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A sufficient condition for pancyclability of graphs

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Abstract

Let G be a graph of order n and S be a vertex set of q vertices. We call G S-pancyclable, if for any integer $3 \leq i \leq q$ there exists a cycle C in G such that $|V(C) \cap S| = i$. For any two nonadjacent vertices u, v of S, we say that u, v is of distance two in S, denoted by $d_S(u,v) = 2$, if there is a path P in G connecting u and v such that $|V(P) \cap S| \leq 3$. In this paper, we will prove that if G is 2-connected and for any two vertices u, v of S with $d_S(u,v) = 2$, $max\{d(u), d(v)\} \geq \frac{n}{2}$, then there is a cycle in G containing all the vertices of S. Furthermore, if for any two vertices u, v of S with $d_S(u,v) = 2$, $max\{d(u), d(v)\} \geq \frac{n+1}{2}$, then G is S-pancyclable unless the subgraph induced by S is in a class of special graphs. This generalizes a result of Fan [2] for the case when $S = V(G)$.

Résumé

Soit G un graphe d'ordre n et S un sous ensemble de $V(G)$ de q sommets. G est dit S-pancyclable si, pour tout entier i, $3 \leq i \leq q$, il existe dans G un cycle C tel que $|V(C) \cap S| = i$. Deux sommets non adjacents u et v de S sont dits à distance deux dans S (notation : $d_S(u,v) = 2$) s'il existe un chemin P dans G connectant u et v tels que $|V(P) \cap S| \leq 3$. Dans cet article nous démontrerons que si G est 2-connexe tel que toute paire de sommets u, v de S non adjacents à distance deux vérifie $max\{d(u), d(v)\} \geq \frac{n}{2}$, alors G possède un cycle qui contient tous les sommets de S. De plus, si toute paire de sommets u, v de S non adjacents à distance deux vérifie $max\{d(u), d(v)\} \geq \frac{n+1}{2}$, alors G est S-pancyclable à moins que le sous graphe induit par S n'appartienne à une classe de graphes spéciaux. Cela généralise un résulatt de Fan [2] pour le cas où $S = V(G)$

Keywords: cycles, hamiltonian graphs, pancyclic graphs, cyclability, pancyclability

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1 Preliminaries and Main Results

We consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$ or just by V; the set of edges by $E(G)$ or just by E. We use $|G|$ (the order of G) as a symbol for the cardinality of $V(G)$. If H and S are subsets of $V(G)$ or subgraphs of G, we denote by $N_H(S)$ the set of vertices in H which are adjacent to some vertex in S, and set $d_H(S) = |N_H(S)|$. In particular, when $H = G$, $S = \{u\}$, then let $N_G(u) = N(u)$ and set $d_G(u) = d(u)$. Paths and cycles in a graph G are considered as subgraphs of G. We use $G[S]$ to denote the subgraph induced by S.

For a cycle C in G with a given orientation and X a subset of $V(C)$, X^+ and X^- are the set of the successors and the predecessors of the vertices of X in C, respectively, and for a and b in C, we define $C[a,b]$ ($C[a,b)$, $C(a,b)$, respectively) to be the subpath of C from a to b (from a to b^-, from a^+ to b^-, respectively). We will write $N_C^+(x)$ for $(N_C(x))^+$. Other notation can be found in [1].

Let S be a vertex set of G; v is called an *S-vertex* if $v \in S$. Following [3, 5], the set S is called *cyclable* in G if all vertices of S belong to a common cycle in G. Following [4], the *S-length* of a cycle in G is defined as the number of the S-vertices that it contains and the graph G is said to be *S-pancyclable*, if it contains cycles of all S-lengths from 3 to $|S|$. Obviously, if G is $V(G)$-pancyclable, then G is pancyclic, i.e., G contains cycles of every length between 3 and n.

For any two nonadjacent vertices u, v of S, we say that u, v is of distance two in S, denoted by $d_S(u,v) = 2$, if there is a path P in G connecting u and v such that $|V(P) \cap S| \leq 3$. If $S = V(G)$, set $d(u,v) = d_{V(G)}(u,v)$.

Given an integer $r \geq 2$, F_{4r} is the graph with $4r$ vertices containing a complete graph K_{2r}, a set of r independent edges, denoted by E_r and a matching between the sets of vertices of K_{2r} and E_r (cf. [2]).

