RAPPORT DE RECHERCHE

L R I

COLOR NEIGHBORHOOD AND THE HETEROCHROMATIC MATCHINGS IN EDGE-COLORED BIPARTITE GRAPHS

LI H / WANG G

Unité Mixte de Recherche 8623
CNRS-Université Paris Sud – LRI

04/2006

Rapport de Recherche N° 1443

CNRS – Université de Paris Sud
Centre d'Orsay
LABORATOIRE DE RECHERCHE EN INFORMATIQUE
Bâtiment 490
91405 ORSAY Cedex (France)

Color neighborhood and heterochromatic matchings in edge-colored bipartite graphs *

Hao Li[1,2],† Guanghui Wang[1,3]

[1] *Laboratoire de Recherche en Informatique*
UMR 8623, C.N.R.S.-Université de Paris-sud
91405-Orsay cedex, France
e-mail: li@lri.fr, wgh@lri.fr

[2] *School of Mathematics and Statistics*
Lanzhou University
730000 Lanzhou, Gansu, China

[3] *School of Mathematics and System Science*
Shandong University
250100 Jinan, Shandong, China

Abstract

Let (G, C) be an (edge-)colored bipartite graph with bipartition (X, Y) and $|X| = |Y| = n$. A heterochromatic matching of G is such a matching in which no two edges have the same color. Let $N^c(S)$ denote a maximum color neighborhood of $S \subseteq V(G)$. In a previous paper, we showed that if $N^c(S) \geq |S|$ for all $S \subseteq X$ or $S \subseteq Y$, then G has a heterochromatic matching with cardinality at least $\lceil \frac{3n-1}{8} \rceil$. In this paper, we improve the result by show that G has a heterochromatic matching with cardinality at least $\lceil \frac{2n}{5} \rceil$.

Keywords: heterochromatic matching, color neighborhood

*This research is supported by the NSFC(60373012 and 10471078), SRSDP(20040422004) and PDSF(2004036402)

†Corresponding author.

1 Introduction and notation

We use [3] for terminology and notations not defined here and consider simple undirected graphs only.

Let $G = (V, E)$ be a graph. An *edge coloring* of G is a function $C : E \to N$(N is the set of nonnegative integers). If G is assigned such a coloring C, then we say that G is an *edge-colored graph*, or simply *colored graph*. Denote by (G, C) the graph G together with the coloring C and by $C(e)$ the *color* of the edge $e \in E$. For a subgraph H of G, let $C(H) = \{C(e) : e \in E(H)\}$.

A subgraph H of G is called *heterochromatic*, or *rainbow*, or *colorful* if its any two edges have different colors. There are many publications studying heterochromatic subgraphs. Very often the subgraphs considered are paths, cycles, trees, etc. The heterochromatic hamiltonian cycle or path problems were studied by Hahn and Thomassen(see [9]), Rödl and Winkler(see [7]), Frieze and Reed, Albert,Frieze and Reed (see [1]), and H. Chen and X.L. Li (see [5]). For more references, see [2, 6, 9].

For an uncolored graph the following theorems are well known in matching theory and have been widely used.

Theorem 1 [10]. Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex of X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Theorem 2 [3]. A bipartite graph G has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V$.

A matching is *heterochromatic* if any two edges of it have different colors. Unlike uncolored matchings for which the maximum matching problem is solvable in polynomial time (see [12]), the maximum heterochromatic matching problem is NP-complete, even for bipartite graphs (see [8]). Heterochromatic matchings have been studied for example in [11] in which by defining $N_c(S)$ (see the definition below) Hu and Li gave some sufficient conditions for the existence of perfect heterochromatic matchings in colored graphs. We have

Let (G, C) be a colored graph. For a vertex v of G, let $CN(v) = \{C(e) : e$ is incident with $v\}$ and $CN(S) = \cup_{v \in S} CN(v)$ for $S \subseteq V$. For $S \in V(G)$, denote $N_c(S)$ as one of the minimum set(s) W satisfying $W \subseteq N(S) \backslash S$ and $[CN(S) \backslash C(G[S])] \subseteq CN(W)$.

Theorem 3 [11]. Let (B, C) be a colored bipartite graph with bipartition X, Y. Then, B contains a heterochromatic matching that saturates every vertex in X, if $|N_c(S)| \geq |S|$, for all $S \subseteq X$.

Theorem 4 [11]. A colored graph (G, C) has a perfect heterochromatic matching, if

(1) $o(G-S) \leq |S|$, where $o(G-S)$ denotes the number of odd components in the remaining graph $G-S$, and

(2) $|N_c(S)| \geq |S|$ for all $S \subseteq V$ such that $0 \leq |S| \leq \frac{|G|}{2}$ and $|N(S) \backslash S| \geq |S|$.

In [13], we define a maximum color neighborhood and study heterochromatic matchings in edge-colored bipartite graphs under a new condition related to maximum color-neighborhoods of subsets of vertices.

Let (G, C) be a colored bipartite graph with bipartition (X, Y). For a vertex set $S \subseteq X$ or Y, a *color neighbourhood* of S is defined as a set $T \subseteq N(S)$ such that there are $|T|$ edges between S and T that are adjacent to distinct vertices of T and have distinct colors. A *maximum color neighborhood* $N^c(S)$ is a color neighborhood of S and $|N^c(S)|$ is maximum. Given a set S and a color neighborhood T of S, denote by $C(S, T)$ a set of $|T|$ distinct colors on the $|T|$ edges between S and distinct vertices of T. Note that there might be more than one such set $C(S, T)$. If there is no ambiguity, let $C(S, T)$ be a fixed color set in the following.

