# Selfish Resource Allocation in Optical Networks*

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**Abstract.** We introduce Colored Resource Allocation Games as a new model for selfish routing and wavelength assignment in multifiber all-optical networks. Colored Resource Allocation Games are a generalization of congestion and bottleneck games where players have their strategies in multiple copies (colors). We focus on two main subclasses of these games depending on the player cost: in Colored Congestion Games the player cost is the sum of latencies of the resources allocated to the player, while in Colored Bottleneck Games the player cost is the maximum of these latencies. We investigate the pure price of anarchy for three different social cost functions and prove tight bounds for each separate case. We first consider a social cost function which is particularly meaningful in the setting of multifiber all-optical networks, where it captures the objective of fiber cost minimization. Additionally, we consider the two usual social cost functions (maximum and average player cost) and obtain improved bounds that could not have been derived using earlier results for the standard models for congestion and bottleneck games.

## 1 Introduction

Potential games are a widely used tool for modeling network optimization problems under a non-cooperative perspective. Initially studied in [1] with the introduction of congestion games and further extended in [2] in a more general framework, they have been successfully applied to describe selfish routing in communication networks (e.g. [3]). The advent of optical networks as the technology of choice for surface communication has introduced new aspects of networks that are not sufficiently captured by the models proposed so far. In this work, we propose a class of potential games which are more suitable for modeling selfish routing and wavelength assignment in multifiber optical networks.

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In optical networks, it is highly desirable that all communication should be carried out *transparently*, that is, each signal should remain on the same wavelength from source to destination. The need for efficient access to the optical bandwidth has given rise to the study of several optimization problems in the past years. The most well-studied among them is the problem of assigning a path and a color (wavelength) to each communication request in such a way that paths of the same color are edge-disjoint and the number of colors used is minimized. Nonetheless, it has become clear that the number of wavelengths in commercially available fibers is rather limited—and will probably remain such in the foreseeable future. Therefore, the use of multiple fibers has become inevitable in large scale networks. In the context of multifiber optical networks several optimization problems have been defined and studied, the objective usually being to minimize either the maximum fiber multiplicity per edge or the sum of these maximum multiplicities over all edges of the graph.

### 1.1 Contribution

We introduce Colored Resource Allocation Games, a class of games that can model non-cooperative versions of routing and wavelength assignment problems in multifiber all-optical networks. They can be viewed as an extension of congestion games where each player has his strategies in multiple copies (colors). When restricted to (optical) network games, facilities correspond to edges of the network and colors to wavelengths. The number of players using an edge in the same color represents a lower bound on the number of fibers needed to implement the corresponding physical link. Having this motivation in mind, we consider the case in which each player's cost is equal to the *maximum* edge congestion encountered on her path (*max* player cost), as well as the case in which each player's cost is equal to the *sum* of edge congestions encountered on her path (*sum* player cost). For our purposes of using Colored Resource Allocation games to model resource allocation in optical networks, it makes sense to restrict our study to the class of identity latency functions.

We use the price of anarchy (PoA) introduced in [4] as a measure of the deterioration of the quality of solutions caused by the lack of coordination. We estimate the price of anarchy of our games under three different social cost functions. The first one ($SC_{\text{fib}}$) is specially designed for the setting of multifiber all-optical networks: it is equal to the sum over all facilities of the maximum color congestion on each facility. Note that in the optical network setting this function represents the total fiber cost needed to accommodate all players; hence, it captures the objective of a well-studied optimization problem ([5–8]). The other two social cost functions are standard in the literature (see e.g. [9]): the first ($SC_{\max}$) is equal to the maximum player cost and the second ($SC_{\text{sum}}$) is equal to the sum of player costs (equivalently, the average player cost).

Let us also note that the $SC_{\max}$ function under the *max* player cost captures the objective of another well known problem, namely minimizing the maximum fiber multiplicity over all edges of the network [7, 10, 11]. In addition, note that our model admits a number of different interpretations as discussed in [12].