People have given different definitions and results on cycles containing certain subsets of vertices and the related papers can be found in [3, 4, 5, 6, 7]. In this paper, we will prove the followings:

Theorem 1. Let G be a 2-connected graph of order n and S be a vertex set of G with $|S| = q \geq 3$. If $max\{d(u), d(v)\} \geq \frac{n}{2}$ holds for any u, v of S with $d_S(u,v) = 2$, then S is cyclable in G.

Theorem 2. Let G be a 2-connected graph of order n and S be a vertex set of G with $|S| = q \geq 3$. If $max\{d(u), d(v)\} \geq \frac{n+1}{2}$ holds for any u, v of S with $d_S(u,v) = 2$, then G is S-pancyclable unless $q = 4r$ and $G[S]$ is a spanning subgraph of F_{4r}.

Theorem 1 generalizes the following result of Fan[2] for the case when $S = V(G)$.

Theorem 3. Let G be a 2-connected graph of order n. If $max\{d(u), d(v)\} \geq \frac{n}{2}$ holds for any u, v of G with $d(u,v) = 2$, then G is hamiltonian.

Notice that $max\{d(v) : v \in V(F_{4r})\} = 2r$. By Theorem 2, we have

Corollary 4. Let G be a 2-connected graph of order n. If $max\{d(u), d(v)\} \geq \frac{n+1}{2}$

holds for any u, v of G with $d(u, v) = 2$, then G is pancyclic.

2 Proof of Theorem 1

Let $T_1 = \{v \in S : d(v) \geq \frac{n}{2}\}$. Since G is 2-connected, it is easy to check that Theorem 1 holds if $|T_1| \leq 1$ (which implies that $G[S - T_1]$ is a clique). Thus we may assume the $|T_1| \geq 2$. In order to prove Theorem 1, we first show the following:

Lemma 1. Let P be a path connecting u and v in G. If $d_P(u) + d_P(v) \geq |P|$, then there exists a cycle C in G such that $V(C) = V(P)$.

Proof. If $uv \in E$, then Lemma 1 holds. If $uv \notin E(G)$, then there exist two consecutive vertices x, y (y is the successor of x on P from u to v) such that $x \in N(v)$ and $y \in N(u)$. Hence there exists a cycle C in G such that $V(P) = V(C)$. $\square$

Lemma 2. Let u, v in T_1 such that $uv \notin E(G)$ and G' be a graph by adding uv to G. If there exists a cycle C' in G' such that $S \subseteq V(C')$, then there exists a cycle C in G such that $S \subseteq V(C)$.

Proof. Let C' be the cycle in G' such that $S \subseteq V(C')$. Then $uv \in E(G'[C'])$, otherwise $C' = C$ is the required cycle in G. Thus there exists a path P starting from u and ending at v in G such that $S \subseteq V(P)$. If $N_{G-P}(u) \cap N_{G-P}(v) \neq \emptyset$, then Lemma 2 holds. If $N_{G-P}(u) \cap N_{G-P}(v) = \emptyset$, then $d_P(u) + d_P(v) \geq |P|$ as $\{u, v\} \subseteq T_1$. Hence Lemma 2 holds by Lemma 1. $\square$

By Lemma 2, we may assume that $G[T_1]$ is a clique of G. Let C be a cycle containing T_1 such that $|V(C) \cap S|$ as large as possible. If $|V(C) \cap S| = q$, then Theorem 1 holds. If $|V(C) \cap S| \leq q-1$, let $u \in S \cap V(G-C)$. Since G is 2-connected, there are two disjoint paths in $G - C$ connecting u and two distinct vertices of C, say w_1 and w_2, respectively. As $T_1 \subseteq V(C)$, we have $u \in S - T_1$. By the choice of C, $V(C(w_1, w_2)) \cap S \neq \emptyset$ and $V(C(w_2, w_1)) \cap S \neq \emptyset$. Let x_1 be the first vertex of $V(C(w_1, w_2)) \cap S$ from w_1 to w_2 and x_2 be the first vertex of $V(C(w_2, w_1)) \cap S$ from w_2 to w_1. If $x_i \notin T_1$ for some $1 \leq i \leq 2$, then $ux_i \in E$, which is impossible by the choice of C. Thus $x_i \in T_1$ for all $1 \leq i \leq 2$. Since $G[T_1]$ is a clique, we can get a cycle C' in G such that $T_1 \subseteq V(C')$ and $|V(C') \cap S| > |V(C) \cap S|$, contrary to the choice of C. Hence Theorem 1 is true.

