On a Non-Cooperative Model for Wavelength Assignment in Multifiber Optical Networks

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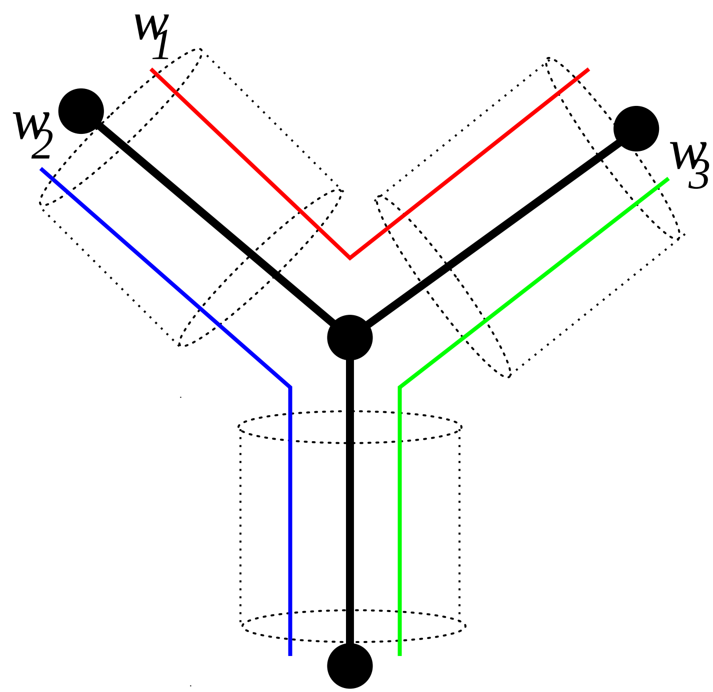
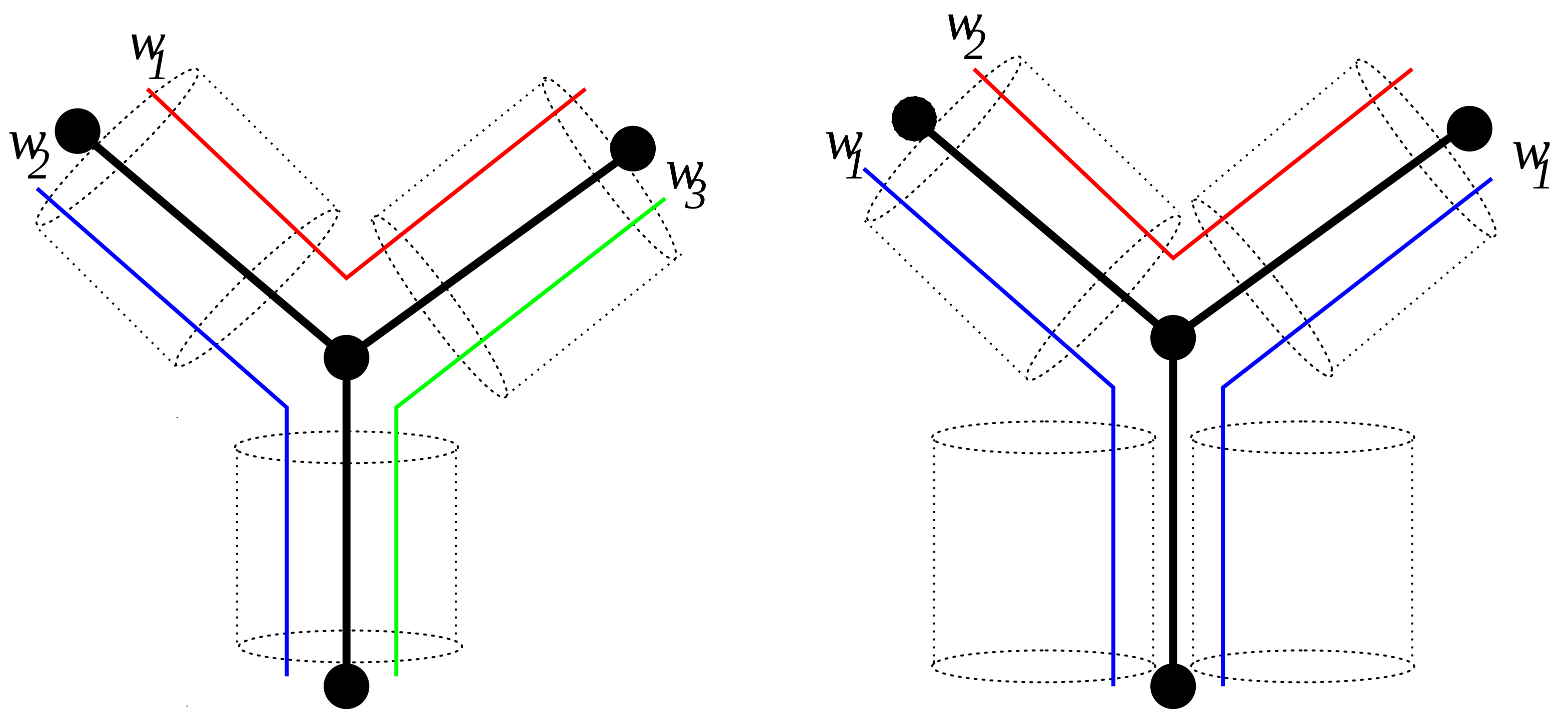
Transparent all-optical networks



