Path multicoloring in spider graphs with even color multiplicity[☆]

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Abstract

We give an exact polynomial-time algorithm for the problem of coloring a collection of paths defined on a spider graph using a minimum number of colors (Min-PMC), while respecting a given even maximum admissible color multiplicity on each edge. This complements a previous result on the complexity of Min-PMC in spider graphs, where it was shown that, for every odd k, the problem is NP-hard in spiders with admissible color multiplicity k on each edge. We also obtain an exact polynomial-time algorithm for maximizing the number of colored paths with a given number of colors (Max-PMC) in spider graphs with even admissible color multiplicity on each edge.

[☆] A preliminary version of this work appears in the Proceedings of the 14^{th} International Conference on Ad-hoc, Mobile, and Wireless Networks, LNCS vol. 9143, pp. 33–47, Springer, 2015.

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Keywords: path multicoloring, spider graph, algorithms

1. Introduction

Path coloring problems have been studied extensively in the context of routing and wavelength assignment in optical networks, as well as in several other applications, including, for example, compiler optimization and vehicle scheduling. In a meaningful generalization, path multicoloring problems were defined and studied in [2, 9, 12, 13, 18]. Note that the term "path multicoloring" means that color collisions are allowed for edge-intersecting paths. This is in contrast to standard path coloring, where edge-intersecting paths must receive distinct colors.

Various optimization objectives have been studied in the context of path multicoloring. In this article, we are interested in two problems that arise from bounding the *admissible color multiplicity* on each edge, i.e., the maximum number of paths that can use this edge and receive the same color. In Min-PMC, one seeks to color all paths with the minimum number of colors. The problem is defined formally as follows:

Problem 1 (Minimum Path MultiColoring, Min-PMC)**.**
Instance*: $\langle G, \mathcal{P}, \mu \rangle$, where $G = (V, E)$ is an undirected graph, $\mathcal{P}$ is a set of undirected simple paths on G, and $\mu : E \to \mathbb{N}$ is a function that maps each edge to its admissible color multiplicity.*
Feasible solution*: a coloring of $\mathcal{P}$ so that, for every edge e, at most $\mu(e)$ paths using e have the same color.*
Goal*: minimize the number of colors used.*

In the maximization version of the problem, the number of available colors is limited and one seeks to maximize the number of paths that are colored. This problem, Max-PMC, is defined formally as follows:

Problem 2 (Maximum Path MultiColoring, Max-PMC)**.**
Instance*: $\langle G, \mathcal{P}, \mu, w \rangle$, where $G = (V, E)$ is an undirected graph, $\mathcal{P}$ is a set of undirected simple paths on G, $\mu : E \rightarrow \mathbb{N}$ is a function that maps each edge to its admissible color multiplicity, and $w \in \mathbb{N}$ is the number of available colors.*
Feasible solution*: a coloring of a subset $\mathcal{Q} \subseteq \mathcal{P}$ with at most w colors, so that, for every edge e, at most $\mu(e)$ paths using e have the same color.*
Goal*: maximize $|\mathcal{Q}|$.*

The restriction of Min-PMC (resp. Max-PMC) to instances with $\mu(e) = 1$ for each edge e is known as the *minimum* (resp. *maximum*) *path coloring* problem and is denoted by Min-PC (resp. Max-PC).

Related work. Path coloring and path multicoloring problems have been extensively studied during the last twenty years (see e.g. [3, 5, 7–9, 16–18, 22] and references therein). Most meaningful variants are NP-hard in general graphs and even in simple topologies, e.g. stars and rings, whereas they can be solved exactly in chains. In addition, most variants are hard to approximate in general graphs within a constant factor. This, however, is possible in stars, rings, and in some other simple topologies.