**Table 1.** The pure price of anarchy of Colored Bottleneck Games (*max* player cost) under different social costs. Results for classical bottleneck games are shown in the right column.

| | Colored Bottleneck Games | Bottleneck Games |
|---|---|---|
| $SC_{\mathrm{fib}}(A) = \sum_{f \in F} \max_{a \in [W]} n_{f,a}(A)$ | $\frac{\lvert E_A \rvert}{\lvert E_{\mathrm{OPT}} \rvert} \left\lceil \frac{N}{W} \right\rceil$ | — |
| $SC_{\max}(A) = \max_{i \in [N]} C_i(A)$ | $\Theta\left(\frac{N}{W}\right)$ | $\Theta(N)$ [13] |
| $SC_{\mathrm{sum}}(A) = \sum_{i \in [N]} C_i(A)$ | $\Theta\left(\frac{N}{W}\right)$ | $\Theta(N)$ [13] |

**Table 2.** The pure price of anarchy of Colored Congestion Games (*sum* player cost) under different social costs. Results for classical congestion games are shown in the right column.

| | Colored Congestion Games | Congestion Games |
|---|---|---|
| $SC_{\mathrm{fib}}(A) = \sum_{f \in F} \max_{a \in [W]} n_{f,a}(A)$ | $\Theta\left(\sqrt{W \lvert F \rvert}\right)$ | — |
| $SC_{\max}(A) = \max_{i \in [N]} C_i(A)$ | $\Theta\left(\sqrt{\frac{N}{W}}\right)$ | $\Theta\left(\sqrt{N}\right)$ [9] |
| $SC_{\mathrm{sum}}(A) = \sum_{i \in [N]} C_i(A)$ | $\frac{5}{2}$ | $\frac{5}{2}$ [9] |

Our main contribution is the derivation of tight bounds on the price of anarchy for Colored Resource Allocation Games. These bounds are summarized in Tables 1 and 2. It can be shown that the bounds for Colored Congestion Games remain tight even for network games.

Observe that known bounds for classical congestion and bottleneck games can be obtained from our results by simply setting $W = 1$. On the other hand, one might notice that our games can be casted as classical congestion or bottleneck games with $W\,\lvert F \rvert$ facilities. However we are able to derive better upper bounds for most cases by exploiting the special structure of the players' strategies.

### 1.2 Related Work

One of the most important solution concepts in the theory of non-cooperative games is the *Nash equilibrium* [14], a stable state of the game in which no player has incentive to change strategy unilaterally. A fundamental question in this theory concerns the existence of *pure* Nash equilibria. For congestion and bottleneck games [1, 2, 15] it has been shown with the use of potential functions that they converge to a pure Nash equilibrium.

In [16] Roughgarden introduces a canonical framework for studying the price of anarchy; in particular he identifies the following canonical sufficient condition, which he calls the "smoothness condition":

$$\sum_{i=1}^{n} C_i(A_i^*, A_{-i}) \leq \lambda SC(A^*) + \mu SC(A) \ .$$

The key idea is that, by showing that a game is $(\lambda, \mu)$-smooth, i.e. that it satisfies the condition above for some choice of $\lambda$ and $\mu$, we immediately get an upper bound of $\frac{\lambda}{1-\mu}$ on the price of anarchy of the game. Hence, bounding the price of anarchy reduces to the problem of identifying $\lambda$ and $\mu$ which minimize the aforementioned quantity, and for which the game is $(\lambda, \mu)$-smooth. From the games and welfare functions that we analyze only colored congestion games from the perpsective of $SC_{\text{sum}}$ are smooth, a property implied by the existing analysis of Christodoulou et al [9] and which we show remains tight even in our setting. On the contrary, our other two social cost functions and our bottleneck game analysis do not seem to admit a similar smoothness argument, and therefore a different approach is required in order to upper bound the price of anarchy for these settings.

Bottleneck games have been studied in [13, 15, 17, 18]. In [13] the authors study atomic routing games on networks, where each player chooses a path to route her traffic from an origin to a destination node, with the objective of minimizing the maximum congestion on any edge of her path. They show that these games always possess at least one optimal pure Nash equilibrium (hence the price of stability is equal to 1) and that the price of anarchy of the game is determined by topological properties of the network. A further generalization is the model of Banner and Orda [15], where they introduce the notion of bottleneck games. In this model they allow arbitrary latency functions on the edges and consider both splittable and unsplittable flows. They show existence, convergence and non-uniqueness of equilibria and they prove that the price of anarchy for these games is exponential in the users' demand.

Since bottleneck games traditionally have price of anarchy that is rather high (proportional to the size of the network in many cases), in [19] the authors study bottleneck games when the utility functions of the players are exponential functions of their congestion, and they show that for this class of *exponential bottleneck games* the price of anarchy is in fact logarithmic. Finally [20] investigate the computational problem of finding a pure Nash equilibrium in bottleneck games, as well as the performance of some natural (de-centralized) improvement dynamics for finding pure Nash equilibria.