Single *vs* multi-fiber



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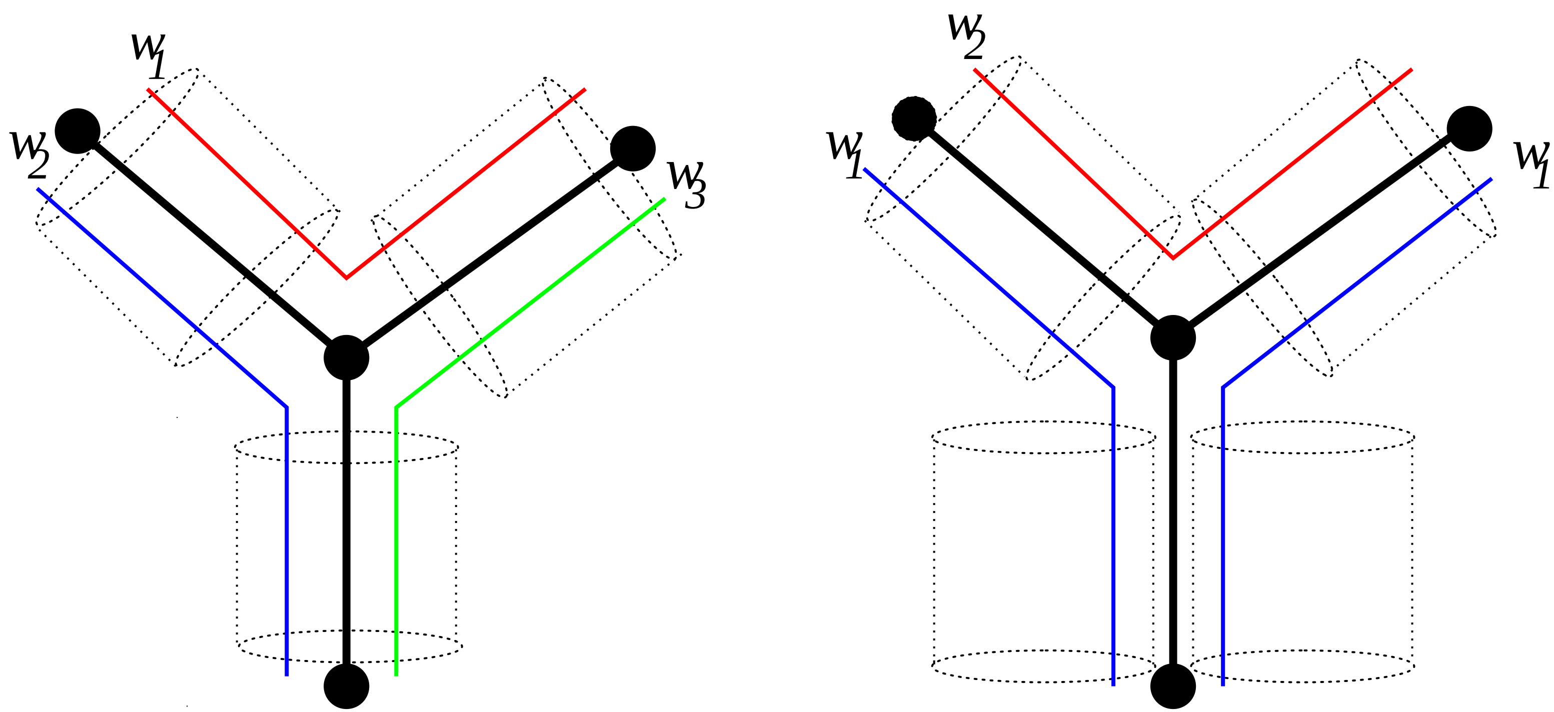

small color multiplicity $\Longrightarrow$ small # fibers

Non-cooperative model

- Large-scale networks: shortage of centralized control
  - provide incentives for users to work for the social good
- Social good: minimize fiber multiplicity
- Reasonable policy: charge users according to the maximum fiber multiplicity incurred by their choice of frequency

