Minimum multiplicity edge coloring via orientation

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Abstract

We study an edge coloring problem in multigraphs, in which each node incurs a cost equal to the number of appearances of the most frequent color among those received by its incident edges. We seek an edge coloring with a given number w of colors, that minimizes the total cost incurred by the nodes of the multigraph. We consider a class of approximation algorithms for this problem, which are based on orienting the edges of the multigraph, then grouping appropriately the incoming and outgoing edges at each node so as to construct a bipartite multigraph of maximum degree w, and finally obtaining a proper edge coloring of this bipartite multigraph. As shown by Nomikos et al. (Inform. Process. Lett. 80 (2001) 249-256), simply choosing an arbitrary edge orientation in the first step yields a 2-approximation algorithm. We investigate whether this approximation ratio can be improved by a more careful choice of the edge orientation in the first step. We prove that, assuming a worst-case bipartite edge coloring, this is not possible in the asymptotic sense, as there exists a family of instances in which any orientation gives a solution with cost at least $2 - \Theta\left(\frac{1}{w}\right)$ times the optimal. On the positive side, we show how to produce an orientation which results in a

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solution with cost within a factor of $2 - \frac{1}{2^w}$ of the optimal, thus yielding an approximation ratio strictly better than 2. This improvement is important in view of the fact that this graph-theoretic problem models, among others, wavelength assignment to communication requests in multifiber optical star networks. In this context, the parameter w corresponds to the number of available wavelengths per fiber, which is limited in practice due to technological constraints.

Keywords: edge coloring, path multicoloring, edge orientation, color multiplicity, optical networks, approximation algorithms

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1. Introduction

Let $G = (V, E)$ be an undirected multigraph without self-loops. Given a coloring of its edges, let $\mu(v, c)$ denote the number of edges incident to v that have received color c and let $\mu(v) = \max_c \mu(v, c)$. We will call $\mu(v, c)$ the *multiplicity of c at v* and $\mu(v)$ the *multiplicity of v*. In the Minimum Multiplicity Edge Multicoloring problem (MinMult-EMC), one seeks an edge coloring with a given number of colors, that minimizes the sum of node multiplicities. Formally:

Problem 1 (MinMult-EMC)**.**
Instance*: $\langle G, w\rangle$, where $G = (V, E)$ is an undirected multigraph and $w \in \mathbb{N}$ is the number of available colors.*
Feasible solution*: a coloring of E with w colors.*
Goal*: minimize $\sum_{v \in V} \mu(v)$.*

There is a large literature on edge coloring, which typically considers the problem from the point of view of minimizing the number of colors used, under various constraints imposed on the obtained coloring. To the best of our knowledge, the MinMult-EMC problem, which has a different objective function, has not been studied as such in the literature. However, in view of the diverse applications of edge coloring in domains such as job scheduling, routing, network resource allocation, etc. [1–5], it is not surprising that MinMult-EMC appears and has, in fact, been considered implicitly in the context of wavelength allocation in multifiber optical networks [6].

We recall some known results and we make some preliminary observations on MinMult-EMC in Section 1.1.

Algorithm 1 A 2-approximation algorithm for MinMult-EMC [6]

Input: an instance $\langle G, w \rangle$ of MinMult-EMC, $G = (V, E)$

Output: a 2-approximate solution

1: Assign an arbitrary direction to each edge of G.

2: For each $v \in V$, group its d_v^{out} outgoing edges into $\left\lceil \frac{d_v^{\text{out}}}{w} \right\rceil$ groups of at most w edges each, and let V_{out} denote the set of all groups of outgoing edges. Similarly, for each $v \in V$, group its d_v^{in} incoming edges into $\left\lceil \frac{d_v^{\text{in}}}{w} \right\rceil$ groups of at most w edges each, and let V_{in} denote the set of all groups of incoming edges.

3: Construct the bipartite multigraph $H = (V_{\text{out}} \cup V_{\text{in}}, A)$, where for each edge in E, A contains one edge joining its outgoing group to its incoming group. The maximum degree of H is bounded by w.

4: Compute a proper edge coloring of H with w colors.

5: Assign to each edge of G the color of the corresponding edge in H.

Notation. Throughout the paper, d_v will denote the degree of a node v in an undirected multigraph, whereas for directed multigraphs we will use d_v^{in} (resp. d_v^{out}) for the in-degree (resp. out-degree) of node v. An *orientation* of an undirected multigraph is a directed multigraph in which each edge $\{u, v\}$ is replaced by one of the arcs (u, v) or (v, u). If G is a graph or a multigraph, $V(G)$ is the node set of G and $E(G)$ is the edge set of G. For $k \geq 2$, C_k denotes the undirected cycle of size k and K_k denotes the clique of size k. We use the binary operation $a \bmod b$ for positive integers a, b, which gives the remainder of the division a/b. If A is an event in a suitable sample space, then $\mathbb{P}[A]$ denotes the probability of A.

1.1. Preliminaries

Fact 1. *Under any edge coloring with w colors, the multiplicity of each node v is at least $\left\lceil \frac{d_v}{w} \right\rceil$, thus the minimum cost for any* MinMult-EMC *instance is at least $\sum_{v \in V} \left\lceil \frac{d_v}{w} \right\rceil$.*

For any fixed $w \geq 3$, MinMult-EMC is NP-hard via a straightforward reduction from the decision version of the classical edge coloring problem on w-regular graphs, which is known to be NP-complete [7, 8]. Nomikos et al. [6] propose a 2-approximation algorithm which we restate as Algorithm 1 in MinMult-EMC terms (the algorithm was originally stated in terms of wavelength allocation in multifiber optical networks). The analysis in [6] is tight, as there exists a family of instances in which Algorithm 1 computes a solution with cost exactly twice the optimum: $\{\langle C_k, w\rangle : \text{even } k \geq 2 \text{ and } w \geq 2\}$. Indeed, if the directions assigned in step 1 are such that each node has in-degree 1 and out-degree 1, then the resulting bipartite multigraph H will contain k edges that can all be colored with the same color. Translated to the original instance, this induces a cost of 2 for each node for a total cost of $2k$, whereas the optimum solution has cost k by coloring the edges with alternating colors around the cycle.