The Min-PMC problem was first considered in [12, 13] in the context of wavelength assignment in multifiber all-optical networks with an equal number of fibers per link. Complexity lower bounds and inapproximability results for Min-PMC follow from the corresponding results for Min-PC. Specifically, Min-PC is NP-hard already in rings and stars and one can exploit the connection of Min-PC in stars with the edge coloring problem in multigraphs to show that Min-PC in stars cannot have a $(\frac{4}{3} - \epsilon)$-approximation algorithm, unless $\mathsf{P} = \mathsf{NP}$ [8, 11]. Furthermore, there exists a constant $c > 0$ such that Min-PC in grids does not admit a $|V|^{\epsilon}$-approximation algorithm for any $\epsilon < c$, unless $\mathsf{P} = \mathsf{NP}$ [15, Corollary 6.1].

Moving on to positive results for Min-PMC, it was shown in [6] that it admits a 4-approximation in trees. A $\frac{3}{2}$-approximation for stars was proposed in [16]; however, an asymptotic $\frac{9}{8}$-approximation can be obtained directly from the equivalence between Min-PMC in stars and f-coloring in multigraphs and the result of Nakano et al. for f-coloring of multigraphs [14]. In [16], the authors also give a 2-approximation algorithm for rings and an exact algorithm for chains. In [20] algorithms for Min-PMC in spiders[1] and caterpillars[2] were proposed, achieving approximation ratios of 2 and 3, respectively. In contrast, it was shown in [20] that the directed version of Min-PMC can be solved exactly in spiders.

A more recent result of Bian and Gu [5] states that Min-PMC can be solved exactly in spiders with *uniform* and even admissible color multiplicity (uniform stands for 'the same on each edge of the graph'). In fact, the same algorithm works under a slightly relaxed uniformity constraint, namely that each leg of the spider has uniform and even admissible color multiplicity, but the multiplicity may vary among different legs. Therefore, Min-PMC can

[1] A *spider* is a tree with at most one node of degree 3 or more.

[2] A *caterpillar* is a tree in which all nodes are within distance 1 of a central path.

also be solved exactly in stars where the admissible color multiplicity is even (not necessarily uniform). However, it is known that Min-PMC is NP-hard when G is a star and $\mu(e) = 1$ for all edges [8, 11]. Bian and Gu also prove that, for every odd k, Min-PMC is NP-hard when G is a star and $\mu(e) = k$ for all edges.

Results for Max-PMC appear in [9], where a 2.54-approximation for trees is proposed, in [21], where the author gives an exact algorithm for chains and 2-approximation algorithms for rings and stars, and in [5], where they provide a 1.58-approximation algorithm for the problem in spiders and an exact algorithm for spiders with uniform and even admissible color multiplicity. The NP-hardness of Max-PMC in rings and stars follows from the NP-hardness of Max-PC in rings [19] and stars [7]. With regard to inapproximability, it is known that for some constant $c > 0$, Max-PC in grids does not admit a $|\mathcal{P}|^{\epsilon}$-approximation algorithm for any $\epsilon < c$, unless $\mathsf{P} = \mathsf{NP}$ [15, Theorem 7.8], and that Max-PC in general graphs does not admit a $|E|^{\frac{1}{2}-\epsilon}$-approximation algorithm for any $\epsilon > 0$, unless $\mathsf{NP} \subseteq \mathsf{ZPP}$ [1].

We summarize the known positive results for Min-PMC and Max-PMC in various topologies in Table 1.

Our contributions. We show in Section 2 that Min-PMC can be solved exactly in spiders with non-uniform even admissible color multiplicity on each edge. This improves the result of Bian and Gu [5], by completely removing the uniformity requirement. Moreover, our result complements the complexity result in [5], where it was shown that allowing odd admissible color multiplicities in a star leads to NP-hardness of the problem.

As a corollary, we obtain in Section 3 an exact algorithm for Max-PMC

Table 1: Known positive results for Min-PMC and Max-PMC in various topologies.

