LONG CYCLES IN GRAPHS WITH SOME LARGE DEGREE VERTICES

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Long cycles in graphs with some large degree vertices

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Abstract

Let $G$ be a graph of order $n$ and $k$ an integer with $3 \leq k \leq n-1$. We obtain that if there are at least $n/2 - 1$ vertices of degree at least $k$ then either the circumference of $G$ is at least $k$ or $G$ has a subgraph isomorphic to the graph obtained from $K_{1,k+1}$ by adding an edge between any pair of vertices in the $k+1$-vertex-part. (Hence the circumference of $G$ is at least $k-1$.) By using above result, we show that the following conjecture of Woodall is true if the graph is 3-connected and $k \geq 25$: if a 2-connected graph of order $n$ has at least $n/2 + k$ vertices of degree at least $k$, then it has a cycle of length at least $2k$. This conjecture was one of the 50 unsolved problems in [2].

1 Introduction and notation

All the graphs considered in this paper are undirected and simple. We use the notation and terminology in [2]. In addition, for a graph $G = (V(G), E(G))$, let $H$ be a subgraph of $G$. Then the neighborhood in $H$ of a vertex $u \in V(G)$ is $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ and the degree of $u$ in $H$ is $d_H(u) = |N_H(u)|$. The minimum degree in $G$ of the vertices in $H$ is denoted by $\delta(H)$. If $X \subseteq V(G)$, let $N_H(X) = \bigcup_{u \in X} (N_H(u) - X)$. In the case $H = G$, we use $N(u), d(u), \delta$ and $N(X)$ instead of $N_G(u), d_G(u), \delta(G)$ and $N_G(X)$, respectively.

If $C = c_1c_2...c_p$ is a cycle, we let $C[c_i, c_j]$, for $i \leq j$, be the subpath $c_ic_{i+1}...c_j$, and $C[c_j, c_i] = c_jc_{j-1}...c_i$, where the indices are taken modulo $p$. We will consider $C[c_i, c_j]$ and $C[c_j, c_i]$ both as paths and as vertex-sets. Define $C(c_i, c_j) = C[c_{i+1}, c_j]$, $C(c_i, c_j) = C[c_i, c_{j-1}]$ and $C(c_i, c_j) = C[c_{i+1}, c_{j-1}]$. For any $i$, we put $c_i^+ = c_{i+1}$, $c_i^- = c_{i-1}$, and for any $j \geq 2$, $c_i^{+j} = c_{i+j}$ and $c_i^{-j} = c_{i-j}$. For $A \subseteq C$, we set $A^+ = \{v^+|v \in A\}, A^- = \{v^-|v \in A\}$,
for any $j \geq 2$, $A^{+j} = \{v^{+j} | v \in A\}$ and $A^{-j} = \{v^{-j} | v \in A\}$. We will use similar definitions for a path.

We denote by $c(G)$ the circumference, i.e. the length of a longest cycle in $G$.

Various longest cycle problems are interesting and important in basic graph theory and have been deeply studied. The main problem studied in this paper is the circumferences of graphs. A classical result is due to Dirac.

**Theorem 1** [3]: If $G$ is a 2-connected graph on $n \geq 3$ vertices, then $c(G) \geq \min \{n, 2\delta\}$.

The above results based on conditions on degrees of all vertices of the graph. It is natural to ask if we can still get a long cycle when the graph contains many vertices of large degrees. We obtain the followings:

**Theorem 2**: Let $G$ be a graph of order $n$ and $k$ an integer with $3 \leq k \leq n-1$. If there are at least $n/2 - 1$ vertices of degree at least $k$ then either the circumference of $G$ is at least $k$ or $G$ has a subgraph isomorphic to the graph $K_{k-1, k+3}^*$ which is obtained from the complete bipartite graph $K_{k-1, k+3}$ by adding an edge between any pair of vertices in the $k-1 \over 2$-vertex-part. (Hence the circumference of $G$ is at least $k-1$).

The following examples are interesting. Let $K_{k-1, k+3}^* := D(X, Y)$ with $|X| = k-1 \over 2$ and $|Y| = k+3 \over 2$. Pick up $q$ copies $D_i(X_i, Y_i), 1 \leq i \leq q$, of $D(X, Y)$ and let $u_i, v_i \in Y_i$. Denote by $H$ the graph obtained by identifying $v_i$ and $u_{i+1}$ for $1 \leq i \leq q-1$. Then $H$ has $q(k-1)/2 + q - 1$ vertices of degree at least $k$ and we have $q(k-1)/2 + q - 1 = q(k-1)/2 + q(k+3)/2 - 1 + k+3/2$. These examples show that the circumference may be less than $k$ even if the number of vertices of degree at least $k$ is at least $n/2 + c$ for any fixed $c$.

As an improvement of Dirac’s theorems, Woodall made the following conjecture in 1975: If a 2-connected graph of order $n$ has at least $n \over 2 + k$ vertices of degree at least $k$, then it has a cycle of length at least $2k$. This conjecture was one of the 50 unsolved problems in the book [2] and has been essentially proved in [5]. But we give a proof of the followings by using Theorem 2.