Selfish path coloring in single fiber all-optical networks has been studied in [21–24]. Bilò and Moscardelli [21] consider the convergence to Nash equilibria of selfish routing and path coloring games. Bilò et al. [22] consider several information levels of local knowledge that players may have and give bounds for the price of anarchy in chains, rings and trees. The existence of Nash equilibria and the complexity of recognizing and computing a Nash equilibrium for selfish routing and path coloring games under several payment functions are considered by Georgakopoulos et al. [23]. In [24] upper and lower bounds for the price of anarchy of selfish path coloring with and without routing are presented under functions that charge a player only according to her own strategy.

Selfish path multicoloring games are introduced in [12] where it is proved that the pure price of anarchy is bounded by the number of available colors and by the length of the longest path; constant bounds for the price of anarchy in specific topologies are also provided. In those games, in contrast to the ones studied here, routing is given in advance and players choose only colors.

## 2 Model Definition

We use the notation $[X]$ for the set $\{1, \ldots, X\}$, where $X$ is a positive natural number.

**Definition 1 (Colored Resource Allocation Games).** *A Colored Resource Allocation Game is defined as a tuple $\langle F, N, W, \{\mathcal{E}_i\}_{i \in [N]} \rangle$ such that:*

1. *$F$ is a set of facilities $f_i$.*
2. *$[W]$ is a set of colors.*
3. *$[N]$ is a set of players.*
4. *$\mathcal{E}_i$ is a set of possible facility combinations for player $i$ such that:*
   a. *$\forall\, i \in [N] : \mathcal{E}_i \subseteq 2^F$,*
   b. *$S_i = \mathcal{E}_i \times [W]$ is the set of possible strategies of player $i$, and*
   c. *$A_i = (E_i, a_i) \in S_i$ is the notation of a strategy for player $i$, where $E_i \in \mathcal{E}_i$ denotes the set of facilities and $a_i \in [W]$ denotes the color chosen by the player.*
5. *$A = (A_1, \ldots, A_N)$ is a strategy profile for the game.*
6. *For a strategy profile $A$, $\forall f \in F$, $\forall c \in [W]$, $n_{f,c}(A)$ is the number of players that use facility $f$ in color $c$ in strategy profile $A$.*

*Depending on the player cost function we define two subclasses of Colored Resource Allocation Games:*

- *Colored Bottleneck Games (CBG), where the player cost is*

$$C_i(A) = \max_{e \in E_i} n_{e,a_i}(A) \ .$$

- *Colored Congestion Games (CCG), where the player cost is*

$$C_i(A) = \sum_{e \in E_i} n_{e,a_i}(A) \ .$$

For each of the above variations we will consider three different social cost functions:

- $SC_{\text{fib}}(A) = \sum_{f \in F} \max_{c \in [W]} n_{f,c}(A)$.
- $SC_{\max}(A) = \max_{i \in [N]} C_i(A)$.
- $SC_{\text{sum}}(A) = \sum_{i \in [N]} C_i(A)$. Note that, in the case of CCG games, the sum social cost can also be expressed as $SC_{\text{sum}}(A) = \sum_{f \in F} \sum_{c \in [W]} n_{f,c}^2(A)$.

From the definition of pure Nash equilibrium we can derive the following two facts that hold in Colored Congestion and Bottleneck Games respectively:

**Fact 1.** *For a pure Nash equilibrium $A$ of a CCG game it holds:*

$$\forall E_i' \in \mathcal{E}_i, \forall c' \in [W] : C_i(A) \leq \sum_{e \in E_i'} (n_{e,c'}(A) + 1) \ . \tag{1}$$

**Fact 2.** *For a pure Nash equilibrium $A$ of a CBG game it holds:*

$$\forall E_i' \in \mathcal{E}_i, \forall c' \in [W] : C_i(A) \leq \max_{e \in E_i'} (n_{e,c'}(A) + 1) \ . \tag{2}$$

*Equivalently:*

$$\forall E_i \in \mathcal{E}_i, \forall c \in [W], \exists e \in E_i : C_i(A) \leq n_{e,c}(A) + 1 \ . \tag{3}$$

In the rest of the paper, we will only deal with pure Nash equilibria and we will refer to them simply as Nash equilibria.