Topology	Best known approximation ratio for: Min-PMC		Max-PMC	
rings	2	(cf. [16])	2	(cf. [21])
trees	4	(cf. [6])	2.54	(cf. [9])
caterpillars	3	(cf. [20])	2.54	(cf. [9])
spiders	2	(cf. [20])	1.58	(cf. [5])
spiders with $\mu(e)$ even and uniform over each leg	exact	(cf. [5])	exact	(cf. [5])
stars	$^3/_2$	(cf. [16])	1.58	(cf. [5])
	$^9/_8$ asympt.	(cf. [14])		
chains	exact	(cf. [16])	exact	(cf. [21])

in spiders with non-uniform even admissible color multiplicity. This result holds for any number of available colors.

Notation. Given an instance of Min-PMC or Max-PMC, we will use the notation L_e for the *load* of edge e with respect to the given path set, i.e., L_e is the number of paths that contain edge e.

2. Minimizing the number of colors

We solve Min-PMC in spiders with (non-uniform) even maximum color multiplicities. Let $\langle G, \mathcal{P}, \mu \rangle$ be an instance of Min-PMC, with $G = (V, E)$.

Fact 1. *Every* Min-PMC *instance requires at least* $w_{\mathrm{lb}} = \max_e \left\lceil \frac{L_e}{\mu(e)} \right\rceil$ *colors.*

We reduce the problem to the *directed* version of MIN-PMC, which is known to be polynomial-time solvable in spiders [20]. Formally, the directed version of the problem is defined as follows: (the problem was called DIRECTED MIN-COLORS-PMC in [20])

Problem 3 (DIRECTED MINIMUM PATH MULTICOLORING, DIR-MIN-PMC)**.**
Instance*: $\langle \vec{G}, \vec{\mathcal{P}}, \mu \rangle$, where $\vec{G} = (V, \vec{E})$ is a directed graph, $\vec{\mathcal{P}}$ is a set of directed simple paths on $\vec{G}$, and $\mu : \vec{E} \to \mathbb{N}$ is a function that maps each arc to its admissible color multiplicity.*
Feasible solution*: a coloring of $\vec{\mathcal{P}}$ so that on every arc e, each color is used by at most $\mu(e)$ paths.*
Goal*: minimize the number of colors.*

In order to perform the reduction, we construct a symmetric directed graph $\vec{G} = (V, \vec{E})$, whose arc set $\vec{E}$ contains both arcs (u, v) and (v, u), for each edge $\{u, v\} \in E$. We further need to decide a direction for each undirected path in the original MIN-PMC instance $\langle G, \mathcal{P}, \mu \rangle$. We accomplish this by adding one unit-length dummy path on each edge with odd load, and then considering the multigraph H which has the same node set as G and contains one edge $\{u, v\}$ for each path with endpoints u and v. The addition of dummy paths ensures that all nodes of H have even degree (see proof of Theorem 1). Therefore, the Euler partition algorithm of [10] will partition the edges of H into closed paths. We then perform an arbitrary orientation of each closed path in the Euler partition. This assigns a direction to each edge of H, which in turn corresponds to a direction for each path in $\mathcal{P}$ and for each dummy path. Finally, we solve the constructed DIR-MIN-PMC instance with a suitably modified arc color multiplicity function μ'.

Figures 1–3 illustrate the process of the path set orientation.

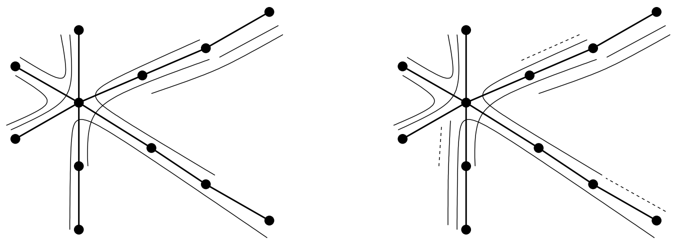

Figure 1: *Left:* The original Min-PMC instance. *Right:* The instance with added dummy paths (dotted lines).