Definition 1. *Let $\langle G, w\rangle$ be an instance of* MinMult-EMC *and fix an orientation of G. We say that a node v is* locally optimal *if the following condition holds:*

$$\big(d_v^{\text{in}} \bmod w = 0\big) \vee \big(d_v^{\text{out}} \bmod w = 0\big) \vee \big((d_v^{\text{in}} \bmod w) + (d_v^{\text{out}} \bmod w) > w\big)$$

The pertinence of locally optimal nodes is revealed by the following lemma, which is implicit in the analysis in [6].

Lemma 2 ([6])**.** *In any solution computed by Algorithm 1, each node v incurs a cost of exactly $\lceil \frac{d_v}{w} \rceil$ if it is locally optimal with respect to the directions assigned during step 1, or at most $\lceil \frac{d_v}{w} \rceil + 1$ if it is not locally optimal.*

In other words, Algorithm 1 incurs an additional cost, with respect to the lower bound of Fact 1, of at most one for each non-locally-optimal node. In fact, as we prove in Section 2 (Lemma 3), for every orientation of the given graph and for every edge grouping that can be chosen in steps 1 and 2 of Algorithm 1, there exists a worst-case proper edge coloring of the resulting bipartite multigraph (step 4 of Algorithm 1) that causes *every* non-locally-optimal node v to contribute a cost of exactly $\lceil \frac{d_v}{w} \rceil + 1$.

If $w = 2$, then the problem can be solved exactly in polynomial time: The Euler partition algorithm in [9] computes a partition of the edges of a multigraph into open and closed paths, with the property that each vertex of odd degree is the extremity of exactly one open path, and each vertex of even degree is the extremity of no open paths. Note, then, that if we color the edges of each path of the Euler partition alternately with the two available colors, the resulting coloring will have the property that the edges incident to each even-degree node will be partitioned into two color classes of equal size, whereas the edges incident to each odd-degree node will be partitioned into two color classes whose sizes differ by exactly one. This implies that the cost incurred by each node v will be exactly $\lceil \frac{d_v}{2} \rceil$, thus the solution is optimal in view of Fact 1. We have thus proved the following:

Theorem 1. *There exists an exact polynomial-time algorithm for* MinMult-EMC *with two available colors.*

1.2. Our contributions

We consider the class of approximation algorithms for MinMult-EMC, in which one computes some orientation of the given multigraph G and then executes steps 2–5 of Algorithm 1. In particular, we investigate the possibility of improving the approximation ratio of Algorithm 1 by choosing a more sophisticated orientation of G in step 1. However, we show in Section 2 that there exists an infinite family of instances in which no orientation whatsoever can guarantee a solution with cost smaller than $2 - \Theta\left(\frac{1}{w}\right)$ times the optimal, under the assumption that step 4 of Algorithm 1 may produce a worst-case edge coloring.

On the positive side, we show in Section 3 how to compute an orientation that yields an approximation ratio of $2 - \frac{1}{2^w}$. This represents an improvement with respect to the algorithm from [6], which has a tight approximation ratio of 2.

Lastly, we explain in Section 4 how our results for MinMult-EMC can be applied to a wavelength allocation problem in multifiber optical star networks. Our algorithm from Section 3 yields a $\left(2 - \frac{1}{2^w}\right)$-approximation algorithm for the wavelength allocation problem, which improves the previously best ratio of 2. It should be noted that, in this context, w represents the number of wavelengths (or optical frequencies) available in each optical fiber in the system and this number is limited in practice due to technological constraints. Therefore, an approximation ratio of $2 - \frac{1}{2^w}$ represents an appreciable improvement in practical terms.

2. The lower bound

Before we present the lower bound, we prove a general lemma concerning the worst-case behavior of Algorithm 1 for non-locally-optimal nodes.

Lemma 3. *Let $\langle G, w \rangle$ be a* MinMult-EMC *instance with $G = (V, E)$, G' be an arbitrary orientation of G, and $H = (V_{\text{out}} \cup V_{\text{in}}, A)$ be a bipartite multigraph as constructed in steps 2 and 3 of Algorithm 1. There exists a proper edge coloring of H with w colors, such that, after every edge in E receives the color of the corresponding edge in A in step 5 of Algorithm 1, each non-locally-optimal node $v \in V$ incurs a cost of $\left\lceil \frac{d_v}{w} \right\rceil + 1$.*

Proof. Let $V^\star$ be the set of non-locally-optimal nodes of G under the orientation G'. For each $v \in V^\star$, let $V_{\text{in}}(v) \subseteq V_{\text{in}}$ (resp. $V_{\text{out}}(v) \subseteq V_{\text{out}}$) be the set of $\lceil \frac{d_v^{\text{in}}}{w} \rceil$ groups of incoming (resp. $\lceil \frac{d_v^{\text{out}}}{w} \rceil$ groups of outgoing) edges of v that are created in step 2 of Algorithm 1 and let $V'_{\text{in}}(v) = \{g \in V_{\text{in}}(v) : |g| \leq w-1\}$ (resp. $V'_{\text{out}}(v) = \{g \in V_{\text{out}}(v) : |g| \leq w-1\}$). Also, let $d_v = a_v w + x_v$, where $a_v \in \mathbb{N}$ and $0 \leq x_v \leq w-1$. Since v is not locally optimal, d_v^{in} and d_v^{out} are not integer multiples of w and also $(d_v^{\text{in}} \bmod w) + (d_v^{\text{out}} \bmod w) \leq w$. We write $d_v^{\text{in}} = b_v w + y_v$, where $b_v \in \mathbb{N}$ and $1 \leq y_v \leq w-1$. For the out-degree, we have $d_v^{\text{out}} = d_v - d_v^{\text{in}} = (a_v - b_v)w + (x_v - y_v)$, and since d_v^{out} is not an integer multiple of w, it must hold that $x_v \neq y_v$.