## 3 Colored Bottleneck Games

By a standard lexicographic argument, one can show that every CBG game has at least one pure Nash equilibrium and that the price of stability [25] is 1.

### 3.1 Price of Anarchy for Social Cost $SC_{\text{fib}}$

**Definition 2.** *We define $E_S$ to be the set of facilities used by at least one player in the strategy profile $S = (A_1, \ldots, A_N)$, i.e., $E_S = E_1 \cup \ldots \cup E_N$.*

**Theorem 1.** *The price of anarchy of any CBG game with social cost $SC_{\text{fib}}$ is at most $\frac{|E_A|}{|E_{\text{OPT}}|} \left\lceil \frac{N}{W} \right\rceil$, where $A$ is a worst-case Nash equilibrium and* OPT *is an optimal strategy profile.*

*Proof.* We exclude from the sum over the facilities, those facilities that are not used by any player since they do not contribute to the social cost. Thus we focus on facilities with $\max_c n_{e,c} > 0$. Let $A$ be a worst-case Nash equilibrium and let $c_{\max}(e)$ denote the color with the maximum multiplicity at facility $e$. Let $P_i$ be a player that uses the facility copy $(e, c_{\max}(e))$. Since $C_i(A) = \max_{e \in E_i} n_{e,a_i}(A)$ it must hold that $n_{e,c_{\max}(e)}(A) \leq C_i(A)$. In fact, we can state the following general property:

$$\forall e \in F, \ \exists i \in [N] : n_{e,c_{\max}(e)} \leq C_i(A) \ . \tag{4}$$

Suppose that there exists a player with cost $\left\lceil \frac{N}{W} \right\rceil + 1$ or more. From Fact 2, at least $\left\lceil \frac{N}{W} \right\rceil$ players must play each of the other colors. By a simple calculation, this implies that there are at least $N+1$ players in the game, a contradiction. We conclude that each player's cost is at most $\lceil \frac{N}{W} \rceil$, thus $C_i(A) \leq \lceil \frac{N}{W} \rceil$. Moreover, it is easy to see that $SC_{\text{fib}}(\text{OPT}) \geq |E_{\text{OPT}}|$. From the above we can conclude:

$$\frac{SC_{\text{fib}}(A)}{SC_{\text{fib}}(\text{OPT})} \leq \frac{|E_A|}{|E_{\text{OPT}}|} \left\lceil \frac{N}{W} \right\rceil \quad . \tag{5}$$

□

**Theorem 2.** *There exists a class of CBG games with social cost $SC_{\text{fib}}$ with* $\text{PoA} = \frac{|E_A|}{|E_{\text{OPT}}|} \lceil \frac{N}{W} \rceil$.

*Proof.* Consider a game in which each player $i$ has the following strategy set: $\mathcal{E}_i = \{\{f_i\}, \{f_1, \ldots, f_M\}\}$, where $M \geq N \geq W$. In the worst-case Nash equilibrium $A$, all players will play the second strategy leading to $SC_{\text{fib}}(A) = M \lceil \frac{N}{W} \rceil = |E_A| \lceil \frac{N}{W} \rceil$. On the other hand in the optimal outcome all players will play the first strategy leading to $SC_{\text{fib}}(\text{OPT}) = N = |E_{\text{OPT}}|$. Thus the price of anarchy for this instance is $\text{PoA} = \frac{|E_A|}{|E_{\text{OPT}}|} \lceil \frac{N}{W} \rceil$. □

### 3.2 Price of Anarchy for Social Cost $SC_{\max}$

**Theorem 3.** *The price of anarchy of any CBG game with social cost $SC_{\max}$ is at most* $\lceil \frac{N}{W} \rceil$.

*Proof.* It is easy to see that $SC_{\max}(\text{OPT}) \geq 1$. We established in the proof of Theorem 1 that the maximum player cost in a Nash equilibrium is $\lceil \frac{N}{W} \rceil$. Therefore, for any worst-case Nash equilibrium $A$, $SC_{\max}(A) \leq \lceil \frac{N}{W} \rceil$. □

**Theorem 4.** *There exists a class of CBG games with social cost $SC_{\max}$ with* $\text{PoA} = \lceil \frac{N}{W} \rceil$.