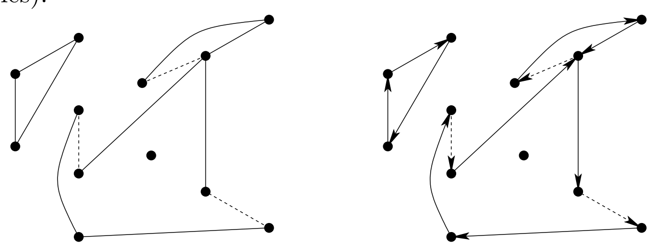

Figure 2: *Left:* The corresponding multigraph H. *Right:* An orientation of the closed paths in the Euler partition of H.

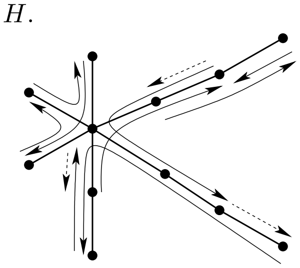

Figure 3: The directed path set in the resulting Dir-Min-PMC instance. Here, the underlying graph is symmetric directed and an edge between two nodes represents a pair of arcs in opposite directions.

Theorem 1. *Algorithm 1 is an exact polynomial-time algorithm for* Min-PMC *in spiders with even admissible color multiplicity.*

Algorithm 1 An exact algorithm for MIN-PMC in spiders with even admissible color multiplicity

Input: an instance $\langle G, \mathcal{P}, \mu \rangle$ of MIN-PMC, where $G = (V, E)$ is a spider and $\mu(e)$ is even for all e

Output: an optimal solution using w_{lb} colors

1: For any edge with odd load, add one unit-length dummy path on that edge. Let $\mathcal{P}'$ be the resulting set of paths.

2: Construct the multigraph $H = (V, D)$, where D contains one edge $\{u, v\}$ for each path in $\mathcal{P}'$ with endpoints u and v.

3: Use the algorithm EP in [10] to find an Euler partition of H. Orient each closed path of the Euler partition in an arbitrary manner, thus obtaining a directed multigraph $\vec{H}$. Assign a direction to each path in $\mathcal{P}'$ according to the orientation of the corresponding edge in $\vec{H}$, thus obtaining a set $\vec{\mathcal{P}}'$ of directed paths.

4: Find an optimal coloring c of the DIR-MIN-PMC instance $\langle \vec{G}, \vec{\mathcal{P}}', \mu' \rangle$, where $\vec{G}$ is the symmetric directed version of G and $\mu'(e) = \frac{\mu(e)}{2}$ for all e, using the algorithm in [20, Theorem 6].

5: Return the coloring c restricted to the original paths in $\mathcal{P}$.

Proof. We first show that the multigraph H contains only even-degree nodes. For any node v in G other than the center, let u and w be its neighbors. If v is the extremity of a leg of the spider, then we imagine that this leg is extended by means of a new dummy edge that is not used by any path, so that v is no longer the extremity of the leg and u and w are both well defined. Note that this extension does not affect the degrees of the original nodes in H. Now, let $\mathcal{P}'$ be the set of paths obtained in step 1 of Algorithm 1 and let x be the number of paths in $\mathcal{P}'$ that have v as an endpoint and contain u, let y be the number of paths in $\mathcal{P}'$ that have v as an endpoint and contain w, and let z be the number of paths in $\mathcal{P}'$ that have v as an internal node. With these definitions, the degree of v in H is given by $x + y$, while $x + z$ is the load of edge $\{u, v\}$ and $y+z$ is the load of edge $\{v, w\}$ with respect to $\mathcal{P}'$. Therefore, since we added unit-length dummy paths on each edge with odd load during step 1, it follows that $x+z$ and $y+z$ are both even, which implies that $x+y$ must also be even. We thus proved that every node v apart from the center of the spider has even degree in H. It follows that all nodes of H have even degree, since there cannot be a single odd-degree node in a graph.

