What are projective subdynamics of 2D sofic subshifts = effective subshifts
- What are projective subdynamics of 2D SFT ? ???
- What are 1D subactions of 3D sofic subshifts ? = effective dynamical systems
- What are subactions of 2D SFT ? ???

Projective Subdynamics and Subactions

# What can be PS of sofic subshifts ? (0)

Trivially, every 1D sofic subshift...

$_{rg} \in \overline{\Sigma}$
x <sub>18</sub> ∈ Σ
$x_{17} \in \Sigma$
x <sub>16</sub> ∈ Σ
$x_{15} \in \Sigma$
$x_{14} \in \Sigma$
$x_{13} \in \Sigma$
$x_{12} \in \Sigma$
$x_{11} \in \Sigma$
x <sub>10</sub> ∈ Σ
$x_{\mathbf{g}} \in \Sigma$
$x_8 \in \Sigma$
$x_7 \in \Sigma$
$x_6 \in \Sigma$
$X_{F} \leq \Sigma$

SFT  $\Sigma^{\mathbb{Z}}$ 

 $X\subset A^{\mathbb{Z}}$  sofic  $\Sigma\subset B^{\mathbb{Z}}$  SFT,  $\Pi:\Sigma o X$  block map

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SFT  $\Sigma^{\mathbb{Z}}$ 

Projective Subdynamics and Subactions

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Trivially, every 1D sofic subshift...

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$\sqcap(x_{\texttt{18}}) \in X$
$\sqcap(x_{17}) \in X$
$\Pi(x_{16}) \in X$
$\Pi(x_{15}) \in X$
$\sqcap(\mathbf{x_{14}}) \in \mathbf{X}$
$\sqcap(x_{13}) \in X$
$\Pi(\mathbf{x_{12}}) \in \mathbf{X}$
$\Pi(\mathbf{x_{11}}) \in \mathbf{X}$
$\Pi(x_{10}) \in X$
$\sqcap(x_{9}) \in X$
$\sqcap(x_8) \in X$
$\sqcap(x_7) \in X$
$\sqcap(x_6) \in X$
$\Pi(x_5) \in Y$

$$X \subset A^{\mathbb{Z}}$$
 sofic  
 $\Sigma \subset B^{\mathbb{Z}}$  SFT,  $\Pi : \Sigma \to X$  block map

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Projective Subdynamics and Subactions

# What can be PS of sofic subshifts ? (0)

Trivially, every 1D sofic subshift...

	$\Pi(x_{19}) \in X$
	$\Pi(x_{18}) \in X$
	$\Pi(x_{17}) \in X$
	$\Pi(\mathbf{x_{16}}) \in \mathbf{X}$
	$\Pi(x_{15}) \in X$
	$\Pi(x_{14}) \in X$
	$\Pi(\mathbf{x_{13}}) \in \mathbf{X}$
SFT $\Sigma^{\mathbb{Z}}$	$\Pi(\mathbf{x_{12}}) \in \mathbf{X}$
	$\Pi(x_{\texttt{11}}) \in X$
	$\Pi(\mathbf{x_{10}}) \in \mathbf{X}$
	$\sqcap(x_{\mathbf{g}}) \in X$
	$\sqcap(x_{8}) \in X$
	$\sqcap(x_{7}) \in X$
	$\sqcap(x_{6}) \in X$
	$\Pi(\neg\neg) \in X$

 $X \subset A^{\mathbb{Z}}$  sofic  $\Sigma \subset B^{\mathbb{Z}} \text{ SFT, } \Pi: \Sigma \to X \text{ block map}$ 

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### Conjecture (Jeandel)

X is sofic  $\Leftrightarrow X^{\mathbb{Z}}$  is sofic.

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# What can be PS of sofic subshifts ? (I)

▶ The 1D subshift  $X_{a^n b^n}$ .

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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	C
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	(
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	C
٥	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	(
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	C
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Projective Subdynamics and Subactions

# What can be PS of sofic subshifts ? (I)

▶ The 1D subshift  $X_{a^n b^n}$ . And even a subaction !



Projective Subdynamics and Subactions

# What can be PS of sofic subshifts ? (II)

▶ The 1D subshift  $X_{a^n b^n c^n}$  (neither sofic nor algebraic).

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Projective Subdynamics and Subactions

# What can be PS of sofic subshifts ? (II)

▶ The 1D subshift  $X_{a^n b^n c^n}$ . And even a subaction !