We construct a new bipartite multigraph $H' = (V'_{\text{in}} \cup V'_{\text{out}}, A \cup A')$, where $V'_{\text{in}} = V_{\text{in}} \cup \{s_v : v \in V^\star\}$, $V'_{\text{out}} = V_{\text{out}} \cup \{t_v : v \in V^\star\}$, and A' contains the following new edges for each $v \in V^\star$:

- for each $g \in V'_{\text{in}}(v)$, $w - |g|$ parallel edges between g and t_v,
- for each $g \in V'_{\text{out}}(v)$, $w - |g|$ parallel edges between g and s_v, and
- the edge $\{s_v, t_v\}$.

Note that the degree of each $g \in V_{\text{in}}(v)$ in H' is w and, therefore, a total of $\lceil \frac{d_v^{\text{in}}}{w} \rceil w$ edges are incident to nodes in $V_{\text{in}}(v)$. Of those, $\lceil \frac{d_v^{\text{in}}}{w} \rceil w - d_v^{\text{in}} = w - (d_v^{\text{in}} \bmod w) = w - y_v$ are newly added edges incident to nodes in $V'_{\text{in}}(v)$, whose other endpoint is t_v. The degree of t_v in H' is, therefore, $1+w-y_v \leq w$.

Similarly, the degree of each $g \in V_{\text{out}}(v)$ in H' is w and a total of $\lceil \frac{d_v^{\text{out}}}{w} \rceil w$ edges are incident to nodes in $V_{\text{out}}(v)$. Of those, $\lceil \frac{d_v^{\text{out}}}{w} \rceil w - d_v^{\text{out}} = w - (d_v^{\text{out}} \bmod w)$ are newly added edges incident to nodes in $V'_{\text{out}}(v)$, whose other endpoint is s_v. The degree of s_v in H' is, therefore, $1 + w - (d_v^{\text{out}} \bmod w)$. If $x_v > y_v$, we have $(d_v^{\text{out}} \bmod w) = x_v - y_v$ and therefore the degree of s_v is at most $1 + w - x_v + y_v \leq w$. On the other hand, if $x_v < y_v$, we have $(d_v^{\text{out}} \bmod w) = w + x_v - y_v$ and therefore the degree of s_v is at most $1 - x_v + y_v \leq w$.

We conclude that the maximum degree of H' is bounded by w and thus there exists a proper edge coloring of H' with w colors. We claim that the restriction of this edge coloring to the edges of H is a proper edge coloring of H with the desired property. Indeed, for $v \in V^\star$, let c_v be the color assigned to the edge $\{s_v, t_v\}$. Since in H' the degree of every node in $V_{\text{in}}(v) \cup V_{\text{out}}(v)$ is w, each of these nodes has an incident edge that is colored with c_v. However, that edge cannot be any of the edges in A', since every edge in A'

that is incident to a node in $V_{\text{in}}(v) \cup V_{\text{out}}(v)$ is also incident to one of the nodes s_v, t_v and thus it cannot be colored with c_v. Therefore, when we restrict the coloring to the edges of H, each node in $V_{\text{in}}(v) \cup V_{\text{out}}(v)$ has an incident edge that is colored with c_v. This implies that, in the edge coloring of G that is returned by the algorithm, $\mu(v, c_v) = |V_{\text{in}}(v) \cup V_{\text{out}}(v)| = \left\lceil \frac{d_v^{\text{in}}}{w} \right\rceil + \left\lceil \frac{d_v^{\text{out}}}{w} \right\rceil$. We will show that $\mu(v, c_v) = \left\lceil \frac{d_v}{w} \right\rceil + 1$, which concludes the proof.

If $x_v > y_v$, then we have $x_v > 0$ and therefore $\left\lceil \frac{d_v}{w} \right\rceil = a_v + 1$. We also have $\mu(v, c_v) = b_v + 1 + (a_v - b_v + 1) = a_v + 2 = \left\lceil \frac{d_v}{w} \right\rceil + 1$.

If $x_v < y_v$, then $\mu(v, c_v) = b_v + 1 + a_v - b_v = a_v + 1$. It remains to show that $\left\lceil \frac{d_v}{w} \right\rceil = a_v$, or, equivalently, that $x_v = 0$. Indeed, from the fact that v is not locally optimal, we have $(d_v^{\text{in}} \bmod w) + (d_v^{\text{out}} \bmod w) \leq w$, which can be rewritten as $y_v + w + x_v - y_v \leq w$, therefore $x_v \leq 0$. □

We are now ready to show the lower bound. For $k \geq 2$, consider the MinMult-EMC instance $\mathcal{I}_k = \langle K_{2k}, 2k-1 \rangle$. In the optimal solution for $\mathcal{I}_k$, K_{2k} is properly edge colored with $w = 2k - 1$ colors, so that the incident edges to any node receive distinct colors. With this edge coloring, every node contributes a cost of $1 = \left\lceil \frac{d_v}{w} \right\rceil$ to the cost of the solution, therefore by Fact 1 this coloring must be an optimal solution with cost $\text{OPT}_k = 2k$.