**Theorem 3**: If $k \geq 25$ and a 3-connected graph of order $n$ has at least $n \over 2 + k$ vertices of degree at least $k$, then it has a cycle of length at least $2k$.

2 Preliminary lemmas

**Lemma 1**: Let $G = (V, E)$ be any 2-connected graph and $B := \{v : d(v) \geq k-1\}$, $3 \leq k \leq n/2$. If $S := G - B$ is independent and if for any set $X \subseteq S$ with a common
neighbor (i.e., $X \subseteq S \cap N(x)$ for some $x \in B$),

$$|N(X)| > \frac{|X|}{2},$$

then $G$ has a cycle of length at least $\min\{\lfloor B \rfloor, k\}$.

**Proof of Lemma 1**: Here we just give a proof for existence of a cycle with at least $\min\{\lfloor B \rfloor, k - 3\}$ vertices. A detailed proof of the lemma can be found in Appendix.

Let $P := v_1v_2...v_p$ be a path in $G$ such that

(a) $v_1, v_p \in B$;
(b) subject to (a) $P$ contains as many as possible vertices of $B$;
(c) subject to the above, $P$ is as long as possible

Firstly we study several properties of the path $P$.

If there is a cycle $C$ containing all the $B$-vertices on $P$, then it is clear that either $C$ contains all $B$-vertices (and hence $|C| \geq \lfloor B \rfloor$) or there is another path containing more $B$-vertices than $P$. We assume that no such cycle exists.

If $v_i \in N(v_1) \cap P$, the cycle $P[v_i, v_{i-1}]v_iv_1$ is of length $i$. Thus there is a cycle of length at least $|N(v_1) \cap P| + 1$. Since $d_G(v_i) \geq k - 1$, without loss of generality we assume that $S_1 := N(v_1) - P \neq \emptyset$ and similarly $S_p := N(v_p) - P \neq \emptyset$. By the choice of $P$ and the independence of $S$, we have $S_1 \subseteq S$, $S_p \subseteq S$, $S_1 \cap S_p = \emptyset$, $N(S_1) \cup N(S_p) \subseteq B \cap P$, $(N(S_1) - \{v_1\})^- \cup (N(S_p) - \{v_p\})^+ \subset S$ and $S_1 \cap S_p^- = S_p \cap S^+ = \emptyset$.

We have

$$N(S_1)^- \cap N(S_p)^+ = \emptyset$$

since $N(S_1)^- \cup N(S_p)^+ \subseteq S$ and $S$ is independent, and

$$(N(v_1) \cup N(S_1))^+ \cap (N(S_p) \cup N(v_p)) = \emptyset$$

since otherwise there is a cycle containing $V(P)$, a contradiction. If $v_s \in N(S_1)^- \cap N(S_p)^+$, then there is a cycle containing $V(P) - \{v_s\}$, a contradiction because $v_s \in S$. It follows that

$$N(S_1)^- \cap N(S_p)^+ = \emptyset.$$

Since $G$ is 2-connected, there exists a vine $Q := \{H_l[v_i, v_j] : 1 \leq l \leq m\}$ on the path $P(v_1, v_p)$, where $H_l[v_i, v_j]$ is a path between $v_i$ and $v_j$, with all internal vertices in $G - P(v_1, v_p)$, such that $1 = i_1 < i_2 < j_1 < i_3 < j_2 < \ldots < i_m < j_{m-1} < j_m = p$. We have the following cycles:

If $m$ is even,

$$C_Q := v_1v_2...v_{i_2}^+H_2P[v_2^+, v_{i_2}^-]H_4P[v_{i_2}^+, v_{i_3}^-]...v_{i_m}^-H_mP[v_{i_m-1}, v_{i_m}^+]$$

$$H_{m-1}P[v_{i_{m-1}}^-, v_{i_{m-2}}^+]...P[v_{i_3}^-, v_{i_2}^+]H_1;$$

where $v_i^+$ and $v_i^-$ denote the endpoints of $H_i[v_i, v_j]$. (Note that these $H_i$ can be empty.)
and if \( m \) is odd,
\[
C_Q := v_1v_2...v_{i_2}H_2P[v_{j_2}, v_{i_3}]H_4P[v_{j_4}, v_{i_5}]...v_1H_m-1P[v_{j_{m-1}}, v_{p-1}]
\]
\[
H_mP[v_{i_1}, v_{j_1}]H_{m-2}P[v_{i_3}, v_{j_3}]...P[v_{i_{m-2}}, v_{j_{m-2}}]v_{j_1}H_1.
\]

Clearly we may choose the vine \( Q \) such that
\[
(N(v_1) \cap P) \cup N(S_l) \subseteq P[v_1, v_{i_1}] \quad \text{and} \quad (N(v_p) \cap P) \cup N(S_p) \subseteq P[v_{j_{m-2}}, v_p].
\]

We will prove that \( |C_Q| \geq k - 3 \).

