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# A NOTE ON ORE CONDITION AND CYCLE STRUCTURE

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# A note on Ore condition and cycle structure

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**Abstract**

For a graph $G$, let $\sigma_2(G)$ denote the minimum degree sum of two nonadjacent vertices (when $G$ is complete, we let $\sigma_2(G) = \infty$). In this paper, we show the following two results: (i) Let $G$ be a graph of order $n \geq 4k+3$ with $\sigma_2(G) \geq n$ and let $F$ be a set of $k$ independent edges of $G$ such that $G - F$ is 2-connected, then $G - F$ is hamiltonian or $G \cong K_2 + (K_2 \cup K_{n-4})$ or $G \cong \overline{K_2} + (K_2 \cup K_{n-4})$; (ii) Let $G$ be a graph of order $n \geq 16k+1$ with $\sigma_2(G) \geq n$ and let $F$ be a set of $k$ edges of $G$ such that $G - F$ is hamiltonian, then either $G - F$ is pancyclic or $G - F$ is bipartite. Examples show the first result is best possible.

## 1 Introduction

In this paper, we only consider finite undirected graphs without loops and multiple edges. For a vertex $x$ of a graph $G$, the neighborhood of $x$ in $G$ is denoted by $N_G(x)$, and $d_G(x) = |N_G(x)|$ is the degree of $x$ in $G$. For a subset $D$ of $V(G)$, the subgraph induced by $D$ is denoted by $G[D]$. For a subset $F$ of $E(G)$, the subgraph with vertex set V(G) and edge set $E(G) \setminus F$ is denoted by $G - F$. For a graph $G$, $|V(G)|$ is the order of $G$, $\delta(G)$ is the minimum degree of $G$, and

$$\sigma_2(G) = \min\{d_G(x) + d_G(y) | x, y \in V(G), x \neq y, xy \notin E(G)\}$$

is the minimum degree sum of nonadjacent vertices. (When $G$ is a complete graph, we define $\sigma_2(G) = \infty$.)

One goal of this paper is to study the graph $G$ which is still hamiltonian after a given set $F$ of edges is deleted. Clearly, if $G$ is such a graph, then $G - F$ must be 2-connected. Working on Ore's classic condition for hamiltonian graphs, we will prove the following result.

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**Theorem 1** *Let $G$ be a graph of order $n \geq 4k+3$ with $\sigma_2(G) \geq n$ and let $F$ be a set of $k$ independent edges of $G$. If $G-F$ is 2-connected, then $G-F$ is hamiltonian or $G \cong K_2 + (K_2 \cup K_{n-4})$ or $G \cong \overline{K_2} + (K_2 \cup K_{n-4})$.*

Going a step further towards the cycle structure, a graph of order $n$ is said to be *pancyclic* if it contains cycles of every length $\ell$, $3 \leq \ell \leq n$. In [1], Bondy suggested the metaconjecture that almost any nontrival condition on graphs which implies that the graph is hamiltonian also implies that the graph is pancyclic (except maybe for a special family of graphs). Many results have been obtained in this problem. Here we will prove the following result.

**Theorem 2** *Let $G$ be a graph of order $n \geq 16k+1$ with $\sigma_2(G) \geq n$ and let $F$ be a set of $k$ edges of $G$ such that $G-F$ is hamiltonian, then either $G-F$ is pancyclic or $G-F$ is bipartite.*

As a consequance of Theorems 1 and 2, we get

**Theorem 3** *Let $G$ be a graph of order $n \geq 16k+1$ with $\sigma_2(G) \geq n$ and let $F$ be a set of $k$ independent edges of $G$. If $G-F$ is 2-connected, then $G-F$ is pancyclic or $G-F$ is bipartite or $G \cong K_2 + (K_2 \cup K_{n-4})$ or $G \cong \overline{K_2} + (K_2 \cup K_{n-4})$.*

The proofs of Theorems 1 and 2 will be placed to sections 2 and 3, respectively. Here we show some examples that demonstrate the sharpness of Theorem 1.

**Example** (a) Let $G := K_{2k} + M$, where $M$ is the graph consisting of $k+1$ independent edges. Then, $\sigma_2(G) = 4k+2 = n$. For each $F \subset E(M)$ with $|F| = k$, $G-F$ is a $2k$-connected non-hamiltonian graph. Hence, the low bound $n \geq 4k+3$ in Theorem 1 is best possible even if $G$ has very large connectivity.