On the other hand, we argue that, under every possible orientation, K_{2k} has at least $2k - 2$ non-locally-optimal nodes. Indeed, the in-degree and the out-degree of each node are between 0 and w (inclusive) and their sum is equal to w. Therefore, none of the nodes can satisfy the local optimality condition $(d_v^{\text{in}} \bmod w) + (d_v^{\text{out}} \bmod w) > w$. Furthermore, if one of them is locally optimal because, say, $d_v^{\text{out}} = w$, then at most one other can be locally optimal (by having $d_{v'}^{\text{in}} = w$). Symmetrically, if one of them has $d_v^{\text{in}} = w$, then at most one other can be locally optimal (by having $d_{v'}^{\text{out}} = w$). We conclude that at most 2 nodes can be locally optimal and at least $2k - 2$ will be non-locally-optimal.

By Lemma 3, for every orientation of K_{2k} and for every possible edge grouping constructed in step 2 of Algorithm 1, there exists an edge coloring of the corresponding bipartite multigraph (step 4) that results in a solution in which each non-locally-optimal node incurs a cost of 2, whereas by Fact 1 each of the rest incurs a cost of at least 1. In total, the solution cost is $\text{SOL}_k \geq 2k + (2k - 2) = 4k - 2$.

We thus have

$$\frac{\text{SOL}_k}{\text{OPT}_k} \geq \frac{4k - 2}{2k} = 2 - \frac{1}{k} \ .$$

Since $w = \Theta(k)$ in this family of instances, we finally have the following:

Theorem 2. *There exists an infinite family of* MinMult-EMC *instances with increasing number of available colors w in which, for every orientation and for every edge grouping computed in steps 1 and 2 of Algorithm 1, there exists a proper edge coloring of the corresponding bipartite multigraph in step 4, such that Algorithm 1 returns a solution with cost at least $2-\Theta\left(\frac{1}{w}\right)$ times the optimal.*

3. Edge orientation to approximate MinMult-EMC

As a first observation, consider the simplest conceivable randomized algorithm for MinMult-EMC, i.e., the one that assigns a random color with uniform probability to each edge. Unfortunately, this algorithm performs quite poorly in the following family of instances: The multigraph contains two nodes with w parallel edges joining them, where w is the number of available colors. As is well known, the maximum multiplicity color will appear $\Theta\left(\frac{\log w}{\log \log w}\right)$ times with high probability [10]. On the other hand, one obtains an optimum solution with cost 2 simply by assigning a different color to each edge. This motivates us to search for a randomized algorithm with a better performance guarantee.

We now consider the algorithm that assigns a random direction to each edge of G and then executes steps 2-5 of Algorithm 1. We first show in Lemma 6 that, given a MinMult-EMC instance $\langle G, w\rangle$, the assignment of random directions to the edges of G yields in expectation at least $\frac{1}{2^w} \cdot n$ locally optimal nodes. Then, we show in Theorem 3 how to derandomize this procedure in order to obtain a deterministic algorithm for MinMult-EMC with approximation ratio at most $2 - \frac{1}{2^w}$.

Definition 2. *$S(d, w)$ denotes the set of integers i such that, if exactly i out of d incident edges to a d-degree node are incoming and $(d-i)$ incident edges are outgoing, then the node is locally optimal assuming we have w colors.*

Lemma 4. *If d is a multiple of w, then $S(d, w) = \{i \cdot w : 0 \leq i \leq \frac{d}{w}\}$.*

Proof. Since every $k \in \{i \cdot w : 0 \leq i \leq \frac{d}{w}\}$ is a multiple of w, a node with k incoming edges and $(d-k)$ outgoing edges is clearly locally optimal.

Now, consider some $k \notin \{i \cdot w : 0 \leq i \leq \frac{d}{w}\}$, i.e., $0 \leq k \leq d$ and $k = t \cdot w + y$, where $0 < y < w$. Since d is a multiple of w, neither k nor $d-k$ are multiples of w. Moreover, $(k \bmod w) + \big((d-k) \bmod w\big) = y + (w-y) = w$. Therefore, a node with k incoming edges and $(d-k)$ outgoing edges cannot be locally optimal. $\square$

Lemma 5. *If $d = r \cdot w + x$, where $0 < x < w$, then $S(d,w)$ is exactly the set: $\bigcup_{i=0}^{r} \big\{ i \cdot w + j : j \in \{0, x, x+1, \ldots, w-1\} \big\} \cap \{0, 1, \ldots, d\}$.*

Proof. Let $k = i \cdot w + j$, where $j \in \{0, x, x+1, \ldots, w-1\}$. If $j = 0$, then k is a multiple of w. If $j = x$, then $d - k = (r-i) \cdot w$. Finally, if $x + 1 \leq j \leq w-1$, then $(k \bmod w) + \big((d-k) \bmod w\big) = j + \big(w - (j-x)\big) > w$. Therefore, in all cases, a node with k incoming edges and $(d-k)$ outgoing edges is locally optimal.