Put
\[
U_1 := \{v_1\} \cup N_P(v_1) \cup (N_P(v_p) - \{v_{i_1}\})^+.
\]

and
\[
U_2 := N(S_1) \cup (N(S_1) - \{v_1, v_{j_1}\})^{-} \cup (N(S_p) - \{v_p, v_{i_1}\})^{+} \cup (N(S_p) - \{v_p, v_{i_1}, v_{j_{m-1}}\})^{+2}.
\]

From the disjoint properties that we have obtained above, it follows that
\[
|C_Q| \geq |U_1| = |N_F(v_1)| + |N_F(v_p)|
\]
and
\[
|C_Q| \geq |U_2| \geq 2|N(S_1)| + 2|N(S_p)| - 6
\]
\[
= |S_1| + |S_2| - 4.
\]

These give
\[
|C_Q| \geq \frac{1}{2}(|U_1| + |U_2|)
\]
\[
\geq \frac{1}{2}(d(v_1) + d(v_p) - 4)
\]
\[
\geq k - 3.
\]

\[ \Box \]

**Lemma 2:** Let \( k \) be an integer with \( 3 \leq k \leq n - 1 \) and \( G \) a connected graph of order \( n \) such that

(a) there are at least \( n/2 - 1 \) vertices of degree at least \( k \),

(b) all vertices of degree less than \( k \) are independent,

(c) any \( B \)-vertex is adjacent to at most one vertex of degree 1 and

(d) there does not exist a vertex \( v \) such that \( G - v \) has at least two components containing vertex of degree at least \( k \),

then either the circumference of \( G \) is at least \( k \) or \( G = K_{\frac{n}{2} - 1, \frac{n}{2} + 1} \) (in this case \( k = n - 1 \) and the circumference is \( n - 2 \)).

**Proof of Lemma 2:** Again put \( B = \{v : d(v) \geq k\} \) and \( S = V(G) - B \).
Let $H$ be the graph obtained from $G$ by deleting all vertices of degree 1. We will show that there is a cycle of length at least $\min\{|B|, k\}$ in $H$.

To prove this, without loss of generality we assume that $H$ is a minimum counter-example. $H$ is 2-connected and every $B$-vertex has degree at least $k - 1$ in $H$. For any subset $S^* \subseteq S \cap H$, since every vertex in $B - N_H(S^*)$ is of degree at least $k - 1$, by the minimality hypothesis, we get $|B - N_H(S^*)| < (|S - S^*|)/2 - 1$, which gives $|S^*| < 2|N_H(S^*)|$. By using Lemma 1, there is a cycle of length at least $\min\{|B|, k\}$ in $H$.

Suppose that $G$ does not contain a cycle of length at least $k$. Hence $k \geq |B| + 1 \geq n/2$.

By a theorem in [6] and [1], $G$ has a cycle containing all $B$-vertices. Let $C$ be a longest cycle in $G$ with $B \subseteq V(C)$. Let $B_0$ be the set of all $B$-vertices such that their processors in $C$ belong to $B$. Since $S$ is independent, $|B_0| = |B| - |S \cap C|$.

If two $B_0$-vertices $b_1$ and $b_2$ have a common neighbor $w \in S - C$, by the definition, $b_1^+$ and $b_2^+$ are in $B$ and thus, have degrees at least $n/2$. By a traditional proof, we can get a longer cycle than $C$, a contradiction. So we assume that any pair of $B_0$-vertices have no common neighbor in $S - C$.

Let $b \in B_0$. Since $b^+$ has degree at least $k$ and $|C| \leq k - 1$, $b^+$ has some neighbor $s \in S - C$ with $d(s) \geq 2$ by (c). So from (b), $s$ has a neighbor $b_1 \in C - \{b, b^+, b^{+2}\}$. By the maximality of $C$, we deduce $b$ is not adjacent to $b_1^-$. So we assume that any $B_0$-vertex has at least one nonadjacency in $C$.

It follows from the above assumptions and $|C| \leq k - 1$ that every $B_0$-vertex has at least three neighbors in $S - C$ and all these neighbors are different. So $|S - C| \geq 3(|B| - |S \cap C|)$. Also $|S| = |S - C| + |S \cap C| \geq 3|B| - 2|S \cap C|$. Hence $|S \cap C| \geq |B| - 1$ since $|B| \geq n/2 - 1$. We have $k - 1 \geq |C| = |B| + |S \cap C| \geq 2|B| - 1 \geq n - 3$.

If $k = n - 1$ it is easy to deduce directly that $G = K_{\frac{n}{2} - 1, \frac{n}{2} - 1}$. If $k = n - 2$, all the equalities holds in the above paragraph. From $|S \cap C| = |B| - 1, B_0 = \{b\}$ for some $b \in B$, by the above augment, $b$ has at least three neighbors in $S - C$. Similarly $b^+$ should have three neighbors in $S - C$. But $S - C$ has at most three vertices. It follows that $b$ and $b^+$ have common neighbor in $S - C$ and $C$ can be extended, a contradiction.

\[ \square \]

3 The main results

Proof of Theorem 2: Let $k$ and $n$ be integers with $3 \leq k \leq n - 1$. Suppose to the contrary, that there is a graph $G$ of order $n$ such that there are at least $n/2 - 1$ vertices of degree at least $k$ and that the circumference of $G$ is less than $k$ and $G$ has no subgraph isomorphic to the graph $K_{\frac{n}{2} - 1, \frac{n}{2} + 1}$.