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What will be the impact on social welfare if we allow users to act freely and selfishly?

Problem formulation

Def. PATH MULTICOLORING problem:

- input: graph $G(V, E)$, path set P, # colors w
- solution: a coloring $c : P \to W$, $W = \{\alpha_1, \ldots, \alpha_w\}$
- goal: minimize the maximum color multiplicity

$$\mu_{\max} \triangleq \max_{e \in E, \alpha \in W} \mu(e, \alpha)$$

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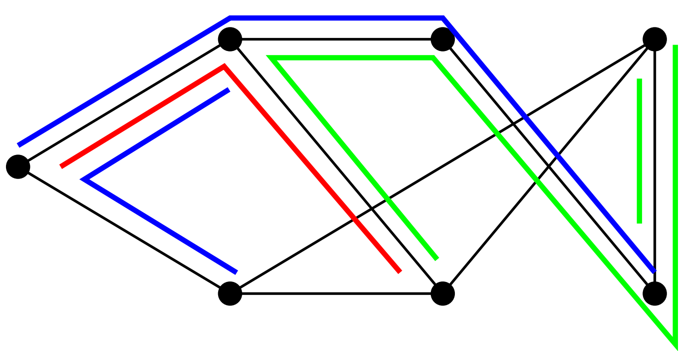

$$L = 3$$

$$\mu_{\max} = 2$$

$$\mu_{\mathrm{OPT}} \geq \left\lceil \frac{L}{w} \right\rceil$$

Game-theoretic formulation

- Def. Given a graph G, path set P and w, define the game $\langle G, P, w \rangle$:
  - players: $p_1, \ldots, p_{|P|} \in P$
  - strategies: each p_i picks a color $c_i \in W$
  - strategy profile: a vector $\vec{c} = (c_1, \ldots, c_{|P|})$
  - disutility functions: $f_i(\vec{c}) = \mu(p_i, c_i)$ (maximum multiplicity of c_i along p_i)
  - social cost: $\mathrm{sc}(\vec{c}) \triangleq \mu_{\max} = \max_{e \in E, \alpha \in W} \mu(e, \alpha)$

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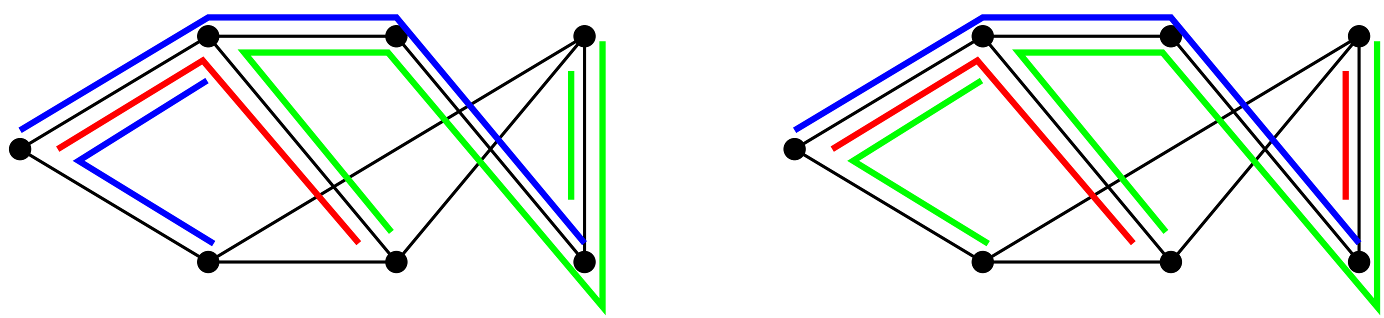
- Def. S-PMC: the class of all $\langle G, P, w \rangle$ games