(b) Let $t$ be an integer with $2 \leq t < \frac{n}{2}$ and let $A$ and $B$ be two complete graphs with $V(A) = \{x_1, x_2, \ldots, x_t\}$ and $V(B) = \{y_1, y_2, \ldots, y_{n-t}\}$. Let $G$ be the graph obtained from $A$ and $B$ by adding the set of edges $\{x_1y_1, x_2y_2, \ldots, x_ty_t\} \cup \{y_1x_2, \ldots, y_{t-1}x_t, y_tx_1\}$. Then, $\sigma_2(G) = n$ and $G - E(A)$ is 2-connected. However, $G - E(A)$ is not hamiltonian. Hence, the $F$ in Theorem 1 cannot be any subset of $E(G)$.

## 2 Proof of Theorem 1

By way of contradiction, assume that Theorem 1 is false. Then, $G' := G - F$ is not hamiltonian. Let $G^* = C_n(G')$ be the $n$-closure of $G'$ (i.e. the graph obtained from $G'$ by recursively joining nonadjacent vertices with degree-sum at least $n$). By Bondy and Chvátal's closure theorem [2], we have

(2.1) $G^*$ is not hamiltonian.

Define

$$A = \{v \in V(G) : v \text{ is not incident with any edge of } F\}.$$

Then for each pair of nonadjacent vertices $x, y$ in $A$, $d_{G'}(x) + d_{G'}(y) = d_G(x) + d_G(y) \geq \sigma_2(G) \geq n$. So, $G^*[A]$ is complete.

Choose $X \subseteq V(G)$ so that

(i) $X \supseteq A$ and $G^*[X]$ is complete;

(ii) Subject to (i), $|X|$ achieves the maximum.

Let $\overline{X} = V(G) \setminus X$. We claim that

(2.2) $uv \in E(G^*) \cup F$ for every two distinct vertices $u$ and $v$ in $\overline{X}$.

Assume, to the contrary, that there exist two distinct vertices $u$ and $v$ in $\overline{X}$ such that $uv \notin E(G^*) \cup F$. Then, $uv \notin E(G)$. So, $d_G(u) + d_G(v) \geq n$. Without loss of generality, assume that $d_G(u) \geq d_G(v)$. Then, $d_{G'}(u) \geq d_G(u) - 1 \geq \frac{n}{2} - 1$. This together with (i) implies that for every $x \in X$

$$\begin{aligned} d_{G^*}(u) + d_{G^*}(x) &\geq (\frac{n}{2} - 1) + (|A| - 1) \\ &= (\frac{n}{2} - 1) + (n - 2k - 1) \\ &\geq n - \frac{1}{2}. \end{aligned}$$

Hence, $G^*[X \cup \{u\}]$ is complete, contrary to the choice of $X$. So, (2.2) is true.

(2.3) $G^*[\overline{X}]$ is not hamilton-connected.

Suppose (2.3) is false. Then, both $G^*[X]$ and $G^*[\overline{X}]$ are hamilton-connected. Since $G^*$ is 2-connected, $G^*$ is hamiltonian, contrary to (2.1).

It follows from (2.3) that $|\overline{X}| \geq 2$. If $|\overline{X}| \geq 5$, then by (2.2) we have $\delta(G^*[\overline{X}]) \geq |\overline{X}| - 2 > \frac{|\overline{X}|}{2}$. This implies that $G^*[\overline{X}]$ is hamilton-connected, contrary to (2.3). Therefore,

(2.4) $2 \leq |\overline{X}| \leq 4$.

(2.5) $|\overline{X}| \neq 4$.

Assume, to the contrary, that $|\overline{X}| = 4$. By (2.2), we have $\delta(G^*[\overline{X}]) \geq |\overline{X}| - 2 \geq \frac{|\overline{X}|}{2}$. Hence, $G^*[\overline{X}]$ is hamiltonian. Let $C = v_1v_2v_3v_4v_1$ be a hamiltonian cycle of $G^*[\overline{X}]$. By the 2-connectivity of $G'$, we may assume that there exist two distinct vertices $u, v \in X$ such that $v_1u, v_iv \in E(G^*)$ for some $i \in \{2, 3, 4\}$. If $i = 2$ or 4, then there is a a hamiltonian $(v_1, v_i)$-path $P[v_1, v_i]$ in $G^*[\overline{X}]$. Let $Q[v, u]$ be a hamiltonian $(v, u)$-path in $G^*[X]$. Then, $P[v_1, v_i]v_ivQ[v, u]uv_1$ is a hamiltonian cycle in $G^*$, contrary to (2.1). Hence, $i = 3$. Similarly, we can derive that $v_2v_4 \notin E(G^*)$