To get a contradiction, we just prove that $G$ satisfies the conditions of Lemma 2. Without loss of generality, we assume that $G$ is a minimum counter-example of the theorem. By the minimality, we may assume that $G$ is connected and $S$ is independent.
Suppose first that there exists some vertex \( v \) such that \( G - v \) has at least two components \( H_1 \) and \( H_2 \) with \( B_1 := H_1 \cap B \neq \emptyset \) and \( B_2 := H_2 \cap B \neq \emptyset \). Put \( G_1 := G[B_1 \cup \{v\}] \) and \( G_2 := G[B_2 \cup \{v\}] \). Since \( |B_1| + |B_2| \geq |B| - |\{v\}| \geq n/2 - 2 \geq \frac{1}{2}(|V(G_1)| + |V(G_2)| - 1) - 2 \geq \frac{1}{2}V(G_1)| - 1 + \frac{1}{2}|V(G_2)| - 1 - \frac{1}{2} \) and hence at least one of \( G_1 \) and \( G_2 \), say \( G_1 \), has at least \( \frac{1}{2}|V(G_1)| - 1 \) vertices of degree at least \( k \). By the minimality hypothesis of \( G_1 \), either the circumference of \( G_1 \) is at least \( k \) or \( G_1 \) has a subgraph isomorphic to the graph \( K_{2,1}^{*}, \frac{5}{2} \). Since \( G_1 \) is a subgraph of \( G \), we have a contradiction. Therefore we assume that there does not exist a vertex \( v \) such that \( G - v \) has at least two components containing vertex of degree at least \( k \).

For any \( v \in B \) such that \( S_0 := \{u \in S \cap N(v) : d(u) = 1\} \). Put \( G_1 := G - S_0 \). Clearly \( G_1 \) has at least \( |B| - |\{v\}| \) vertices of degrees at least \( k \). By the minimality of \( G \), we deduce that \( |B| - 1 < \frac{1}{2}|G_1| - 1 = \frac{1}{2}(n - |S_0|) - 1 \) and hence \( |S_0| \leq 1 \).

We have shown that \( G \) satisfies the conditions (a),(b),(c) and (d) of Lemma 2 and so by Lemma 2, either the circumference of \( G \) is at least \( k \) or \( G \) has a subgraph isomorphic to the graph \( K_{2,1}^{*}, \frac{5}{2} \). This contradiction completes the proof.

\[ \square \]

**Proof of Theorem 3**: Suppose that \( G \) is a 3-connected graph of order \( n \) such that at least \( \frac{n}{2} + k \) vertices are of degree at least \( k \), \( k \geq 25 \) and \( G \) does not contain a cycle of length at least \( 2k \). Denote by \( B = \{u \in V(G) : d(u) \geq k\} \) and \( S = V(G) - B \).

Let \( C = c_1c_2...c_pc_1 \) be a longest cycle in \( G \). Since \( G - C \) contains at least \( \frac{n}{2} + k - (2k - 1) = \frac{n}{2} - k + 1 \geq |S| + 1 \) vertices in \( B \). Hence there exists a component \( H \) of \( G - C \) such that \( |H \cap B| \geq |S\cap H| + 1 \). Let \( d = k - \max\{|N(u) \cap C| : u \in H \cap B\} \) and \( N(u_f) \cap C = \{c_{m_1}, c_{m_2}, ..., c_{m_{d-1}}\} \subseteq C \). Then every vertex of \( H \cap B \) has degree at least \( d \) in \( H \).

By Theorem 2, either \( H \) admits a longest cycle \( C_H \) of \( q \geq d \) vertices or \( H \) has a subgraph \( C_H \) isomorphic to \( K_{2,1}^{*}, \frac{5}{2} \).

We claim that the longest cycle \( C_H = u_1u_2...u_qu_1 \) in \( H \) has at least 8 vertices.

If \( H \cap B = \{u\} \) then \( H = \{u\} \) and by the maximality of \( C \), \( |C| \geq 2d(u) \geq 2k \). Assume that \( |H \cap B| \geq 2 \) and \( q \leq 7 \). For any vertex \( u \in H \cap B - \{u_f\} \), by the maximality of \( C \), we have \( |C| \geq |N_C(u_f)| + |N_C(u_f)|^{+2} + |N_C(u)| \geq 2(k - d) + |N_C(u)| \) and hence \( |N_C(u)| \leq 2d - 1 \) and \( |N_H(u)| \geq k - 2d + 1 \). It follows that in the subgraph \( H - \{u_f\} \), there are at least \( \frac{|H - \{u_f\}|}{2} \) vertices of degree at least \( k - 2d \). By Theorem 2, \( H - \{u_f\} \) has a cycle of at least \( \min\{k - 2d - 1, d - 1\} \) vertices. Then \( k - 2d - 1 \leq 7 \) and \( d - 1 \leq 7 \), contrary to \( k \geq 25 \). The claim holds.