3 Proof of Theorem 2

By Theorem 1, there exists a cycle in G containing all the vertices of S. Choose such a cycle C with as few vertices as possible and give C an arbitary orientation. Put $R = G - C$. Let $x_1, x_2, \cdots, x_q$ be the vertices of $V(C) \cap S$, the order $1, 2, \cdots, q$ respecting the orientation of C, and consider the subscripts modulo q. Two S-vertices x_i and x_{i+1} are said to be *S-consecutive.* We use C_l for a cycle of S-length l in G.

In [4], it was proved:

Theorem 5. Let G be a graph of order n, S a subset of $V(G)$ such that S is cyclable in G, and let C be a shortest cycle through all the vertices of S. If $d_C(x) + d_C(y) \geq |C|+1$ for some pair of S-consecutive vertices x and y in C, then G is S-pancyclable.

By using the same method as that used in the proof of Theorem 5 in [4], we can get

Lemma 3. Let G be a graph of order n, S a subset of $V(G)$ such that S is cyclable in G, and let C be a shortest cycle through all the vertices of S. If there exist some $1 \leq i \leq q$ such that $x_{i-1}x_{i+1} \in E$ and $d_C(x_i) \geq \frac{|C|+1}{2}$, then G is S-pancyclable.

The following lemma is easy to check.

Lemma 4. Let P be a path connecting two vertices u_1 and u_t of S and $V(P) \cap S = \{u_1, u_2, \cdots, u_t\}$ (the order $1, \cdots, t$ respecting the orientation of P from u_1 to u_t). If there exists some $1 \leq i \leq t-p$ ($p \leq t-3$) such that there exists a path connecting u_i and u_{i+p+1} with the internal vertices, if any, in $V(G) - (S \cup V(P))$, then there exists a path P' connecting u_1 and u_t in G such that $|V(P') \cap S| = t - p$.

Now, let $T_2 = \{v \in S : d(v) \geq \frac{n+1}{2}\}$. It is easy to see the following

Remark 1. If there is no any pair of S-consecutive vertices x, y in $C[x_i, x_j]$ ($i \neq j$) such that $\{x, y\} \subseteq T_2$, then $G[V(C[x_i, x_j]) \cap (S - T_2)]$ is a clique of G.

If there exists at most one pair of S-consecutive vertices which are both in T_2, then it is easy to check that G is S-pancyclable as $G[S - T_2]$, by Remark 1, is a clique unless $|S| = 4$ and $G[S]$ is a spanning subgraph of F_4. Thus Theorem 2 is true. Hence we may assume that $|T_2| \geq 3$ and there exist at least two pairs of S-consecutive vertices which are all in T_2. Without loss of generality, let $\{x_q, x_1\} \subseteq T_2$ such that

$$|N_R(x_1) \cap N_R(x_q)| = min\{|N_R(x) \cap N_R(y)| : x, y \in T_2 \text{ and } x, y \text{ are } S\text{-consecutive}\}.$$

If $d_C(x_1) + d_C(x_q) \geq |C| + 1$, then Theorem 2 holds. Thus in the rest of the proof, we assume that $d_C(x_1) + d_C(x_q) \leq |C|$ and let $M_1 = N_R(x_1) \cap N_R(x_q)$. We first show the following lemmas.

Lemma 5. If there is a path $P = u_1 \cdots u_2 \cdots u_{p-1} \cdots u_p$ in $G[V(C)]$ such that $|V(P) \cap S| = l+1 \geq 4$, $\{u_1, u_2, u_{p-1}, u_p\} \subseteq T_2$ and $\{u_1, u_2\}$, $\{u_{p-1}, u_p\}$ are two pairs of S-consecutive vertices on C, then there exists a C_l in G.