and $(N_{G^*}(v_2) \cup N_{G^*}(v_4)) \cap X = \emptyset$. This together with (2.2) implies $v_2v_4 \in F$ and $(N_G(v_2) \cup N_G(v_4)) \cap X = \emptyset$. Let $x$ be any vertex of $X$. Since $xv_2 \notin E(G)$, $n \le d_G(x)+d_G(v_2) \le |(X\setminus\{x\})\cup\{v_1,v_3\}|+3 = n$. So, $v_1x \in E(G)$ for every $x \in X$, implying that $N_{G^*}[X \cup \{v_1\}]$ is complete, contrary to the choice of $X$. Hence, (2.5) is true.

(2.6) $|\overline{X}| \ne 3$.

Suppose (2.6) is false. Since $F$ is a set of independent edges, by (2.2) and (2.3), we may assume that $\overline{X} = \{v_1, v_2, v_3\}$, $E(G^*[\overline{X}]) = \{v_1v_2, v_2v_3\}$ and $v_1v_3 \in F$. Since $G'$ is 2-connected, $N_{G^*}(v_i) \cap X \ne \emptyset$ for $i = 1,3$. Say $v_1x, v_3y \in E(G^*)$ for some $x, y \in X$. If $x \ne y$, then $xv_1v_2v_3y$ together with the hamiltonian $(y,x)$-path in $G^*[X]$ forms a hamiltonian cycle of $G^*$, a contradiction. Therefore, $N_{G^*}(v_1) \cap X = N_{G^*}(v_3) \cap X = \{x\}$. Let $z$ be any vertex of $X \setminus \{x\}$. Then, $zv_1, zv_3 \notin E(G^*) \cup F$. So, $n \le d_G(v_1) + d_G(z) \le 3 + |(X \setminus \{z\}) \cup \{v_2\}|$. This implies $v_2z \in E(G)$ for every $z \in X \setminus \{x\}$. Since $d_{G^*}(x) + d_{G^*}(v_2) \ge |(X \setminus \{x\}) \cup \{v_1, v_3\}| + 2 = n$, we have $xv_2 \in E(G^*)$. So, $d_{G^*}(v_2) \ge |X \cup \{v_1, v_3\}| - 1 = n - 2$. This together with (i) implies that for every $u \in X$

$$d_{G^*}(v_2) + d_{G^*}(u) \ge (n-2) + |X \setminus \{u\}| \ge n.$$

Hence, $G^*[X \cup \{v_2\}]$ is complete, a contradiction. So, (2.6) is true.

It follows from (2.4)-(2.6) that $|\overline{X}| = 2$. By (2.2) and (2.3), we may assume that $\overline{X} = \{v_1, v_2\}$ and $v_1v_2 \in E(F) \setminus E(G^*)$. Since $G'$ is 2-connected, $|N_{G'}(v_i) \cap X| \ge 2$ for $i = 1, 2$. Suppose $x_1, x_2 \in N_{G'}(v_1)$. If $N_{G'}(v_2) \cap X \ne \{x_1, x_2\}$, then since $G^*[X]$ is complete, we can easily get a hamiltonian cycle of $G^*$. This contradiction shows $N_{G'}(v_2) \cap X = \{x_1, x_2\}$. Similarly, we have $N_{G'}(v_1) \cap X = \{x_1, x_2\}$. By the degree sum condition, we can derive that $G \cong K_2 + (K_2 \cup K_{n-4})$ or $G \cong \overline{K_2} + (K_2 \cup K_{n-4})$. This contradiction completes the proof of Theorem 1. □

# 3 Proof of Theorem 2

By way of contradiction, assume that Theorem 2 is false. Put $H := G - F$. Let $C = v_1v_2 \dots v_nv_1$ be a hamiltonian cycle of $H$. Since $H$ is not pancyclic, $H$ misses a cycle of length $\ell$ for some $\ell$, $3 \le \ell \le n-1$, $\ell$ being fixed until the end of the paper. Clearly

(3.1) For every $i$, $v_iv_{i+\ell-1} \notin E(H)$, where the indices are taken modulo $n$.