Since \( G \) is 3-connected, there are three disjoint paths \( P_1, P_2, P_3 \) between three distinct vertices \( c_i, c_j, c_m \in C \) and three distinct vertices \( u_{i'}, u_{j'}, u_{m'} \in C_H \) respectively.
Assume first that \(d \geq k - 2\). By the maximality of \(C\), if \(C_H^g\) is a cycle of \(q \geq d\) vertices, we have

\[
|C(c_i, c_j)| \geq |C_H^g[u_j, u_{j'}]C_H^g(u_{j'}, u_{j''})|, |C(c_j, c_m)| \geq |C_H^g[u_{j'}, u_{j''}]C_H^g(u_{j''}, u_{j'''})| \text{ and } |C(c_m, c_i)| \geq |C_H^g[u_{j''}, u_{j'''}]C_H^g(u_{j'''}, u_{j'})|.
\]

\[
|C| \geq |\{c_i, c_j, c_m\}| + |C(c_i, c_j)| + |C(c_j, c_m)| + |C(c_m, c_i)| \geq 3 + 2|C_H^g[u_{j'}, u_{j''}]| + 2|C_H^g(u_{j''}, u_{j'''})| + 2|C_H^g(u_{j'''}, u_{j'})| + 3|\{u_{j'}, u_{j''}, u_{j'''}\}|
\]

\[
\geq 2k.
\]

When \(C_H^g = K_{\frac{k+1}{2}, \frac{k+1}{2}}\), then clearly \(|C(c_i, c_j)| \geq d - 2\), \(|C(c_j, c_m)| \geq d - 2\) and \(|C(c_m, c_i)| \geq d - 2\). It follows that when \(k \geq 9\),

\[
|C| \geq |\{c_i, c_j, c_m\}| + |C(c_i, c_j)| + |C(c_j, c_m)| + |C(c_m, c_i)| \geq 3k - 9.
\]

Then we assume that \(d \leq k - 3\). Then clearly \(|C(c_m, c_{m+1})| \geq 1\) for any \(g\).

Without loss of generality we may choose the paths \(P_1, P_2, P_3\) such that if \(u_f \in C_H^g\), \(u_f = u_{m'}\) and if \(u_f \notin C_H^g\), there is a path \(P_4\) between \(u_f\) and the vertex \(u_{m'}\) such that \(P_3 = P_4(u_{m'}, u_f)C_H^g(u_f, c_m)\).

When \(C(c_i, c_j) \cap N(u_f) \neq \emptyset\) and \(C(c_j, c_i) \cap N(u_f) \neq \emptyset\), let \(c_{m'}, c_{m''} \in N(u_f) \cap C(c_i, c_j)\) and \(c_{m''}, c_{m''} \in N(u_f) \cap C(c_j, c_i)\) such that \((C(c_{m'}, c_i) \cup C(c_i, c_{m''})) \cap (N(u_f) \cup \{c_j\}) = \emptyset\) and \((C(c_{m''}, c_j) \cup C(c_j, c_{m''})) \cap (N(u_f) \cup \{c_i\}) = \emptyset\) (i.e., \(c_{m''}\) is the last vertex of \(N(u_f) \cap C\) before \(c_i\), \(c_{m''}\) is the first vertex of \(N(u_f) \cap C\) before \(c_i\), and \(c_{m''}\) is the first vertex of \(N(u_f) \cap C\) after \(c_i\)). If \(C_H^g\) is a cycle, by the maximality of \(C\), we have \(|C(c_{m''}, c_j)| \geq |C_H^g[u_{m''}, u_f]C_H^g(u_f, u_{j'})|\) and \(|C(c_{m''}, c_{m'''})| \geq |C_H^g[u_{m''}, u_f]|\).

These give \(|C(c_{m''}, c_{m'''})| \geq |C_H^g| + 3\). Similarly we have \(|C(c_{m''}, c_{m'''})| \geq |C_H^g| + 3\). It follows that when \(q \geq d\)

\[
|C| \geq |N(u_f)| - 2 + |N(u_f)| - 4 + 2(|C_H^g| + 3)
\]

\[
\geq 2(k - d) - 6 + 2q + 6
\]

\[
\geq 2k,
\]
a contradiction. It follows that \(8 \leq q \leq d - 1\) and \(C_H^g = K_{\frac{k+1}{2}, \frac{k+1}{2}}\). Clearly \(|C(c_m, c_i)| \geq d - 2\), \(|C(c_i, c_{m''})| \geq d - 2|C(c_m, c_j)|\) and \(|C(c_j, c_{m''})| \geq d - 2|C(c_{m''}, c_j)|\). Then we obtain

\[
|C| \geq |N(u_f)| + |N(u_f)| - 4 + 4(d - 2)
\]

\[
\geq 2(k - d) - 4 + 4d - 8
\]

\[
\geq 2k + 2d - 12
\]

\[
\geq 2k,
\]
a contradiction.
Assume then that at least one of \(C(c_i, c_j) \cap N(u_f)\) and \(\overline{C}(c_i, c_j) \cap N(u_f)\), say \(C(c_i, c_j) \cap N(u_f) = \emptyset\).