Proof. If $N_R(u_1) \cap N_R(u_{p-1}) \neq \emptyset$ or $N_R(u_2) \cap N_R(u_p) \neq \emptyset$, then Lemma 5 holds. If $N_R(u_1) \cap N_R(u_{p-1}) = \emptyset$ and $N_R(u_2) \cap N_R(u_p) = \emptyset$, noting that $\{u_1, u_2, u_{p-1}, u_p\} \subseteq T_2$, we have

$$d_C(u_1) + d_C(u_2) + d_C(u_{p-1}) + d_C(u_p) \geq 2(|C| + 1).$$

Thus either $d_C(u_1) + d_C(u_2) \geq |C| + 1$ or $d_C(u_{p-1}) + d_C(u_p) \geq |C| + 1$. By Theorem 5, G is S-pancyclable. Hence Lemma 5 holds. □

Lemma 6. Let $P = u_1 \cdots u_p$ in G such that $|V(P) \cap S| = l \geq 3$. If $\{u_1, u_p\} \subseteq T_2$ and there is no any C_l in G, then we have

(i) $|(N(u_1) \cap N(u_p) - V(P)) \cap (V(G) - S)| = \emptyset$;

(ii) $|N(u_1) \cap N(u_p) \cap S \cap (V(G) - V(P))| \geq 2$; and there exist a C_4 and a C_{l+1} which contains P as its subpath;

(iii) when $P = C[x_i, x_j]$ for some $j = l + i - 1$ $(3 \leq l \leq q - 1)$ and $\{x_i, x_j\} \subseteq T_2$, then there exists a pair of S-consecutive vertices y and z in $V(C(x_j, x_i))$ such that $y \in N(x_i)$ (or $y \in N(x_j)$ and $z \in N(x_j)$ (or $z \in N(x_i)$), and there exists a C_{l+2} which contains $C[x_i, x_j]$ as its subpath.

Proof. Since there is no any C_l in G, (i) is obvious and $|N(u_1) \cap V(P)| + |N(u_p) \cap V(P)| \leq |V(P)| - 1$ by Lemma 1. As $d(u_1) + d(u_p) \geq n + 1$, by (i), it is easy to check that (ii) holds.

(iii) As $d(x_i)+d(x_j) \geq n+1$, by Lemma 1 and (i), we have $|N(x_i) \cap V(C(x_j, x_i)) \cap S| + |N(x_j) \cap V(C(x_j, x_i)) \cap S| \geq |V(C(x_j, x_i)) \cap S| + 2$. Thus (iii) holds. □

Lemma 7. Let $P = u_1u_2 \cdots u_p$ be a path in $G[V(C)]$ such that $V(P) \cap S = \{v_1, v_2, \cdots, v_l\}$, where $v_1 = u_1$, $v_l = u_p$ and the order $1, 2 \cdots, l$ respects the orientation of P from u_1 to u_p. Suppose that $l \geq 5$ and there is no any C_l in G. If there exist a C_{l+m} and a C_{l+m+1} in G $(m \in \{1, 2\})$, both of which contain P as their subpath and $|V(C_{l+m}) \cap S - V(C_{l+m+1}) \cap S| \leq 1$, then $\{v_i, v_{i+m+2}\} \cap (S - T_2) \neq \emptyset$ for any $1 \leq i \leq l - m - 2$.

Proof. Let $C' = C_{l+m+1}$ and $C^* = C_{l+m}$. Since P is a subgraph of both C' and C^*, we have $C'[v_i, v_{i+m+2}] = C^*[v_i, v_{i+m+2}] = P[v_i, v_{i+m+2}]$. Since there is no any C_l in G and $i \leq l - m - 2$, we obtain $N_R(v_i) \cap N_R(v_{i+m+2}) = \emptyset$ and $(N(v_i) \cap V(C'(v_{i+2}, v_{i+m+2}))) \cup (N(v_{i+m+2}) \cap V(C'(v_i, v_{i+2}))) = \emptyset$, which implies $|(N(v_i) \cup N(v_{i+m+2})) \cap V(C'(v_i, v_{i+m+2}))| \leq |V(C'(v_i, v_{i+m+2}))|$. Notice that $P' = C'[v_{i+m+2}, v_i]$ is a path with $|V(P') \cap S| = l$. By Lemma 1, $d_{C'}(v_i) + d_{C'}(v_{i+m+2}) < |C'|$.