Now, consider some k such that $k = i \cdot w + j$, where $1 \leq j \leq x - 1$. We have that k is not a multiple of w, $(d-k) \bmod w = (x-j) \bmod w > 0$, and $(k \bmod w) + ((d-k) \bmod w) = j + (x-j) = x < w$. Therefore, a node with k incoming edges and $(d-k)$ outgoing edges cannot be locally optimal. $\square$

Lemma 6. *Let $\langle G, w \rangle$ be an instance of* MinMult-EMC *and let $\vec{G}$ be a random orientation of G in which each edge receives each of the two possible directions with probability $\frac{1}{2}$. The expected number of locally optimal nodes in $\vec{G}$ is at least $\frac{1}{2^w} \cdot n$.*

Proof. For any fixed node v, let X_v denote the indicator random variable which takes the value 1 if v is locally optimal, and takes the value 0 otherwise. Note that, by linearity of expectation, the expected number of locally optimal nodes in $\vec{G}$ is exactly $\sum_{v \in V} \mathbb{P}[X_v = 1]$. We now prove that $\mathbb{P}[X_v = 1] \geq \frac{1}{2^w}$.

By Definition 2, v is locally optimal if and only if $d_v^{\text{in}} \in S(d,w)$, where d is the degree of v. Therefore, $\mathbb{P}[X_v = 1] = \sum_{s \in S(d,w)} \mathbb{P}\left[d_v^{\text{in}} = s\right]$. However, note that $\mathbb{P}\left[d_v^{\text{in}} = s\right]$ is simply given by $\frac{1}{2^d}\binom{d}{s}$. We conclude that we can write $\mathbb{P}[X_v = 1]$ as follows:

$$\mathbb{P}[X_v = 1] = \frac{1}{2^d} \sum_{s \in S(d,w)} \binom{d}{s}$$

It suffices, therefore, to show that $\sum_{s \in S(d,w)} \binom{d}{s} \geq 2^{d-w}$. We distinguish the following cases:

Case 1: $d = r \cdot w$, for $r \geq 2$. By Lemma 4, we have:

$$\sum_{s \in S(d,w)} \binom{d}{s} = \sum_{i=0}^{r} \binom{r \cdot w}{i \cdot w} \tag{1}$$

By repeated applications of the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, we can derive the inequality:

$$\binom{r \cdot w}{i \cdot w} \geq \sum_{j=0}^{w} \binom{(r-1) \cdot w}{(i-1) \cdot w + j} \text{ , for } 1 \leq i \leq r-1 \tag{2}$$

The combination of Eq. 1 and 2 yields:

$$\begin{aligned}\sum_{s \in S(d,w)} \binom{d}{s} &> \sum_{i=1}^{r-1} \sum_{j=0}^{w} \binom{(r-1) \cdot w}{(i-1) \cdot w + j} \\ &\geq \sum_{i=0}^{(r-1) \cdot w} \binom{(r-1) \cdot w}{i} \\ &= 2^{(r-1) \cdot w} = 2^{d-w}\end{aligned}$$

Case 2: $d = r \cdot w + x$, for $r \geq 1$ and $0 < x < w$. By Lemma 5, we have:

$$\sum_{s \in S(d,w)} \binom{d}{s} > \sum_{i=0}^{r-1} \sum_{j=x}^{w} \binom{r \cdot w + x}{i \cdot w + j} \tag{3}$$

Again, by repeated applications of the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, it is possible to derive the following:

$$\sum_{j=x}^{w} \binom{r \cdot w + x}{i \cdot w + j} \geq \sum_{j=0}^{w} \binom{r \cdot w}{i \cdot w + j} \tag{4}$$

The combination of Eq. 3 and 4 yields:

$$\begin{aligned}\sum_{s \in S(d,w)} \binom{d}{s} &> \sum_{i=0}^{r-1} \sum_{j=0}^{w} \binom{r \cdot w}{i \cdot w + j} \\ &\geq \sum_{i=0}^{r \cdot w} \binom{r \cdot w}{i} \\ &= 2^{r \cdot w} = 2^{d-x} > 2^{d-w}\end{aligned}$$

Case 3: $d \leq w$. In this case, $2^{d-w} \leq 1$ and $\sum_{s \in S(d,w)} \binom{d}{s} = 2 > 2^{d-w}$, since, by Lemmas 4 and 5, we have that $S(d, w) = \{0, d\}$. □

Theorem 3. *There exists a* MinMult-EMC *algorithm which computes an orientation of the given multigraph and then executes steps 2-5 of Algorithm 1, that has an approximation ratio of at most* $2 - \frac{1}{2^w}$.

Proof. By Lemma 6, if we assign random directions to the edges of G, we get at least $\frac{1}{2^w} \cdot n$ locally optimal nodes in expectation. This algorithm can be derandomized by a standard application of the method of conditional expectations. Indeed, let's assume that the orientation of a subset of the edges has already been fixed. Then, we can compute in polynomial time the probability that a fixed node v of degree d will be locally optimal if we assign the rest of the directions randomly, as follows: If a of its incident edges have already been oriented as incoming to v, and b of its incident edges have already been oriented as outgoing from v, then the probability that the node will be locally optimal is $\frac{1}{2^{d-a-b}} \cdot \sum_{s \geq a : s \in S(d,w)} \binom{d-a-b}{s-a}$.

Therefore, the algorithm which examines edges in an arbitrary order, and to each edge assigns the direction which maximizes the expected number of locally optimal nodes under the current partial orientation, runs in deterministic polynomial time and produces an orientation with at least $\frac{1}{2^w} \cdot n$ locally optimal nodes. Taking also into account Lemma 2, this implies that, if we execute steps 2-5 of Algorithm 1 on this orientation, we will obtain a solution with cost SOL that can be expressed as follows: (let $\mathcal{O}$ denote the set of locally optimal nodes)

$$\begin{aligned} \text{SOL} &\leq \sum_{v \in \mathcal{O}} \left\lceil \frac{d_v}{w} \right\rceil + \sum_{v \in V \setminus \mathcal{O}} \left(\left\lceil \frac{d_v}{w} \right\rceil + 1 \right) \\ &= \sum_{v \in V} \left\lceil \frac{d_v}{w} \right\rceil + |V \setminus \mathcal{O}| \\ &\leq \sum_{v \in V} \left\lceil \frac{d_v}{w} \right\rceil + \left(1 - \frac{1}{2^w}\right) \cdot n \end{aligned} \tag{5}$$