Nash Equilibria

- Def. A strategy profile is a Nash Equilibrium (NE) if no player can reduce her disutility by changing strategy unilaterally:

$$\forall p_i \in P, \forall c_i' \in W : \; f_i(\vec{c}; c_i) \leq f_i(\vec{c}; c_i')$$



Efficiency of Nash Equilibria

- Def. The price of anarchy (PoA) of an S-PMC game:

$$\mathrm{PoA} = \frac{\max_{\vec{c}\text{ is NE}} \mathrm{sc}(\vec{c})}{\mu_{\mathrm{OPT}}} \triangleq \frac{\hat{\mu}}{\mu_{\mathrm{OPT}}}$$

- Def. The price of stability (PoS) of an S-PMC game:

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- Rate of convergence to some NE?
  - by repeatedly changing some player's strategy to improve her disutility (Nash dynamics)

Results in this work

- Any Nash dynamics converges in at most $4^{|P|}$ steps
- Efficient computation of NE:
  - optimal NE for S-PMC(ROOTED-TREE)
  - $\frac{1}{2}$-approximate NE for S-PMC(STAR)
- Upper and lower bounds for the PoA:
  - # colors
  - minimum length of any path that contributes to the cost of some worst-case NE
  - matching lower bounds for graphs with $\Delta \geq 3$

Results in this work (Cont'd)

- The PoA on graphs with degree 2:
  - if $L = \Omega(w^2)$, $\mathrm{PoA} = O(1)$
  - if $L = o(w^2)$, PoA is unbounded

Related work

- Minimization problem with the $\mu_{\max}$ objective [AZ04]
- Minimization problem with the $\sum_{e \in E} \max_{\alpha \in W} \mu(e, \alpha)$ objective [NPZ01]
- Bottleneck network games
  - player cost: MAX of delays along her path
  - players pick among several possible routings [BM06]
  - latency functions on edges [BO06]
- Congestion games [MS96, Ros73]

Convergence to NE

Thm. Any Nash dynamics converges in at most $4^{|P|}$ steps

- consider the vector

$$(d_L(\vec{c}), d_{L-1}(\vec{c}), \ldots, d_1(\vec{c}))$$

- lexicographic-order argument (attributed to Mehlhorn in [FKK$^+$02])
- $\mathrm{PoS} = 1$

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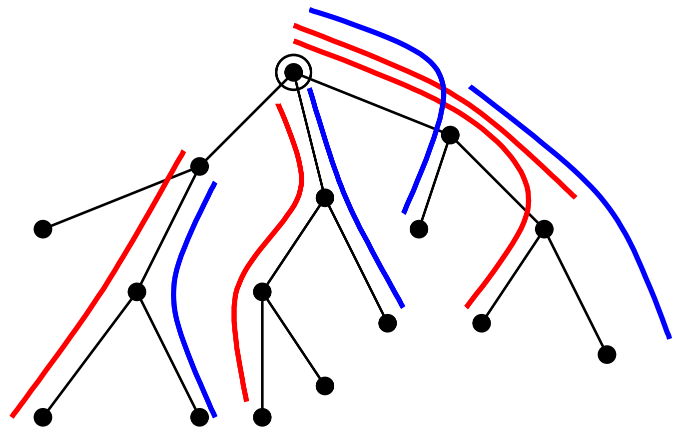
- how many such vectors?