If $\{v_i, v_{i+m+2}\} \subseteq T_2$, then there exists at least two vertices, say x and y in $N(v_i) \cap N(v_{i+m+2}) \cap (V(C) - V(C'))$. When $x \notin S$ or $y \notin S$, then there is a C_l which contains $(V(C') - V(C'(v_i, v_{i+m+2})))$ and x (or y), a contradiction. When $\{x, y\} \subseteq S$, then $|\{x, y\} \cap V(C^*)| \leq 1$, as $\{x, y\} \subseteq V(C) - V(C')$ and $|V(C^*) \cap S - V(C') \cap S| \leq 1$. Thus we can also get a C_l in G, a contradiction. Hence $\{v_i, v_{i+m+2}\} \cap (S - T_2) \neq \emptyset$ and Lemma 7 holds. □

Lemma 8. If there exists some $i > 1$ such that $\{x_i, x_{i+1}\} \subseteq T_2$ and $d_C(x_i) + d_C(x_{i+1}) \leq |C|$, then

(i) $|(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_i) \cap N_R(x_{i+1})| \geq 1$;

(ii) there exist a cycle C_3 and a cycle C_4 in G.

Proof. (i) Recall that $M_1 = N_R(x_1) \cap N_R(x_q)$. By the choice of x_1 and x_q, we have $|M_1| \leq |N_R(x_i) \cap N_R(x_{i+1})|$. Thus

$|R| + 1 \leq |N_R(x_1) \cup N_R(x_q)| + |M_1| \leq |N_R(x_1) \cup N_R(x_q)| + |N_R(x_i) \cap N_R(x_{i+1})| = |(N_R(x_1) \cup N_R(x_q)) \cup (N_R(x_i) \cap N_R(x_{i+1}))| + |(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_i) \cap N_R(x_{i+1})| \leq |R| + |(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_i) \cap N_R(x_{i+1})|$.

¿From the inequalities above, we can easily check that (i) holds.

(ii) Suppose that there is no any C_l in G for $l = 3$ or $l = 4$. Since $(N_R(x_1) \cup$

$N_R(x_q))\cap N_R(x_i)\cap N_R(x_{i+1}) \neq \emptyset$, without loss of generality, we may choose a vertex, say v, in $N_R(x_q)\cap N_R(x_i)\cap N_R(x_{i+1})$. Notice that $\{x_q, x_1, x_i, x_{i+1}\} \subseteq T_2$. Applying Lemma 6(ii) to the path $C[x_i, x_{i+1}]vx_q$ or the path $C[x_i, x_{i+1}]vC[x_q, x_1]$, we can get a C_3 and a C_4 in G, a contradiction. □

Lemma 9. If there is no any C_l in G for some integer $l \geq 3$, then $l = q-1$.

Proof. By contradiction, assume that $3 \leq l \leq q-2$. Then by Theorem 5, for any pair of S-consecutive vertices x and y in C, we have $d_C(x) + d_C(y) \leq |C|$.

Thus by the assumption and Lemma 5 , $M_1 \neq \emptyset$, $d_C(x_1) + d_C(x_q) \leq |C|$, and $|\{x_{l-1}, x_l\} \cap T_2| \leq 1$.

Case 1. $x_l \in T_2$.

Then $x_{l-1} \notin T_2$. If $x_{l+1} \notin T_2$, then $x_{l-1}x_{l+1} \in E$ and there exists a C_3 in G. By Lemma 3, $d_R(x_l) \geq \frac{|R|+1}{2}$. Since $N_R(x_l)\cap N_R(x_1) = \emptyset$ and $d_R(x_1)+d_R(x_q) \geq |R|+1$, we have

$2|R| + |N_R(x_q) \cap N_R(x_l)| \geq |N_R(x_1) \cup N_R(x_l)| + |N_R(x_q) \cup N_R(x_l)| + |N_R(x_q) \cap N_R(x_l)| \geq d_R(x_1) + d_R(x_q) + 2d_R(x_l) \geq 2|R| + 2$,