*Proof.* Consider the following class of CBG games. We have $N$ players and $N$ facilities. Each player $P_i$ has two possible strategies: $\mathcal{E}_i = \{\{f_i\}, \{f_1, \ldots, f_N\}\}$. In a worst-case Nash equilibrium, all players choose the second strategy and they are equally divided in the W colors. This leads to player cost $\lceil \frac{N}{W} \rceil$ for each player and thus to a social cost $\lceil \frac{N}{W} \rceil$. In the optimal strategy profile, all players would choose their first strategy leading to player and social cost equal to 1. Thus the price of anarchy for this instance is $\lceil \frac{N}{W} \rceil$. □

### 3.3 Price of Anarchy for Social Cost $SC_{\text{sum}}$

**Theorem 5.** *The price of anarchy of any CBG game with social cost $SC_{\text{sum}}$ is at most* $\lceil \frac{N}{W} \rceil$.

*Proof.* As before, we know that the maximum player cost in a Nash equilibrium is $\lceil \frac{N}{W} \rceil$, therefore the social cost is at most $N \cdot \lceil \frac{N}{W} \rceil$. Moreover, $SC_{\text{sum}}(\text{OPT}) \geq N$. Thus the price of anarchy is bounded by $\lceil \frac{N}{W} \rceil$. □

The instance used in the previous section can also be used here to prove that the above inequality is tight for a class of CBG games.

## 4 Colored Congestion Games

### 4.1 Price of Anarchy for Social Cost $SC_{\mathrm{fib}}$

**Theorem 6.** *The price of anarchy of any CCG game with social cost* $SC_{\mathrm{fib}}$ *is at most* $\mathcal{O}\left(\sqrt{W\,|F|}\right)$.

*Proof.* We denote by $\overline{n_e(S)}$ the vector $[n_{e,c_1}(S), \ldots, n_{e,c_W}(S)]$. We can rewrite the social cost as $SC_{\mathrm{fib}}(S) = \sum_{e \in F} \max_{c \in [W]} n_{e,c}(S) = \sum_{e \in F} \|\overline{n_e(S)}\|_\infty$. From norm inequalities, we have:

$$\frac{\|\overline{n_e(S)}\|_2}{\sqrt{W}} \leq \|\overline{n_e(S)}\|_\infty \leq \|\overline{n_e(S)}\|_2 \ , \tag{6}$$

hence:

$$SC_{\mathrm{fib}}(S) = \sum_{e \in F} \|\overline{n_e(S)}\|_\infty \leq \sum_{e \in F} \sqrt{\sum_c n_{e,c}^2(S)} \leq \sqrt{|F|} \sqrt{\sum_{e \in F} \sum_c n_{e,c}^2(S)} \ , \tag{7}$$

where the last inequality is a manifestation of the norm inequality $\|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2$, where $\boldsymbol{x}$ is a vector of dimension $n$. Now, from the first inequality of (6) we have:

$$SC_{\mathrm{fib}}(S) \geq \frac{1}{\sqrt{W}} \sum_{e \in F} \sqrt{\sum_c n_{e,c}^2(S)} \geq \frac{1}{\sqrt{W}} \sqrt{\sum_{e \in F} \sum_c n_{e,c}^2(S)} \ . \tag{8}$$

Combining (8) and (7), we get:

$$\frac{1}{\sqrt{W}} \sqrt{SC_{\mathrm{sum}}(S)} \leq SC_{\mathrm{fib}}(S) \leq \sqrt{|F|} \sqrt{SC_{\mathrm{sum}}(S)} \ . \tag{9}$$

From [9] we know that the price of anarchy with social cost $SC_{\mathrm{sum}}(S)$ is at most $5/2$. Let $A$ be a worst-case Nash equilibrium under social cost $SC_{\mathrm{fib}}$ and let OPT be an optimal strategy profile. From (9) we know that $SC_{\mathrm{fib}}(A) \leq \sqrt{|F|}\sqrt{SC_{\mathrm{sum}}(A)}$ and $SC_{\mathrm{fib}}(\mathrm{OPT}) \geq \frac{1}{\sqrt{W}} \sqrt{SC_{\mathrm{sum}}(\mathrm{OPT})}$. Thus:

$$\mathrm{PoA} = \frac{SC_{\mathrm{fib}}(A)}{SC_{\mathrm{fib}}(\mathrm{OPT})} \leq \sqrt{W\,|F|} \sqrt{\frac{SC_{\mathrm{sum}}(A)}{SC_{\mathrm{sum}}(\mathrm{OPT})}} \leq \sqrt{W\,|F|} \sqrt{\frac{5}{2}} \ . \tag{10}$$

□

**Theorem 7.** *There exists a class of CCG games with social cost* $SC_{\mathrm{fib}}$ *with* $\mathrm{PoA} = \sqrt{W\,|F|}$.