Let \(c_{m_h}, c_{m_g} \in N(u_f) \cap C(c_j, c_i)\) such that \((C(c_{m_h}, c_i) \cup C(c_j, c_{m_g})) \cap N(u_f) = \emptyset\) (i.e., \(c_{m_h}\) is the last vertex of \(N(u_f) \cap C\) before \(c_i\), \(c_{m_g}\) is the first vertex of \(N(u_f) \cap C\) after \(c_j\)).

Let \(C(c_{m_h}, c_i) \neq \emptyset\) and \(C(c_{m_h}, c_i) \cap (N(u_f) \cup \{c_j\}) = \emptyset\) (i.e., \(c_{m_h}\) is the last vertex of \(N(u_f) \cap C\) before \(c_i\)) and let \(C(c_j, c_{m_g}) \neq \emptyset\) and \(C(c_j, c_{m_g}) \cap (N(u_f) \cup \{c_i\}) = \emptyset\) (i.e., \(c_{m_g}\) is the first vertex of \(N(u_f) \cap C\) after \(c_j\)).

If \(C_H\) is a cycle, by the maximality of \(C\), we have \(|C(c_{m_h}, c_i)| \geq |C_H(u_f', u_j')C_H[u_f', u_{m'}]|\)

\(|C(c_i, c_j)| \geq |\overline{C_H}[u_f', u_{m'}]C_H[u_{m'}, u_j']|\) and \(|C(c_j, c_{m_g})| \geq |\overline{C_H}[u_f', u_{m}]C_H[u_{f'}, u_{m'}]|\). These give

\[
|C| \geq |N(u_f)| + |N(u_f)| - 3 + 2|C_H| + 3 \\
\geq 2(k - d) + 2q,
\]

a contradiction when \(q \geq d\). It implies that \(8 \leq q \leq d - 1\) and \(C_H = K^*_{d-3, \frac{d+3}{2}}\). Since \(d \geq 9\) and \(|C(c_{m_h}, c_i)| \geq d - 2\), \(|C(c_i, c_j)| \geq d - 2\) and \(|C(c_j, c_{m_g})| \geq d - 2\), we obtain

\[
|C| \geq |N(u_f)| + |N(u_f)| - 3 + 3(d - 2) \\
\geq 2(k - d) + 3d - 9 \\
\geq 2k + d - 9 \\
\geq 2k,
\]

a contradiction.

The proof is complete.

\[\square\]

4 Appendix: Proof of Lemma 1

For any vertex \(v\) and a condition \(A\), let \(\theta(v : A) = \{v\}\) if \(A\) is satisfied or \(\theta(v : A) = \emptyset\) if \(A\) is not satisfied.

**Proof of Lemma 1:** Let \(P := v_1v_2...v_p\) be a path in \(G\) such that

(a) \(v_1, v_p \in B\);

(b) subject to (a) \(P\) contains as many as possible vertices of \(B\);

(c) subject to the above, \(P\) is as long as possible and

(d) subject to the above, \(\max\{i : v_iv_1 \in E(G)\}\) is as large as possible.

Firstly we study several properties of the path \(P\).
If there is a cycle $C$ containing all the $B$-vertices on $P$, then it is clear that either $C$ contains all $B$-vertices (and hence $|C| \geq |B|$) or there is another path containing more $B$-vertices than $P$. We assume that no such cycle exists.

If $v_i \in N(v_1) \cap P$, the cycle $P[v_1, v_{i-1}]v_iv_1$ is of length $i$. Thus there is a cycle of length at least $|N(v_1) \cap P| + 1$. Since $d_G(v_1) \geq k - 1$, without loss of generality we assume that $S_1^0 := N(v_1) - P \neq \emptyset$ and similarly $S_p^0 := N(v_p) - P \neq \emptyset$. Put $S_1 := S_1^0 \cup \theta(v_2 : v_2 \in N(S_1^0)^-)$ and $S_p := S_p^0 \cup \theta(v_{p-1} : v_{p-1} \in N(S_p^0)^+)$.

By the choice of $P$ and the independence of $S$, we have $S_1 \subseteq S$, $S_p \subseteq S$, $S_1 \cap S_p = \emptyset$, $N(S_1) \cup N(S_p) \subseteq B \cap P$, $(N(S_1) - \{v_1\}) \cup (N(S_p) - \{v_p\})^+ \subseteq C$ and $S_1 \cup S_1^- = S_p \cup S_p^+ = \emptyset$.

We have

$$N(S_1)^- \cap N(S_p)^{+2} = \emptyset$$

since $N(S_1)^- \cup N(S_p)^+ \subseteq S$ and $S$ is independent, and

$$(N(v_1) \cup N(S_1))^- \cap (N(S_p) \cup N(v_p)) = \emptyset$$

since otherwise there is a cycle containing $V(P)$, a contradiction. If $v_s \in N(S_1)^- \cap N(S_p)^+$, then there is a cycle containing $V(P) - \{v_s\}$, a contradiction because $v_s \in S$. It follows that

$$N(S_1)^- \cap N(S_p)^+ = \emptyset.$$

For any $w^* \in S_1 \cap N(v_1)$ and $w^{**} \in S_p \cap N(v_{p-2})$, define a path $P(w^*, w^{**}) := v_1 v_2 v_3 \ldots v_{p-2} w^{**} v_p$ which has the same properties as $P$.