which implies $|N_R(x_q) \cap N_R(x_l)| \geq 2$ and there exist a C_{l+1} and a C_{l+2}, both of which contain $C[x_q, x_{l-1}]$ as their subpath and $V(C_{l+1}) \cap S \subseteq V(C_{l+2})$. As $\{x_1, x_l\} \subseteq T_2$, by Lemma 6(ii), we have $l \geq 5$ and by Lemma 7, we have $\{x_3, x_4\} \subseteq S - T_2$ which implies $x_2 \in T_2$. When $l \geq 6$, then $x_5 \in S - T_2$ by Lemma 7 which implies $x_3x_5 \in E$ and we can get a C_l in G, a contradiction. When $l = 5$, that is, $x_5 \in T_2$, since there is no any C_5 in G, we obtain $N(x_2) \cap V(C(x_3, x_5)) = \emptyset$ and $N(x_5) \cap V(C[x_2, x_4)) = \emptyset$. As $d(x_2) + d(x_5) \geq n+1$, there exits some vertex, say v in $N(x_2) \cap N(x_5) - V(C[x_2, x_5))$. Thus we can get a C_5 which contains either $V(C[x_2, x_6]) \cup \{v\}$ whenever $v \notin S$ or $V(C[x_2, x_5]) \cup \{v\}$ whenever $v \in S$, a contradiction. Hence we have $x_{l+1} \in T_2$.

Since there is no any C_l in G, we have $N_R(x_1) \cap N_R(x_l) = \emptyset$ and by Lemma 1, $d_C(x_l)+d_C(x_{l+1}) \leq |C|$. Thus we obtain that $|N_R(x_q)\cap N_R(x_l)\cap N_R(x_{l+1})| \geq 1$ and $l \geq 5$ by Lemmaa 8. Hence there exist a C_{l+1} and a C_{l+2}, which contain $C[x_q, x_l]$ as their subpath.

Since $l \geq 5$, by the assumption and Lemma 7, we obtain $\{x_3, x_4\} \subseteq S - T_2$. By Lemma 5, $x_2 \in S - T_2$ which implies $x_2x_4 \in E$. Thus we can get a C_l in G, a contradiction.

Case 2. $x_l \notin T_2$, $x_{l-1} \in T_2$.

By Lemmas 3 and 5, $x_{l-2} \in T_2$ (otherwise $x_lx_{l-2} \in E$, $d_C(x_{l-1}) \geq \frac{|C|+1}{2}$, since $(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_{l-1}) = \emptyset$ by the assumption and $|N_R(x_1) \cup N_R(x_q)| \geq \frac{|R|+1}{2}$). Noting that $N_R(x_q) \cap N_R(x_{l-1}) = \emptyset$, by Lemma 8, $|N_R(x_1) \cap N_R(x_{l-1}) \cap N_R(x_{l-2})| \geq 1$ and $l \geq 5$. As $\{x_q, x_{l-1}\} \subseteq T_2$, by Lemma 6(ii) and 6(iii), we have $l \geq 6$ and there exist a C_{l+1} and a C_{l+2} which contain $C[x_q, x_{l-1}]$ as their subpath and $|V(C_{l+1})\cap S - V(C_{l+2})\cap S| \leq 1$. Thus by Lemma 7, we can get $\{x_3, x_4\} \subseteq S-T_2$ and $\{x_2, x_5\} \cap (S - T_2) \neq \emptyset$. Thus we can get a C_l in G, a contradiction.

Case 3. $\{x_l, x_{l-1}\} \cap T_2 = \emptyset$.

Case 3.1. There is no any pair of S-consective vertices x and y in $V(C[x_{l+1}, x_{q-1}])$ such that $\{x, y\} \subseteq T_2$.

Then $G[V(C[x_{l-1}, x_{q-1}]) \cap (S - T_2)]$ is a clique by Remark 1. Since $l \leq q-2$, $|V(C[x_{l-1}, x_{q-1}]) \cap S| \geq 3$.

If $x_{q-1} \notin T_2$, then $x_{l-1}x_{q-1} \in E$ and $x_l x_{q-1} \in E$. Thus there exist a C_3, and two cycles C_{l+1}, C_{l+2} in $G[V(C)]$, which contain $C[x_{q-1}, x_{l-1}]$ as their subpath. Thus $l \geq 4$ and $\{x_{l-2}, x_{l-3}\} \subseteq T_2$. By Lemma 7, we have $\{x_3, x_4\} \subseteq S - T_2$ which implies $x_2 \in T_2$. When $l \geq 5$, by Lemma 7, $x_5 \in S - T_2$ implying $x_3x_5 \in E$ and we can get a C_l in G, a contradiction. Thus $l = 4$. Since there is no any C_4 in G, we have $x_q x_2 \notin E$, $N(x_q) \cap V(C(x_1, x_3]) = \emptyset$ and $N(x_2) \cap V(C[x_{q-1}, x_1)) = \emptyset$, as $\{x_3, x_4\} \subseteq N(x_{q-1})$. Thus by Lemma 1, $|N(x_q) \cap V(C[x_{q-1}, x_4])| + |N(x_2) \cap V(C[x_{q-1}, x_4])| \leq |V(C[x_{q-1}, x_4])|$. Since $d(x_q) + d(x_2) \geq n + 1$, we obtain $N(x_2) \cap N(x_q) - V(C[x_4, x_{q-1}]) \neq \emptyset$. Let w in $N(x_2) \cap N(x_q) - V(C[x_4, x_{q-1}])$ and we can get a C_4 in G, which contains $V(C[x_q, x_2]) \cup \{w\}$ when $w \in S$ or $V(C[x_{q-1}, x_q]) \cup V(C[x_2, x_3]) \cup \{w\}$ when $w \notin S$, a contradiction.