This guarantees that the Euler partition computed in Step 3 contains only closed paths [10], thus at the end of Step 3 each path in $\mathcal{P}'$ has been assigned a direction. For every edge e, let L_e be its load with respect to $\mathcal{P}$ and let L'_e be its load with respect to $\mathcal{P}'$. It is not hard to see that after Step 3, the paths that use that edge are equipartitioned into $\frac{L'_e}{2}$ paths in each direction. Indeed, each closed path in the Euler partition contributes, for every edge e, an equal number of paths using it in either direction. To see why, consider an edge e in G and the cut (V_1, V_2) obtained by removing e. The corresponding cut in H is crossed by any closed path an equal number of times in each direction. Each crossing of the cut in H in the direction $V_1 \to V_2$ (resp. $V_2 \to V_1$) corresponds to a distinct path that traverses e in G in the direction $V_1 \to V_2$ (resp. $V_2 \to V_1$). Therefore, e is traversed by an equal number of paths in each direction. By the properties of the algorithm for Dir-Min-PMC presented in [20], the coloring computed in Step 4 uses $\max_e \left\lceil \frac{\frac{L'_e}{2}}{\frac{\mu(e)}{2}} \right\rceil = \max_e \left\lceil \frac{L'_e}{\mu(e)} \right\rceil$ colors.

Finally, note that, for every edge e, we have either $L'_e = L_e$ or $L'_e = L_e + 1$. In the latter case, L_e must be odd and we can write it as $L_e = q \cdot \mu(e) + r$, where $0 < r < \mu(e)$ and r is odd, since $\mu(e)$ is even by assumption. Thus, $\left\lceil \frac{L'_e}{\mu(e)} \right\rceil = \left\lceil \frac{L_e}{\mu(e)} \right\rceil = q + 1$. We conclude that, for all $e \in E$, $\left\lceil \frac{L'_e}{\mu(e)} \right\rceil = \left\lceil \frac{L_e}{\mu(e)} \right\rceil$. Therefore, the solution produced by Algorithm 1 uses $\max_e \left\lceil \frac{L_e}{\mu(e)} \right\rceil$ colors, which is optimal by Fact 1. □

3. Maximizing the number of satisfied requests

We obtain as a corollary of Theorem 1 that the Max-PMC problem is also exactly solvable in polynomial time in spiders with (non-uniform) even admissible color multiplicity.

In [5], Bian and Gu propose a deterministic polynomial-time algorithm for Max-PMC in spiders with uniform even admissible color multiplicity, which works as follows:

1. First, it computes a maximum-size subset $\mathcal{Q}$ of $\mathcal{P}$ such that every edge e of the spider is used by at most $w\mu(e)$ paths in $\mathcal{Q}$. This is accomplished by solving exactly an instance of the Call Control problem on spiders, via an algorithm developed in [5].
2. Then, $\mathcal{Q}$ is colored with w colors using the algorithm for Min-PMC in spiders with uniform even admissible color multiplicity that is proposed in [5].

By plugging Algorithm 1 instead of the one in [5] in the above Step 2, we can solve Max-PMC in spiders with non-uniform even admissible color multiplicity. Indeed, if L_e is the load of edge e with respect to path set $\mathcal{Q}$, then

Algorithm 1 will use $\max_e \left\lceil \frac{L_e}{\mu(e)} \right\rceil \leq \max_e \left\lceil \frac{w\mu(e)}{\mu(e)} \right\rceil = w$ colors. Thus, we have the following:

Theorem 2. *There exists an exact polynomial-time algorithm for* MAX-PMC *in spiders with even admissible color multiplicity.*

Acknowledgment

We wish to thank two anonymous referees for their careful reading of the manuscript and for their suggestions to improve the presentation of our results.

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