If OPT is the cost of an optimal solution, then Eq. 5, Fact 1, and the observation that $n \leq \sum_{v \in V} \left\lceil \frac{d_v}{w} \right\rceil$ imply: $\text{SOL} \leq \left(2 - \frac{1}{2^w}\right) \cdot \sum_{v \in V} \left\lceil \frac{d_v}{w} \right\rceil \leq \left(2 - \frac{1}{2^w}\right) \cdot \text{OPT}$. □

4. Application to wavelength allocation in multifiber optical star networks

In a multifiber optical network, physical links are implemented by multiple parallel optical fibers and communication requests between nodes are represented as simple (undirected) paths on the network. The network operator needs to assign a wavelength (optical frequency, also referred to as *color*) to each request for transmission through an optical fiber. A single optical fiber supports a given number of wavelengths for transmission of requests. If two requests are assigned the same wavelength, then they cannot use the same optical fiber on any edge. Therefore, if an edge carries f requests that are assigned the same wavelength, then the operator will need to deploy at least f parallel optical fibers on that edge in order to accommodate these requests. The fiber demand on a given edge is determined by the wavelength that is used the most among those assigned to paths using that edge. It thus makes sense to strive for a wavelength assignment that minimizes the sum of maximum color loads over all edges, since this directly affects the total number of fibers that need to be deployed and, consequently, the cost of the network. We call this problem MINIMUM MULTIPLICITY PATH MULTICOLORING (MINMULT-PMC).[1]

MINMULT-PMC was defined in [6], where an exact algorithm for chains was presented, as well as 2-approximation algorithms for rings and stars. These algorithms were later extended to a generalized version of MINMULT-PMC with non-uniform multiplicity costs [11]. Path multicoloring problems with different objective functions were defined and studied in [6, 12–14]. Path multicoloring problems were also studied in a non-cooperative setting in [15, 16].

Let us introduce some further notation before we define the problem formally. We will use L_e for the *load* of edge e with respect to a given path set in a simple graph, i.e., L_e is the number of paths which contain edge e. Given, additionally, a coloring of the path set, $f(e)$ will denote the maximum color multiplicity on edge e, i.e., the number of paths in the largest color class among the paths that use edge e. The formal definition of MINMULT-PMC is as follows:

Problem 2 (MINMULT-PMC)**.**
Instance*: $\langle G, \mathcal{P}, w \rangle$, where $G = (V, E)$ is an undirected simple graph, $\mathcal{P}$ is a set of undirected simple paths on G, and $w \in \mathbb{N}$ is the number of available colors.*

[1]The term "path multicoloring" refers to the fact that color collisions are allowed between edge-intersecting paths. This is in contrast to standard path coloring, where edge-intersecting paths must receive distinct colors.

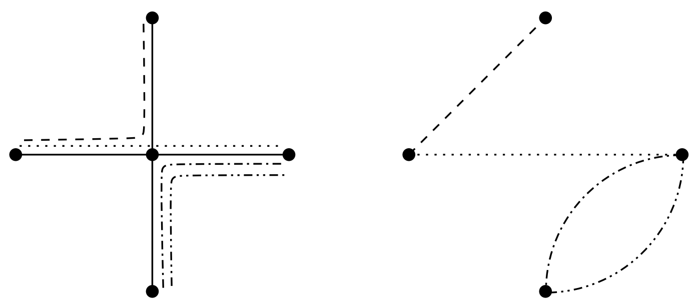

Figure 1: A graph G with a set $\mathcal{P}$ of paths of length 2 (left) and the corresponding multigraph G' (right). Every node of G corresponds to a node of G' (with the exception of the central node of the star) and each edge of G' corresponds to a path in $\mathcal{P}$ that joins its two endpoints. It is straightforward to verify that the line graph of G' is isomorphic to the conflict graph of $\mathcal{P}$.

Feasible solution*: a coloring of* $\mathcal{P}$ *with* w *colors.*
Goal*: minimize* $\sum_{e \in E} f(e)$.

It has been pointed out before in the literature [17, 18] that there exists an easily computed bijection between pairs $(G, \mathcal{P})$ on the one hand, where G is a star and $\mathcal{P}$ is a set of paths of length 2 on G, and multigraphs G' on the other hand, such that the conflict graph of $\mathcal{P}$ (i.e., a graph with vertex set $\mathcal{P}$ and edges connecting two nodes if the corresponding paths are not edge-disjoint in G) is isomorphic to the line graph of G'. Figure 1 illustrates the bijection.

Consequently, MinMult-PMC in star networks with paths (requests) of length 2 can be equivalently cast as MinMult-EMC. We now prove that any approximation algorithm for MinMult-EMC can also be used for general instances of MinMult-PMC in stars (i.e., instances that may also contain paths of length 1), under the assumption that the algorithm performs well with respect to the lower bound of Fact 1.