*Proof.* Consider a colored congestion game with N players, $|F| = N$ facilities and $W = N$ colors. Each player has as strategies the singleton sets consisting of one facility: $\mathcal{E}_i = \{\{f_1\}, \{f_2\}, \ldots, \{f_N\}\}$.

The above instance has a worst-case equilibrium with social cost N when all players choose a different facility in an arbitrary color. On the other hand in the optimum strategy profile players fill all colors of the necessary facilities. This needs $\frac{N}{W}$ facilities with maximum capacity over their colors 1. Thus the optimum social cost is $\frac{N}{W}$ leading to a $\mathrm{PoA} = \sqrt{W\,|F|}$. □

### 4.2 Price of Anarchy for Social Cost $SC_{\max}$

**Theorem 8.** *The price of anarchy of any CCG game with social cost $SC_{\max}$ is at most $\mathcal{O}\left(\sqrt{\frac{N}{W}}\right)$.*

*Proof.* Let $A$ be a Nash equilibrium and let OPT be an optimal strategy profile. Without loss of generality, we assume that player 1 is a maximum cost player: $SC_{\max}(A) = C_1(A)$. Thus, we need to bound $C_1(A)$ with respect to the optimum social cost $SC_{\max}(\text{OPT}) = \max_{j\in[N]} C_j(\text{OPT})$.

Since A is a Nash equilibrium, no player benefits from changing either her color or her choice of facilities. We denote by $\text{OPT}_1 = (E_1^\star, a_1^\star)$ the strategy of player $P_1$ in OPT. Since $A$ is a Nash equilibrium it must hold:

$$\forall c \in [W]: \; C_1(A) \leq \sum_{e\in E_1^\star} (n_{e,c}(A) + 1) \leq \sum_{e\in E_1^\star} n_{e,c}(A) + C_1(\text{OPT}) \; . \tag{11}$$

The second inequality holds since any strategy profile cannot lead to a cost for a player that is less than the size of her facility combination.

Let $I \subset [N]$ be the set of players that, in $A$, use some facility $e \in E_1^\star$. The sum of their costs is:

$$\begin{aligned} \sum_{i\in I} C_i(A) \geq \sum_{e\in E_1^\star} \sum_{c\in[W]} n_{e,c}^2(A) \geq \frac{(\sum_{e\in E_1^\star}\sum_{c\in[W]} n_{e,c}(A))^2}{|E_1^\star|W} \geq \\ \frac{(W \min_{c\in[W]} \sum_{e\in E_1^\star} n_{e,c}(A))^2}{|E_1^\star|W} \geq \frac{W(\min_{c\in[W]}\sum_{e\in E_1^\star} n_{e,c}(A))^2}{|E_1^\star|} \; . \end{aligned} \tag{12}$$

The first inequality holds since a player in $I$ might use facilities $(e, c)$ not in $E_1^\star$ and the second inequality holds from the Cauchy-Schwarz inequality. Denoting by $c_{\min}$ the color $\arg\min_{c\in[W]} \sum_{e\in E_1^\star} n_{e,c}(A)$, we have:

$$\left(\sum_{e\in E_1^\star} n_{e,c_{\min}}(A)\right)^2 \leq \frac{|E_1^\star|}{W} \sum_{i\in I} C_i(A) \; . \tag{13}$$

We know from [9] that:

$$\sum_{i\in[N]} C_i(A) \leq \frac{5}{2} \sum_{i\in[N]} C_i(\text{OPT}) \; . \tag{14}$$

Combining the above two inequalities we have:

$$\left(\sum_{e\in E_1^\star} n_{e,c_{\min}}(A)\right)^2 \leq \frac{|E_1^\star|}{W}\sum_{i\in I} C_i(A) \leq \frac{|E_1^\star|}{W}\sum_{i\in[N]} C_i(A) \leq \frac{5}{2}\frac{|E_1^\star|}{W}\sum_{i\in[N]} C_i(\text{OPT}) \tag{15}$$

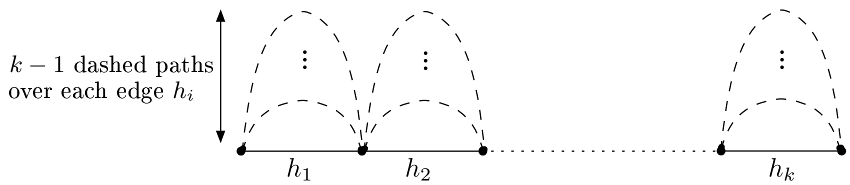


**Fig. 1.** A worst-case instance that proves the asymptotic tightness of the upper bound on the price of anarchy of CCG games with social cost $SC_{\max}$, depicted as a network game. A dashed line represents a path of length $k$ connecting its two endpoints.