$$\binom{|P| + L - 1}{|P|} \leq 2^{|P|+L-1} < 4^{|P|}$$

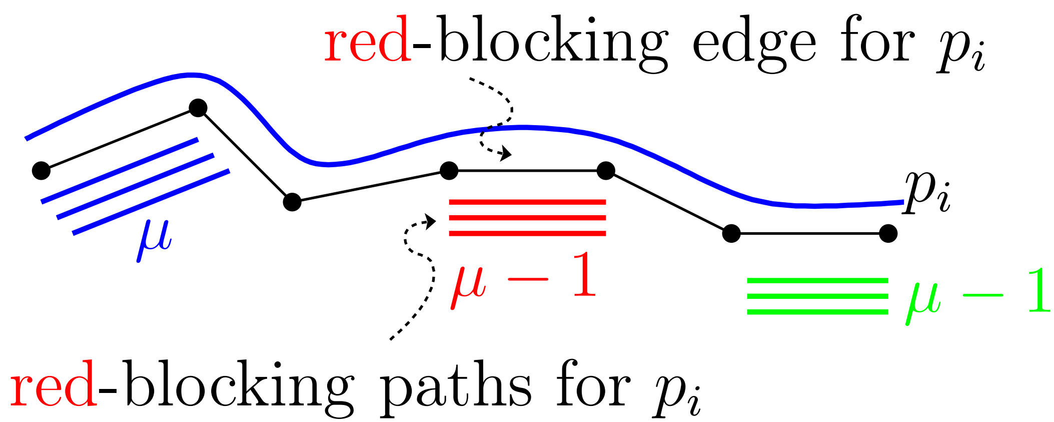
Efficient computation of optimal NE



- $\langle G, P, w\rangle$ is in S-PMC(ROOTED-TREE) if $\exists r$ s.t. each path in P lies entirely on some simple path from r to a leaf
- consider edges in BFS order: color paths with min-multiplicity color in the partial solution

A structural property of NE

- If $\vec{c}$ is a NE, then for any $p_i \in P$ and for any $\alpha \in W$ there is an $e \in p_i$ s.t. $\mu(e, \alpha) \geq f_i(\vec{c}) - 1$



An upper bound on the PoA

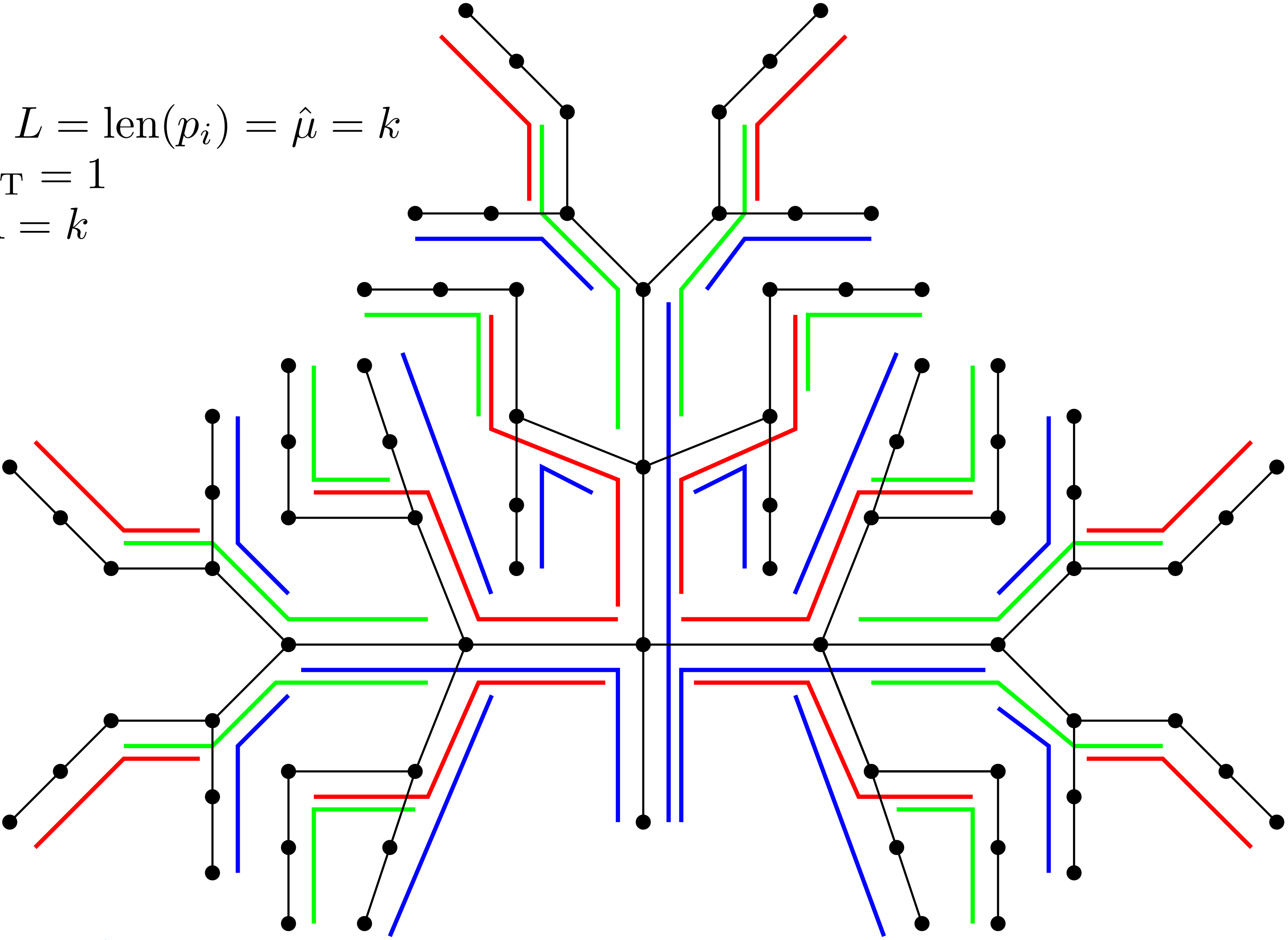
Thm. If $\vec{c}$ is a NE and $\mathrm{sc}(\vec{c}) = f_i(\vec{c}) = \hat{\mu}$ then $\mathrm{PoA} \leq \mathrm{len}(p_i)$