Theorem 4. *Suppose that there is a polynomial-time algorithm that, given a* MinMult-EMC *instance* $\langle H, w \rangle$*, produces a solution with cost at most* $\alpha\big(|V(H)|, |E(H)|, w\big) \cdot \sum_{v \in V(H)} \lceil \frac{d_v}{w} \rceil$*, where* α *is increasing in the second argument. Then, there exists a polynomial-time algorithm that, given a* MinMult-PMC *instance* $\langle G, \mathcal{P}, w \rangle$ *where* G *is a star, computes a solution with cost within a factor* $\alpha\big(|E(G)|, |\mathcal{P}|, w\big)$ *of the optimal.*

Proof. Let $\mathcal{I} = \langle G, \mathcal{P}, w \rangle$ be an instance of MinMult-PMC where $G = (V, E)$ is a star, let $\mathcal{P}'$ be the subset of $\mathcal{P}$ that contains only the paths of length 2, and let H be the multigraph corresponding to $(G, \mathcal{P}')$ via the bijection explained above. Let $\mathcal{A}$ be the stipulated approximation algorithm for MinMult-EMC and consider the following algorithm $\mathcal{B}$ for MinMult-PMC:

1. Execute algorithm $\mathcal{A}$ on the MinMult-EMC instance $\langle H, w\rangle$.
2. Obtain a coloring of $\mathcal{P}'$ by assigning to each path in $\mathcal{P}'$ the color of the corresponding edge in H.
3. Complete the coloring produced in the previous step as follows: For each path of length 1 on edge e (in an arbitrary order), color it with the color of smallest multiplicity on edge e in the current partial coloring.

Let L'_e be the load of edge e in G with respect to path set $\mathcal{P}'$, let $f'(e)$ be the cost on edge e after the partial coloring of step 2, and let $\delta_e = f'(e) - \lceil \frac{L'_e}{w} \rceil$. By the properties of algorithm $\mathcal{A}$ and the bijection between $(G, \mathcal{P}')$ and H, we know that $\sum_{e \in E} f'(e) \leq \alpha\big(|E(G)|, |\mathcal{P}'|, w\big) \cdot \sum_{e \in E} \lceil \frac{L'_e}{w} \rceil$. Substituting $f'(e)$ and denoting $\hat{\alpha} = \alpha\big(|E(G)|, |\mathcal{P}'|, w\big)$, we obtain:

$$\sum_{e \in E} \delta_e \leq (\hat{\alpha} - 1) \cdot \sum_{e \in E} \left\lceil \frac{L'_e}{w} \right\rceil \tag{6}$$

Furthermore, let L_e be the load of edge e with respect to path set $\mathcal{P}$, let $f(e)$ be the cost on edge e after step 3, and let $E^+ \subseteq E$ be the set of edges e with $f(e) > f'(e)$. Consider an edge $e \in E^+$. The fact that the cost of e increased with respect to the partial coloring of step 2 implies that the paths on e are now partitioned equitably into w color classes. Therefore, $f(e) = \lceil \frac{L_e}{w} \rceil$. On the other hand, if $e \in E \setminus E^+$, we have $f(e) = f'(e) = \delta_e + \lceil \frac{L'_e}{w} \rceil \leq \delta_e + \lceil \frac{L_e}{w} \rceil$, because $L'_e \leq L_e$. The cost of the solution returned by algorithm $\mathcal{B}$ is, therefore:

$$\begin{aligned} \sum_{e \in E} f(e) &= \sum_{e \in E^+} f(e) + \sum_{e \in E \setminus E^+} f(e) \\ &\leq \sum_{e \in E^+} \left\lceil \frac{L_e}{w} \right\rceil + \sum_{e \in E \setminus E^+} \left(\delta_e + \left\lceil \frac{L_e}{w} \right\rceil\right) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{e \in E} \left\lceil \frac{L_e}{w} \right\rceil + \sum_{e \in E} \delta_e \\
&\leq \sum_{e \in E} \left\lceil \frac{L_e}{w} \right\rceil + (\hat{\alpha} - 1) \cdot \sum_{e \in E} \left\lceil \frac{L'_e}{w} \right\rceil && \text{(by Eq. 6)} \\
&\leq \hat{\alpha} \cdot \sum_{e \in E} \left\lceil \frac{L_e}{w} \right\rceil && \text{(since } L'_e \leq L_e\text{)}
\end{aligned}$$

Since $|\mathcal{P}'| \leq |\mathcal{P}|$ and α is increasing in the second argument, we have $\hat{\alpha} \leq \alpha\big(|E(G)|, |\mathcal{P}|, w\big)$. Moreover, $\sum_{e \in E} \left\lceil \frac{L_e}{w} \right\rceil$ is a lower bound for the cost of the optimal solution of $\mathcal{I}$, therefore $\mathcal{B}$ satisfies the desired property. □

By our analysis, the algorithm that we propose in Theorem 3 satisfies the condition of Theorem 4. Consequently, MinMult-PMC in stars admits an approximation algorithm with ratio $2 - \frac{1}{2^w}$.

5. Concluding remarks

We investigated the possibility of improving Algorithm 1 for MinMult-EMC, by choosing a more appropriate edge orientation in step 1 instead of taking an arbitrary orientation that yields a 2-approximation algorithm [6]. Note that it is possible to compute an orientation that yields an approximation ratio of $2 - \Theta\big(\frac{1}{n}\big)$, where n is the number of nodes of the multigraph, simply by taking an orientation that creates at least one source and one sink, which is always feasible. In Section 3, we gave an algorithm with approximation ratio $2 - \frac{1}{2^w}$.

The comparative advantage of our algorithm is highlighted in view of its application to the MinMult-PMC problem in stars, as we explained in Section 4. The naive MinMult-EMC algorithm with ratio $2 - \Theta\big(\frac{1}{n}\big)$ is translated by Theorem 4 to a MinMult-PMC algorithm with ratio $2 - \Theta\big(\frac{1}{m}\big)$, where m is the number of edges in the star network. It is crucial to note that, in the context of optical networks, the parameter w represents the number of available wavelengths per fiber, which is limited in practice due to technological constraints (see, e.g., [19, Section 1.8.3]). On the other hand, the parameter m represents the size of the network, which may be arbitrarily large. Consequently, from a practical standpoint, a $\big(2 - \frac{1}{2^w}\big)$-approximation algorithm for MinMult-EMC is arguably more desirable than an algorithm with ratio $2 - \Theta\big(\frac{1}{n}\big)$.