If $x_{q-1} \in T_2$, then $x_{q-2} \in S - T_2$ as there is no any pair of S-consecutive vertices in $V(C[x_{l+1}, x_{q-1}]) \cap T_2$, and $x_{l-1}x_{q-2} \in E$, $x_l x_{q-2} \in E$. Thus there exist a C_3 and a C_{l+2} in G which contains $C[x_{q-2}, x_{l-1}]$ as its subpath. When $x_{l-2} \notin T_2$, then $x_{l-2}x_{q-2} \in E$ and there exist a C_4 and a C_{l+1} in G which contains $C[x_{q-2}, x_{l-2}]$ as its subpath. When $x_{l-2} \in T_2$, since $x_{q-1} \in T_2$, by Lemma 6(ii), there exist a C_4 and C_{l+1} in G which contains $C[x_{q-1}, x_{l-2}]$ as its subpath. Thus in both subcases, we have $l \geq 5$ and $|V(C_{l+1}) \cap S - V(C_{l+2}) \cap S| \leq 1$. By using Lemma 7, we can get $\{x_2, x_3, x_4\} \subseteq S - T_2$ and $x_2x_4 \in E$ which implies there exists a C_l in G, a contradiction.

Case 3.2. There exists a pair of S-consecutive vertices x and y in $V(C[x_{l+1}, x_{q-1}])$ such that $\{x, y\} \subseteq T_2$.

Choose $q - 1 > t \geq l + 1$ such that x_t and x_{t+1} is a pair of S-consecutive vertices and t as small as possible. Then by Remark 1, we have $G[V(C[x_{l-1}, x_t)) \cap S]$ is a clique of G which implies $x_{t-1}x_{l-1} \in E$. Let $P = C[x_1, x_{l-1}]x_{t-1}$. By Lemma 8, $|(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_t) \cap N_R(x_{t+1})| \geq 1$ and $l \geq 5$. We distinguish the following two subcases.

Case 3.2.1. $|(N_R(x_1) \cap N_R(x_t) \cap N_R(x_{t+1})| \geq 1$.

Then we can get a C_{l+1} and a C_{l+2} in G, both of which contain P as their subpath. Notice that $l \geq 5$. By Lemma 7, we have $x_4 \in S - T_2$ which implies $x_2 \in T_2$ by the assumption. Using Lemma 5 to the path $P' = C[x_2, x_{l-1}]C[x_{t-1}, x_{t+1}]$, we have $x_3 \in S - T_2$ as $\{x_2, x_{t+1}, x_t\} \in T_2$. Thus by Lemma 7, $x_j \subseteq S - T_2$ and $x_3 x_j \in E$, where $j = 5$ when $l \geq 6$ or $j = t - 1$ when $l = 5$. Hence there is a C_l in G, a contradiction.

Case 3.2.2. $|(N_R(x_1) \cap N_R(x_t) \cap N_R(x_{t+1})| = 0$.

By Lemma 8(i), we have $|(N_R(x_q) \cap N_R(x_t) \cap N_R(x_{t+1})| \geq 1$. Then there exist a C_{l+2} and a C_{l+3} which satisfy the conditions of Lemma 7. Since $l \geq 5$, we have $\{x_4, x_j\} \subseteq S - T_2$ ($j = 5$ when $l > 5$ and $j = t - 1$ when $l = 5$), which implies $x_2 \in T_2$. For the same reason as above, we have $x_3 \notin T_2$ by Lemma 5 and $x_3 x_j \in E$.