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 $\mathcal{A}^{\mathbb{Z}^2}$ 

Projective Subdynamics and Subactions

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# What can be PS of sofic subshifts ? (III)

$x_{19} \in \mathcal{A}^{*}$	
$x_{18} \in \mathcal{A}^{\mathbb{Z}}$	
$x_{17} \in \mathcal{A}^{\mathbb{Z}}$	
$\mathbf{x_{16}} \in \mathcal{A}^{\mathbb{Z}}$	
$x_{15} \in \mathcal{A}^{\mathbb{Z}}$	
$x_{14} \in A^{\mathbb{Z}}$	
$x_{13} \in A^{\mathbb{Z}}$	
$x_{12} \in A^{\mathbb{Z}}$	
$x_{11} \in A^{\mathbb{Z}}$	
$x_{10} \in A^{\mathbb{Z}}$	
$x_{9} \in \mathcal{A}^{\mathbb{Z}}$	
$x_{8} \in \mathcal{A}^{\mathbb{Z}}$	
$x_7 \in \mathcal{A}^{\mathbb{Z}}$	
$x_6 \in \mathcal{A}^{\mathbb{Z}}$	
$\mathbf{x}_{\mathbf{E}} \in \mathcal{A}^{\mathbb{Z}}$	

Projective Subdynamics and Subactions

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# What can be PS of sofic subshifts ? (III)

 $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$  $x \in A^{\mathbb{Z}}$  $x \in \mathcal{A}^{\mathbb{Z}}$  $x \in A^{\mathbb{Z}}$  $x \in \mathcal{A}^{\mathbb{Z}}$  $x \in \mathcal{A}^{\mathbb{Z}}$  $x \in A^{\mathbb{Z}}$  $x \in \mathcal{A}^{\mathbb{Z}}$  $x \in \mathcal{A}^{\mathbb{Z}}$  $x \in \mathcal{A}^{\mathbb{Z}}$  $x \in \mathcal{A}^{\mathbb{Z}}$ 



Projective Subdynamics and Subactions

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# What can be PS of sofic subshifts ? (III)



Projective Subdynamics and Subactions

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Projective Subdynamics and Subactions

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Projective Subdynamics and Subactions

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# What can be PS of sofic subshifts ? (III)

▶ Any effective subshift *X* that contains a uniform configuration.

On the third layer:

- The Turing Machine  $\mathcal{M}$  works on the first tape and enumerates forbidden patterns for X (initialization thanks to the second layer).
- Each time a forbidden patterns is produced, it is copied out on the two other tapes.
- Patterns written on the second (resp. third) tape are shifted to the left (resp. right) at each step of computation.

Projective Subdynamics and Subactions

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- If a pattern written on the two last tapes matches with the corresponding pattern in x, then the configuration is forbidden (intercation by local rules with the first layer).
- If a forbidden pattern for X appears in x, it will eventually be detected and the configuration is rejected.
- If  $x \in X$ , the configuration is accepted.

Projective Subdynamics and Subactions

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# What can be PS of sofic subshifts ? (III)



Projective Subdynamics and Subactions

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Projective Subdynamics and Subactions

# What can be PS of sofic subshifts ? (III)

▶ Any effective subshift *X* that contains a uniform configuration.

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$\infty_a \infty$
$\infty_a \infty$
$\mathbf{x} \in \mathbf{X}$
$\infty_a \infty$
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 $\Rightarrow X$  is a PS of a sofic subshift

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### Projective Subdynamics and Subactions

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# Hochman's result

### Theorem (Hochman 2008)

- Any effective  $\mathbb{Z}^d$ -subshift may be obtained as the subaction of a  $\mathbb{Z}^{d+2}$  sofic subshift.
- Any effective Z<sup>d</sup> dynamical system may be obtained as the subaction of a Z<sup>d+2</sup> sofic subshift.

The proof is based on

- the use of Turing machines as SFT,
- substitutive tilings to construct computation zones in 3D.

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# Conclusion of Part I

- Challenging question: characterize soficness in higher dimension.
- Projective subdynamics and subaction: decrease dimension to better understand 2D subshifts.
- Complete characterization of PS/subactions of sofic subshifts (Hochman)
- Coming soon:
  - Sketch of Hochman's proof...
  - that can be improved to dimension d + 1 !
  - Some other results about PS/subactions of SFT

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# Conclusion of Part I

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# Thank you for your attention !