We also showed, in Theorem 2, a lower bound of $2-\Theta\left(\frac{1}{w}\right)$ on the approximation ratio of any algorithm which computes an orientation of the given multigraph and then executes steps 2-5 of Algorithm 1, under the assumption that step 4 of Algorithm 1 may produce a worst-case edge coloring. Therefore, we may need to design a special edge coloring procedure or to employ different techniques in order to achieve an approximation ratio asymptotically better than 2.

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References

[1] G. F. Lev, N. Pippenger, L. G. Valiant, A fast parallel algorithm for routing in permutation networks, IEEE Trans. Computers 30 (2) (1981) 93–100. doi:10.1109/TC.1981.6312171.

[2] E. G. Coffman Jr., M. R. Garey, D. S. Johnson, A. S. LaPaugh, Scheduling file transfers, SIAM J. Comput. 14 (3) (1985) 744–780. doi:10.1137/0214054.

[3] S. Nakano, X. Zhou, T. Nishizeki, Edge-coloring algorithms, in: J. van Leeuwen (Ed.), Computer Science Today: Recent Trends and Developments, Vol. 1000 of Lecture Notes in Computer Science, Springer, 1995, pp. 172–183. doi:10.1007/BFb0015243.

[4] C. Hsu, P. Liu, D. Wang, J. Wu, Generalized edge coloring for channel assignment in wireless networks, in: 2006 International Conference on Parallel Processing (ICPP 2006), 14-18 August 2006, Columbus, Ohio, USA, IEEE Computer Society, 2006, pp. 82–92. doi:10.1109/ICPP.2006.45.

[5] S. Gandham, M. Dawande, R. Prakash, Link scheduling in wireless sensor networks: Distributed edge-coloring revisited, J. Parallel Distrib. Comput. 68 (8) (2008) 1122–1134. doi:10.1016/j.jpdc.2007.12.006.

[6] C. Nomikos, A. Pagourtzis, S. Zachos, Routing and path multicoloring, Information Processing Letters 80 (5) (2001) 249–256. doi:10.1016/S0020-0190(01)00167-3.

[7] I. Holyer, The NP-completeness of edge-colouring, SIAM Journal on Computing 10 (4) (1981) 718–720. doi:10.1137/0210055.

[8] D. Leven, Z. Galil, NP completeness of finding the chromatic index of regular graphs, Journal of Algorithms 4 (1) (1983) 35–44. doi:10.1016/0196-6774(83)90032-9.

[9] H. N. Gabow, Using Euler partitions to edge color bipartite multigraphs, International Journal of Computer and Information Sciences 5 (4) (1976) 345–355. doi:10.1007/BF00998632.

[10] G. H. Gonnet, Expected length of the longest probe sequence in hash code searching, Journal of the ACM 28 (2) (1981) 289–304. doi:10.1145/322248.322254.

[11] C. Nomikos, A. Pagourtzis, K. Potika, S. Zachos, Routing and wavelength assignment in multifiber WDM networks with non-uniform fiber cost, Computer Networks 50 (1) (2006) 1–14. doi:10.1016/j.comnet.2004.11.028.

[12] G. Li, R. Simha, On the wavelength assignment problem in multifiber WDM star and ring networks, in: INFOCOM, 2000, pp. 1771–1780. URL http://www.ieee-infocom.org/2000/papers/222.ps

[13] T. Erlebach, A. Pagourtzis, K. Potika, S. Stefanakos, Resource allocation problems in multifiber WDM tree networks, in: H. L. Bodlaender (Ed.), Graph-Theoretic Concepts in Computer Science, 29th International Workshop, WG 2003, Elspeet, The Netherlands, June 19-21, 2003, Revised Papers, Vol. 2880 of Lecture Notes in Computer Science, Springer, 2003, pp. 218–229.

[14] M. Andrews, L. Zhang, Minimizing maximum fiber requirement in optical networks, J. Comput. Syst. Sci. 72 (1) (2006) 118–131. doi:10.1016/j.jcss.2005.08.001.

[15] E. Bampas, A. Pagourtzis, G. Pierrakos, K. Potika, On a noncooperative model for wavelength assignment in multifiber optical networks,

IEEE/ACM Trans. Netw. 20 (4) (2012) 1125–1137. doi:10.1109/TNET.2011.2173948.

[16] E. Bampas, A. Pagourtzis, G. Pierrakos, V. Syrgkanis, Selfish resource allocation in optical networks, in: P. G. Spirakis, M. J. Serna (Eds.), Algorithms and Complexity, 8th International Conference, CIAC 2013, Barcelona, Spain, May 22-24, 2013. Proceedings, Vol. 7878 of Lecture Notes in Computer Science, Springer, 2013, pp. 25–36. doi:10.1007/978-3-642-38233-8.

[17] M. C. Golumbic, R. E. Jamison, The edge intersection graphs of paths in a tree, Journal of Combinatorial Theory, Series B 38 (1) (1985) 8–22. doi:10.1016/0095-8956(85)90088-7.

[18] T. Erlebach, K. Jansen, The complexity of path coloring and call scheduling, Theoretical Computer Science 255 (1-2) (2001) 33–50. doi:10.1016/S0304-3975(99)00152-8.

[19] R. Ramaswami, K. N. Sivarajan, G. H. Sasaki, Optical Networks: a practical perspective, 3rd Edition, Morgan Kaufmann, San Francisco, CA, USA, 2009.