Since $\{x_2, x_t\} \subseteq T_2$, applying Lemma 6(ii) to the path $P^* = C[x_2, x_{l-1}]C[x_{t-1}, x_t]$, we have $N(x_2) \cap N(x_t) \cap (V(G) - V(P^*)) \cap S \neq \emptyset$. Notice that $x_3x_j \in E$, we can get a C_l in G, a contradiction. $\square$

Now, we turn to prove Theorem 2. By Lemma 9, there exists a C_l in G for $3 \leq l \leq q-2$. If there exists some $1 \leq i \leq q$ such that $x_{i-1}x_{i+1} \in E$ or $N_R(x_{i-1}) \cap N_R(x_{i+1}) \neq \emptyset$, then there exists a C_{q-1} and Theorem 2 holds. Thus we may assume that for any $1 \leq i \leq q$, $x_{i-1}x_{i+1} \notin E$ and $N_R(x_{i-1}) \cap N_R(x_{i+1}) = \emptyset$. Hence for any $1 \leq i \leq q$, we have $\{x_{i-1}, x_{i+1}\} \cap T_2 \neq \emptyset$.

If there exists some $1 \leq i \leq q$ such that $\{x_{i-1}, x_{i+1}\} \subseteq T_2$, then $d_C(x_{i-1}) + d_C(x_{i+1}) \geq |C|+1$ which implies $N_C^+(x_{i-1}) \cap N(x_{i+1}) \cap V(C[x_{i+1}, x_{i-1}]) \neq \emptyset$. Hence we can get a C_{q-1} in G and Theorem 2 holds.

If for any $1 \leq i \leq q$, $|\{x_{i-1}, x_{i+1}\} \cap T_2| = 1$, since $\{x_q, x_1\} \subseteq T_2$, we obtain that $q = 4r$, $\{x_2, x_3\} \subseteq S - T_2$, $\{x_{4t}, x_{4t+1}\} \subseteq T_2$ and $\{x_{4t+2}, x_{4t+3}\} \subseteq S - T_2$ implying that $x_{4t+2}x_{4t+3} \in E$ for any $1 \leq t \leq r-1$ by the choice of C.

When there exist some m and t with $0 \leq m < t \leq r-1$ such that $(N(x_{4m+2}) \cup N(x_{4m+3})) \cap \{x_{4t+2}, x_{4t+3}\} \neq \emptyset$, then $G[\{x_{4m+2}, x_{4m+3}, x_{4t+2}, x_{4t+3}\}]$ is a clique. Let $P = C[x_{4m+4}, x_{4t+2}]x_{4m+2}x_{4t+3}C(x_{4t+3}, x_{4m+1}]$. Then $|V(P) \cap S| = q-1$. Since $\{x_{4m+1}, x_{4m+4}\} \subseteq T_2$, we have either $d_P(x_{4m+1}) + d_P(x_{4m+4}) \geq |P|$ or $N_{G-P}(x_{4m+1}) \cap N_{G-P}(x_{4m+4}) - \{x_{4m+3}\} \neq \emptyset$. Thus we can get a C_{q-1} in G by Lemma 1 or Lemma 6(i).

When for any m and t with $0 \leq m < t \leq r-1$, $(N(x_{4m+2}) \cup N(x_{4m+3})) \cap \{x_{4t+2}, x_{4t+3}\} = \emptyset$, then we can derive that $G[S]$ is a spanning subgraph of F_{4r} and Theorem 2 holds.

Therefore, the proof of Theorem 2 is complete.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, Now York, 1976.

[2] G. Fan, New sufficient conditions for cycles in graphs, J. Combin. Theory B 37 (1984) 221-227.

[3] O. Favaron, E. Flandrin, H. Li, Y-P. Liu, F. Tian and Z-S. Wu, Sequences, claws and cyclability of graphs, J. Graph Theory 21 (4) (1996) 357-369.

[4] O. Favaron, E. Flandrin, H. Li and F. Tian, An Ore-type condition for pancyclability, Disrete Math. 206 (1999) 139–144.

[5] K. Ota, Cycles through prescribed vertices with large degree sum, Discrete Math. 145 (1995) 201-210.

[6] L. Stacho, Locally pancyclic graphs, J. Combin. Theory B 76 (1999) 22–40.

[7] M.E. Watkins and D.M. Mesner, Cycles and connectivity in graphs, Can. J. Math. 19 (1967) 1319-1328.

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