Combining with (11) for $c_{\min}$, we get

$$C_1(A) \leq C_1(\mathrm{OPT}) + \sqrt{\frac{5}{2}\frac{|E_1^\star|}{W}\sum_{i\in[N]} C_i(\mathrm{OPT})} \ . \tag{16}$$

Since $|E_1^\star| \leq C_1(\mathrm{OPT})$ and $C_i(\mathrm{OPT}) \leq SC_{\max}(\mathrm{OPT})$ for any $i \in [N]$, we get

$$C_1(A) \leq \left(1 + \sqrt{\frac{5}{2}\frac{N}{W}}\right) SC_{\max}(\mathrm{OPT}) \ . \tag{17}$$

□

**Theorem 9.** *There exists a class of CCG games with social cost* $SC_{\max}$ *with* $\mathrm{PoA} = \Theta\left(\sqrt{\frac{N}{W}}\right)$.

*Proof.* Given integers $k > 1$ and $W > 0$, we will describe the lower bound instance as a network game. The set of colors is $[W]$. The network consists of a path of $k+1$ nodes $n_0, \ldots, n_k$. In addition, each pair of neighboring nodes $n_i, n_{i+1}$ is connected by $k-1$ edge-disjoint paths of length $k$. Figure 1 provides an illustration.

In this network, $W$ major players want to send traffic from $n_0$ to $n_k$. For every $i$, $0 \leq i \leq k-1$, there are $(k-1)W$ minor players that want to send traffic from node $n_i$ to node $n_{i+1}$. In the worst-case equilibrium $A$ all players choose the short central edge, leading to social cost $SC_{\max}(A) = k^2$. In the optimum the minor players are equally divided on the dashed-line paths and the major players choose the central edge. This leads to $SC_{\max}(\mathrm{OPT}) = k$, and the price of anarchy is therefore:

$$\mathrm{PoA} = k = \Theta\left(\sqrt{\frac{N}{W}}\right) \ . \tag{18}$$

□

### 4.3 Price of Anarchy for Social Cost $SC_{\mathrm{sum}}$

The price of anarchy of CCG games with social cost $SC_{\mathrm{sum}}$ is upper-bounded by 5/2, as proved in [9]. For the lower bound, we use a slight modification of the instance described in [9]. We have $NW$ players and $2N$ facilities. The facilities are separated into two groups: $\{h_1, \ldots, h_N\}$ and $\{g_1, \ldots, g_N\}$. Players are divided into N groups of W players. Each group $i$ has strategies $\{h_i, g_i\}$ and $\{g_{i+1}, h_{i-1}, h_{i+1}\}$. The optimal allocation is for all players in the $i$-th group to select their first strategy and be equally divided in the $W$ colors, leading to $SC_{\mathrm{sum}}(\mathrm{OPT}) = 2NW$. In the worst-case Nash equilibrium, players choose their second strategy and are equally divided in the $W$ colors, leading to $SC_{\mathrm{sum}}(A) = 5NW$. Thus, the price of anarchy of this instance is 5/2 and the upper bound remains tight in our model as well.

## 5 Discussion

In this paper we introduced Colored Resource Allocation Games, a class of games which generalize both congestion and bottleneck games. The main feature of these games is that players have their strategies in multiple copies (colors). Therefore, these games can serve as a framework to describe routing and wavelength assignment games in multifiber all-optical networks. Although we could cast such games as classical congestion games, it turns out that the proliferation of resources together with the structure imposed on the players' strategies allows us to prove better upper bounds.

Regarding open questions, it would be interesting to consider more general latency functions. This would make sense both in the case where fiber pricing is not linear in the number of fibers, and also in the case where the network operator seeks to determine an appropriate pricing policy so as to reduce the price of anarchy. Another interesting direction is to examine which network topologies result in better system behavior.

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