Since $G$ is 2-connected, there exists a vine $Q := \{H_i[v_i, v_j] : 1 \leq i \leq m\}$ on the path $P(w^*, w^{**})$, where $H_i[v_i, v_j]$ is a path between $v_i$ and $v_j$, with all internal vertices in $G - P(w^*, w^{**})$, such that $1 = i_1 < i_2 < j_1 \leq i_3 < j_2 \leq \ldots < i_m < j_{m-1} < j_m = p$. We have the following cycles:

If $m$ is even,

$$C_Q := v_1 w^* \ldots v_{i_4}^{-} H_2 P[v_{j_4}^+, v_{i_4}^-] H_4 P[v_{j_4}^+, v_{i_4}^-] \ldots v_m^{-} H_m P[w^{**}, v_{j_{m-1}}^+] H_{m-1}^{-1} P[v_{i_{m-1}}^+, v_{j_{m-3}}^+] \ldots P[w_{i_2}^+, v_{i_1}^-] H_1^{-1}$$

and if $m$ is odd,

$$C_Q := v_1 w^* \ldots v_{i_4}^{-} H_2 P[v_{j_4}^+, v_{i_4}^-] H_4 P[v_{j_4}^+, v_{i_4}^-] \ldots v_m^{-} H_m P[v_{j_1}^+, v_{j_{m-1}}^{-1}] H_{m-1}^{-1} P[v_{j_{m-1}}^+, w^{**}] H_m^{-1} P[v_{i_{m-1}}^+, v_{j_{m-3}}^+] \ldots P[w_{i_2}^+, v_{i_1}^-] H_1^{-1}.$$

We note that in the above cases, the paths $H_1$ and $H_m$ are contained in the cycles.

We may choose a vine $Q (N(v_1) \cap P) \cup N(S_1) \subseteq P[v_1, v_{i_3}]$ and $(N(v_p) \cap P) \cup N(S_p) \subseteq P[v_{j_{m-2}}, v_p]$.

Let $v^*$ be the first vertex on $P$ that is adjacent to $v_p$. Put

$$U_1 := \{v_1\} \cup N_P(v_1) \cup (N_P(v_p) - \{v_{i_m}\})^+ \cup \theta(v_{j_{m-1}} : v_p v_{j_{m-1}}^{-1} \notin E(G) \text{ and } v_1 v_{j_{m-1}} \notin E(G)) \cup \theta(v^* : v^* \notin N(v_1)) \cup \theta(w_1 : v_1 \in N(S_1)) \cup \theta(v_2 : v_2 \in N(S_p))$$

9
and

\[ U_2 := N(S_1) \cup ((N(S_1) - \{v_1\}) - \theta(v_{i-1} : v_{j-1} \in N(S_1))
\cup((N(S_p) - \{v_p\})^+ - \theta(v_{i+1} :,v_{i-1} \in N(S_p) - \{v_{j-1}^-\}))) \cup ((N(S_p) - \{v_p\})^+^2
\cup(\theta(v_{i+1}^- : v_{i-1} \in N(S_p) - \{v_{j-1}^-\}) - \theta(v_{i+2} : v_{i-1} \in N(S_p) - \{v_{j-1}^-\})))
\cup(\theta(v_{j+1} : v_{j-1} \in N(S_1)) \cup (\theta(v_2 : v_{i-1} \in N(S_p)) \cup (\theta(v_2 : v \in N(S_1) \cup \theta(v : v \notin N(S_0)))))
\cup(\theta(v : v \notin N(S_p))),
\]

where \( u_1 \in N(v_{j-1}) \cap N(S_1) \) and \( v_2 \in N(v_{i-1}) \cap N(S_1) \).

It follows that

\[ |C_Q| \geq |U_1| = |N_p(v_1)| + |N_p(v_2)| - |N(v_{j-1}) : v \in N(S_1) \notin E(G)\text{ and } v_1 v_{j-1} \notin E(G)|
\cup(\theta(v^* : v^* \notin N(v_p))) + |\theta(w_1 : v_{j-1} \in N(S_1))| + |\theta(w_2 : v_{i-1} \in N(S_p))|.
\]