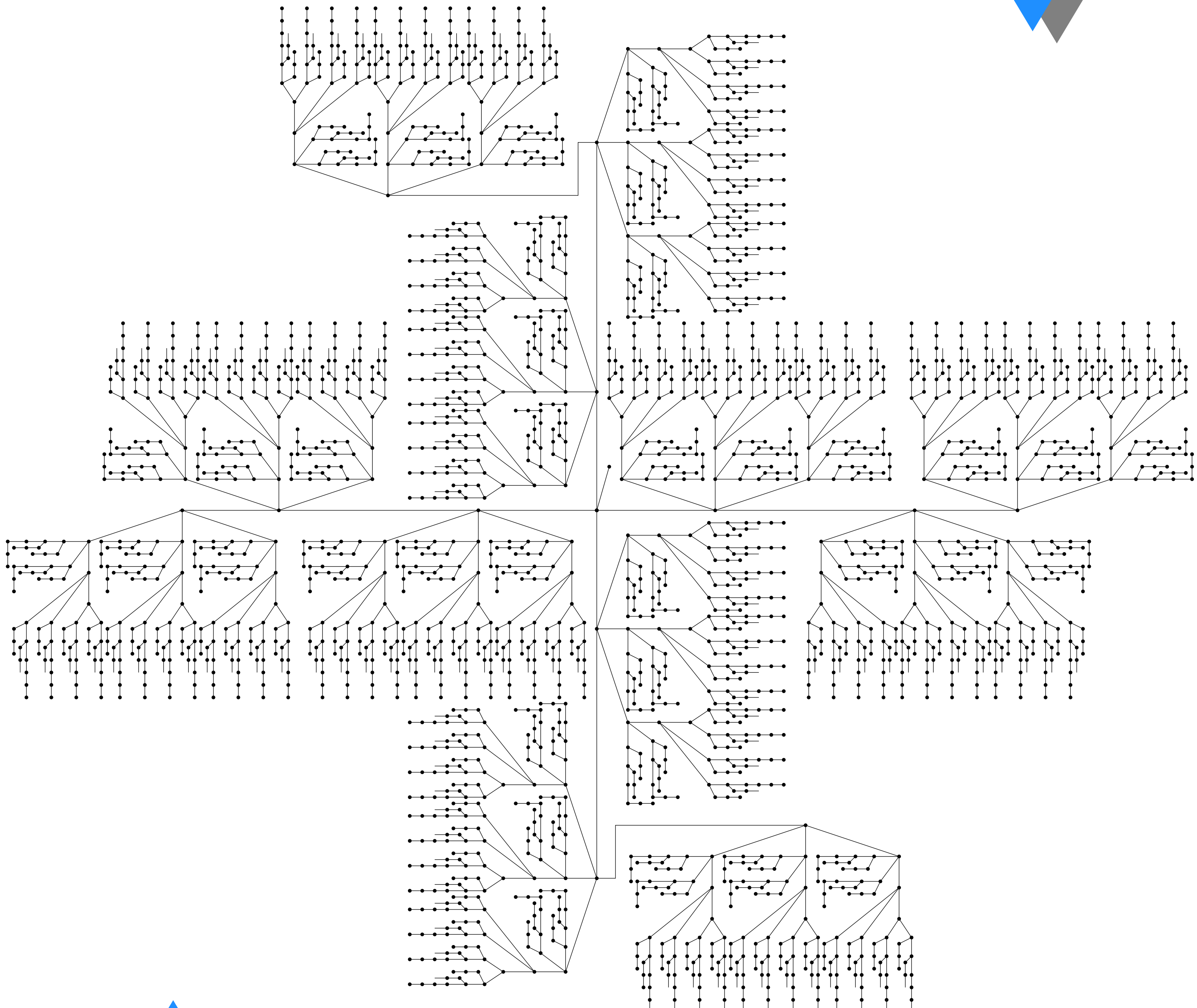
Proof.