(3.2) There exists an $i$, $1 \le i \le n$, such that none of $v_i$, $v_{i+1}$, $v_{i+2}$, $v_{i+3}$, $v_{i+\ell-1}$, $v_{i+\ell}$, $v_{i+\ell+1}$ and $v_{i+\ell+2}$ is incident to any edges of $F$.

Assume, to the contrary, that (3.2) is false. For convenience, we let $d_F(x)$ denote the degree of $x$ in the graph $(V(G), F)$. Then,

$$\sum_{j=0}^{3} (d_F(v_{i+j}) + d_F(v_{i+j+\ell-1})) \geq 1$$

for every $i$, $1 \leq i \leq n$. So

$$16|F| = \sum_{i=1}^{n} \sum_{j=0}^{3} (d_F(v_{i+j}) + d_F(v_{i+j+\ell-1})) \geq n,$$

contrary to $n \geq 16k + 1$. Hence, (3.2) is true.

It follows from (3.2) that there exists an $i$, $1 \leq i \leq n$, such that $d_F(v_i) = d_F(v_{i+1}) = d_F(v_{i+2}) = d_F(v_{i+3}) = d_F(v_{i+\ell-1}) = d_F(v_{i+\ell}) = d_F(v_{i+\ell+1}) = d_F(v_{i+\ell+2}) = 0$. This together with (3.1) implies $d_H(v_j) = d_G(v_j)$ for $j = i, i+1, i+2, i+3, i+\ell-1, i+\ell, i+\ell+1, i+\ell+2$ and $v_i v_{i+\ell-1}, v_{i+1} v_{i+\ell}, v_{i+2} v_{i+\ell+1}, v_{i+3} v_{i+\ell+2} \notin E(G)$. By $\sigma_2(G) \geq n$, we get

$$\sum_{j=0}^{3} (d_H(v_{i+j}) + d_H(v_{i+j+\ell-1})) = \sum_{j=0}^{3} (d_G(v_{i+j}) + d_G(v_{i+j+\ell-1})) \geq 4n.$$

So, the following statement is true.

(3.3) There exist four consecutive vertices in $C$ that have degree sum in $H$ at least $2n$ and none of which is incident to any edges of $F$.

Without loss of generality, we can choose $v_n, v_1, v_2$ and $v_3$ as consecutive vertices in $C$ that satisfies (3.3). Then,

(3.4) $d_H(v_n) + d_H(v_1) + d_H(v_2) + d_H(v_3) \geq 2n$.

By (3.4), we may assume, without loss of generality, that $d_H(v_n) + d_H(v_1) \geq n$. We will use the following Theorem.

**Theorem 4** *(Schmeichel and Hakimi [4]). Let $H$ be a graph with a hamiltonian cycle $C := v_1 v_2 \cdots v_n v_1$ with $n \geq 3$. Suppose $d_H(v_n) + d_H(v_1) \geq n$, then*
*(i) $H$ is pancyclic or*
*(ii) $H$ is bipartite or*
*(iii) $H$ contains cycles of all lengths except an $(n-1)$-cycle.*
*Moreover, if (iii) holds, then $d_H(v_{n-2}), d_H(v_{n-1}), d_H(v_2), d_H(v_3) < \frac{n}{2}$.*

By our assumption, case (iii) occurs. So $d_H(v_2) + d_H(v_3) < n$. On the other hand, since $H$ contains no $(n-1)$-cycle, we have $\{i : 1 \leq i \leq n, v_{i-2} v_n \in E(H)\} \cap \{i : 1 \leq i \leq n, v_i v_1 \in E(H)\} = \emptyset$. So, $d_H(v_n) + d_H(v_1) \leq n$. This together with (3.4) implies $d_H(v_2) + d_H(v_3) \geq n$, a contradiction. Hence, Theorem 2 is true. □

# References

[1] J.A. Bondy, Pancyclic graphs: Recent results, Colloquia Mathematica Societatis János Bolyai (1973), 181-187.

[2] J.A. Bondy and V. Chvátal, A method in graph theory, Discrete Math. **15** (1976), no. 2, 111-135.

[3] O. Ore, Note on hamilton circuits, Amer. Math. Monthly **67** (1960), 55.

[4] E.F. Schmeichel and S.L. Hakimi, A cycle structure theorem for hamiltonian graphs, J. Combin. Theory **B 45** (1988), 99-107.

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