Since \( N(S_p) \cap N(S_p)^+ = \emptyset \), \( |\theta(v_{i+1} : v_{i-1} \in N(S_p) - \{v_{j-1}^-\})| + |\theta(v_{i+2} : v \in N(S_p) - \{v_{j-1}^-\})| \leq 1. \) Because \( |N(v_1) - P| = |S_1| - |\theta(v_1 : v \in N(S_1) - P)| \)
and \( |N(v_2) - P| = |S_p| - |\theta(v_{p-1} : v_{p-1} \in N(S_1) - P)| \), we obtain

\[ |C_Q| \geq |U_2| = 2|N(S_1)| - 1 - |\theta(v_{i-1} : v_{j-1} \in N(S_1))| + 2|N(S_p)| - 2 - |\theta(v_{i+1} : v_{i-1} \in N(S_p))|
\cup|\theta(v_{i+2} : v \in N(S_p) - \{v_{j-1}^-\})| + |\theta(w_1 : v_{j-1} \in N(S_1))| + |\theta(w_2 : v \in N(S_p))| + |\theta(v_2 : v \notin N(S_0))|
\cup(\theta(v \notin N(S_p)^+))|
\]

\[ = S_1 + |S_p| - 2 - |\theta(v_{i+1} : v_{j-1} \in N(S_1))| - |\theta(v_{i+1} : v \in N(S_p))|
\cup|\theta(w_1 : v_{j-1} \in N(S_1))| + |\theta(w_2 : v \in N(S_p))| + |\theta(v_2 : v \notin N(S_0))|
\cup(\theta(v \notin N(S_p)^+))|
\]

\[ = |N(v_1) - P| + |N(v_2) - P| - 2 - |\theta(v_1 : v \in N(S_1) - P)|
\cup|\theta(v_{p-1} : v_{p-1} \in N(S_1) - P)| + |\theta(v_2 : v \notin N(S_0))| + |\theta(v : v \notin N(S_p)^+)|
\]

\[ = |N(v_1) - P| + |N(v_2) - P|.
\]

It follows that

\[ |C_Q| \geq \frac{1}{2}(|U_1| + |U_2|)
\]

\[ \geq \frac{1}{2}(|N(v_1) - P| + |N(v_2) - P| + |N_p(v_1)| + |N_p(v_2)| + |\theta(v^* : v^* \notin N(v_1))|
\cup|\theta(v_{j-1} : v \in N(S_1) - P)\notin E(G)\text{ and } v_1 v_{j-1} \notin E(G)| + |\theta(w_1 : v_{j-1} \in N(S_1))| + |\theta(w_2 : v \in N(S_p))|
\cup|\theta(v^* : v^* \notin N(v_1))| + |\theta(v_{j-1} : v \in N(S_1) - P)\notin E(G)\text{ and } v_1 v_{j-1} \notin E(G)|
\]

\[ \geq k - 1 + \frac{1}{2}(|\theta(v^* : v^* \notin N(v_1))| + |\theta(v_{j-1} : v \in N(S_1) - P)\notin E(G)\text{ and } v_1 v_{j-1} \notin E(G)|
\cup|\theta(w_1 : v_{j-1} \in N(S_1))| + |\theta(w_2 : v \in N(S_p))|).
\]

Then we assume that \( |C_Q| = k - 1 \) and deduce that \( C_Q = U_1 = U_2, v^* \notin N(v_1) \cap N(v_2) \)
(which implies \( 2 \leq m \leq 3 \)), \( v_{j-1} \notin N(S_1) \), \( v_{i-1} \notin N(S_p) \) and either \( v_p v_{j-1}^- \in E(G) \) or
\( v_1 v_{j-1}^- \in E(G) \). One of \( v_{i-1} \) and \( v_{j-1} \) is in \( N(S_p) \). By symmetric, we may get either
\( v_{i-1} \in E(G) \) or \( v_{i-1} \in E(G) \). If \( m = 3 \), then \( v_{j-1} \notin E(G) \) and \( v_p v_{i-1} \notin E(G) \). Thus
\( v_p v_{i-1} \in E(G) \) and \( v_1 v_{i-1} \in E(G) \). It implies that \( V(P) \subseteq C_Q \), a contradiction. So we
assume that $m = 2$. It follows that $v_j^{-1}v_p \notin E(G)$ and hence $v_1v_j \in E(G)$. Similarly $v_pv_{i_2} \in E(G)$.

If $v_{i_2}^{-2} \in N(S_p)$, then $v_{i_2} \notin N(v_1)$ and since $V(C_Q) = U_1$, $v_i^\perp \in N(v_p)$. Then let $w^* \in S_p \cap N(v_{i_2}^{-2})$ and put

$$P^* = P[v_1, v_{i_2}^{-2}]w^*v_pv_{i_2}^{-1}P[v_{i_2}, v_{p-2}]{v_{p-1}}.$$

If $v_{i_2}^{-1} \in N(S_p)$. Let $w^* \in S_p \cap N(v_{i_2}^{-1})$ and put

$$P^{**} = P[v_1, v_{i_2}^{-2}]v_{i_2}w^*v_pv_{i_2}P(v_{i_2}, v_{p-2}){v_{p-1}}.$$

Where $v_{p-1}$ is in $P^*$ or $P^{**}$ if and only if it is in $B \setminus \{w^*\}$. $P^*$ and $P^{**}$ satisfy the hypotheses (a)(b) and (c), but are contrary to (d) because $N(v_1) \cap P[v_j, v_{p-2}] \neq \emptyset$.

\begin{flushright}$\Box$\end{flushright}

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