- all w colors are blocked along p_i
- some edge of p_i must block at least $\left\lceil \frac{w}{\mathrm{len}(p_i)} \right\rceil$ colors
- max load is $L \geq 1 + \left\lceil \frac{w}{\mathrm{len}(p_i)} \right\rceil (\hat{\mu} - 1)$
- $\mu_{\mathrm{OPT}} \geq \left\lceil \frac{L}{w} \right\rceil$
- $\mathrm{PoA} = \frac{\hat{\mu}}{\mu_{\mathrm{OPT}}} \leq \frac{\hat{\mu}}{\left\lceil \frac{1+\left\lceil \frac{w}{\mathrm{len}(p_i)} \right\rceil (\hat{\mu}-1)}{w} \right\rceil} \leq \mathrm{len}(p_i)$

A matching lower bound

$w = L = \mathrm{len}(p_i) = \hat{\mu} = k$
$\mu_{\mathrm{OPT}} = 1$
$\mathrm{PoA} = k$





What about graphs with degree 2?

A more involved structural property:

$P(e, \alpha_i)$: the set of paths using edge e that are colored with α_i.

Lem. In a NE of an S-PMC(RING) game, $\forall$ edge e and $\forall \alpha_i$ there is an arc s.t.:

- $\forall \alpha_j \neq \alpha_i$ the arc contains an edge which is an α_j-blocking edge for at least half of the paths in $P(e, \alpha_i)$, and
- $\forall e'$ in the arc, $|P(e', \alpha_i) \cap P(e, \alpha_i)| \geq \left\lceil \frac{|P(e,\alpha_i)|}{2} \right\rceil$

Establishing an edge with high load

Repeated application of the previous Lemma yields:

Lem. In every S-PMC(RING) game $\langle G, P, w \rangle$ with $\hat{\mu} \geq w$ there is an edge with load at least $\frac{\hat{\mu} w}{4}$

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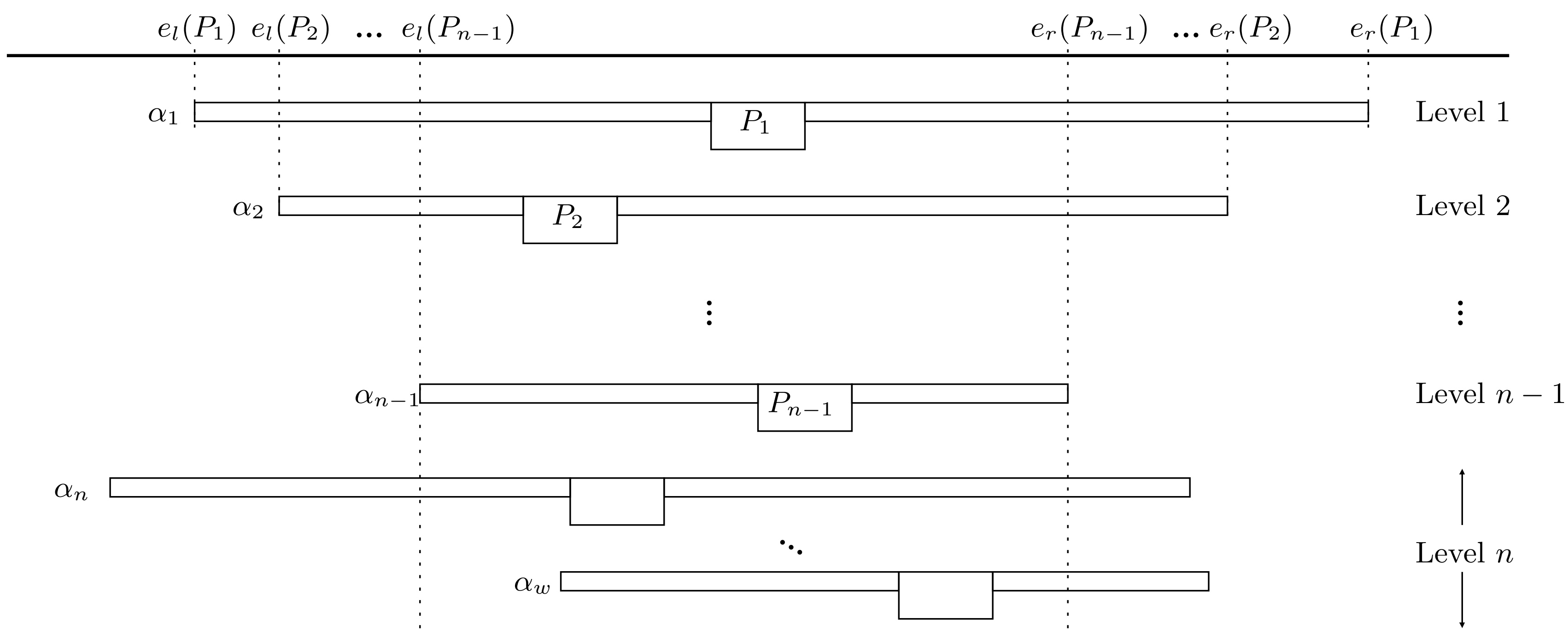

Constant PoA for $L = \Omega(w^2)$

Thm. For any S-PMC(RING: $L = \Omega(w^2)$) game, $\mathrm{PoA} = O(1)$

Proof.

- If $\hat{\mu} \geq w$, then $L \geq \frac{\hat{\mu}w}{4} \Rightarrow \mu_{\mathrm{OPT}} \geq \frac{\hat{\mu}}{4} \Rightarrow \mathrm{PoA} \leq 4$
- If $\hat{\mu} < w$, then:

$$\mathrm{PoA} = \frac{\hat{\mu}}{\mu_{\mathrm{OPT}}} \leq \frac{\hat{\mu}w}{L} < \frac{w^2}{L} = O(1)$$

Unbounded PoA for $L = o(w^2)$

Thm. For any $\varepsilon > 0$ there is an infinite family of S-PMC(CHAIN: $L = \Theta(w^{2-\varepsilon})$) games with $\mathrm{PoA} = \Omega(w^{\frac{\varepsilon}{2}})$.

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Proof (sketch). For any $\varepsilon > 0$ and any $\rho \geq 4$, we can construct a game and a strategy profile thereof with:

$$w = \left\lceil \rho^{1+\frac{\varepsilon}{2-\varepsilon}} \right\rceil, \;\; L = \Theta(\rho^2), \;\; \mu_{\max} = \rho \;\; .$$

The PoA of this game is therefore:

$$\mathrm{PoA} = \frac{\hat{\mu}}{\mu_{\mathrm{OPT}}} > \frac{\hat{\mu}}{\frac{L}{w} + 1} = \frac{w \cdot \mu_{\max}}{L + w} = \Omega(w^{\frac{\varepsilon}{2}}) \;\; .$$

Further work

- Bounds for convergence
- Complexity of computing Nash Equilibria
- Selfish routing and wavelength assignment

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... Thank you!

[NPZ01] C. Nomikos, A. Pagourtzis, S. Zachos: Routing and path multicoloring. Inf. Process. Lett. 80(5): 249-256 (2001)

[AZ04] M. Andrews, L. Zhang: Wavelength Assignment in Optical Networks with Fixed Fiber Capacity. ICALP 2004: 134-145

[KP99] E. Koutsoupias, C.H. Papadimitriou: Worst-case Equilibria. STACS 1999: 404-413

[ADK+04] E. Anshelevich, A. Dasgupta, J.M. Kleinberg, É. Tardos, T. Wexler, T. Roughgarden: The Price of Stability for Network Design with Fair Cost Allocation. FOCS 2004: 295-304

[MS96] D. Monderer, L.S. Shapley: Potential games. Games and Economic Behavior 14 (1996) 124-143

[Ros73] R.W. Rosenthal: A class of games possessing pure-strategy nash equilibria. Int. J. Game Theory 2 (1973) 65-67

[BM06] C. Busch, M. Magdon-Ismail: Atomic Routing Games on Maximum Congestion. AAIM 2006: 79-91

[BO06] R. Banner, A. Orda: Bottleneck Routing Games in Communication Networks. INFOCOM 2006

[FKK+02] D. Fotakis, S.C. Kontogiannis, E. Koutsoupias, M. Mavronicolas, P.G. Spirakis: The Structure and Complexity of Nash Equilibria for a Selfish Routing Game. ICALP 2002: 123-134