Let M be a heterochromatic matching of G, we denote $b_M = |\{e \mid e \in E(G - V(M))$ and $C(e) \in C(M)\}|$ and denote by $(X_M \cup Y_M)$ with $X_M \in X$, $Y_M \in Y$, the set of vertices that is incident with the edges in M.

In [13], we gain the following theorem.

Theorem 5 [13]. Let (G, C) be a colored bipartite graph with bipartition (X, Y) and $|X| = |Y| = n$. If $|N^c(S)| \geq |S|$ for all $S \subseteq X$ or $S \subseteq Y$, then G has a heterochromatic matching of cardinality at least $\lceil \frac{3n-1}{8} \rceil$.

We improve the bound of the above theorem and gain the following main result of this paper.

Theorem 6. Let (G, C) be a colored bipartite graph with bipartition (X, Y) and $|X| = |Y| = n$. If $|N^c(S)| \geq |S|$ for all $S \subseteq X$ or $S \subseteq Y$, then G has a heterochromatic matching of cardinality at least $\lceil \frac{2n}{5} \rceil$.

Under the conditions of Theorem 6, the following example shows that the best bound can not be better than $\lceil \frac{n}{2} \rceil$. Let $G = (X, Y)$ with $X = \{x_1, x_1, \cdots, x_{2s}\}$ and $Y = \{y_1, y_2, \cdots, y_{2s}\}$ be a bipartite graph such that $E(G) = \{x_i y_i | i = 1, 2, \cdots, 2s\} \cup \{x_{2i-1} y_{2i} | i = 1, 2, \cdots, s\} \cup \{x_{2i} y_{2i-1} | i = 1, 2, \cdots, s\}$. The edge coloring C of G is given by $C(x_{2i-1} y_{2i-1}) = C(x_{2i} y_{2i}) = 2i - 1$ and $C(x_{2i-1} y_{2i}) = C(x_{2i} y_{2i-1}) = 2i$ for $i = 1, 2, \cdots, s$. Clearly the cardinality of the maximum heterochromatic matching of (G, C) is $s = \lceil \frac{2s}{2} \rceil$. This example shows that the bound in Theorem 6 is not very far away from the best.

2 Proof of Theorem 6

Let M be a maximum heterochromatic matching of G with $t := |M|$ such that b_M is maximum. Assume to the contrary that $t < \frac{2n}{5}$.

Let $C(M) = \{c_1, c_2, \cdots, c_t\}$. Put $S_x = X - X_M$ and $S_y = Y - Y_M$. Let $N^c(S_x)$ and $N^c(S_y)$ be a maximum color neighborhood of S_x and S_y, respectively. Set $N^c(S_x) = Y_P \cup Y_Q (Y_P \cap Y_Q = \phi)$ where $C(S_x, Y_P) \cap C(M) = \phi$, $C(S_x, Y_Q) \subseteq C(M)$ and let $N^c(S_y) = X_P \cup X_Q (X_P \cap X_Q = \phi)$ where $C(S_y, X_P) \cap C(M) = \phi$, $C(S_y, X_P) \subseteq C(M)$. Clearly $|Y_Q| \leq t$, $|X_Q| \leq t$.

Claim 1. $Y_P \subseteq Y_M$, $X_P \subseteq X_M$.

Proof. Otherwise, there is an edge $e \in E(S_x, S_y)$ and $C(e) \notin C(M)$, then we can obtain a heterochromatic matching $M + e$ with cardinality $t + 1$, a contradiction. □

An *alternating* 4-cycle AC is a cycle $e_1 e_2 e_3 e_4 e_1$ such that $e_1 \in E(M)$, $e_3 \in E(G - V(M))$ and $C(e_1) = C(e_3)$, $C(e_2) = C(e_4) \notin C(M)$. Given two alternating 4-cycles $AC = e_1 e_2 e_3 e_4 e_1$ and $AC' = e'_1 e'_2 e'_3 e'_4 e'_1$, AC is *different* from AC', we mean that $e_1 \neq e'_1$, $e_3 \neq e'_3$ and $C(e_2) \neq C(e'_2)$.

Claim 2. There exists an alternating 4-cycle in G.

Proof. Since $|N^c(S_x)| = |Y_P| + |Y_Q| \geq |S_x| = n - t$, it follows that $|Y_P| \geq n - t - |Y_Q| \geq n - 2t$. Similarly $|X_P| \geq n - t - |X_Q| \geq n - 2t$. Hence $|X_P| + |Y_P| \geq 2(n - 2t) = 2n - 4t > t = |X_M| = |Y_M|$. Then there exists an edge $xy \in E(M)$ such that x is adjacent with a vertex $y' \in S_y$, $C(xy') \notin C(M)$ and y is adjacent with a vertex $x' \in S_x$, $C(x'y) \notin C(M)$. Clearly $C(xy') = C(x'y)$, otherwise we obtain a new heterochromatic matching $M' = M \cup xy' \cup x'y - xy$ with $|M'| = |M| + 1 > M$, a contradiction.

Then there exists an edge $e \in E(G - V(M))$ such that $C(e) = C(xy)$. Otherwise $M'' = M \cup xy' - xy$ is a heterochromatic matching with $|M''| = |M|$ and $b_{M''} \geq b_M + 1$, contradicting with the choice of M. If $e \neq x'y'$, without loss of generality, assume that y' is not incident with e, then $M''' = M \cup e \cup xy' - xy$ is a heterochromatic matching with $|M'''| = |M| + 1$, a contradiction. □

Suppose that the maximum number of the vertex-disjoint pairwise different alternating 4-cycles in G is l. Clearly $1 \leq l \leq t$. Assume that the alternating 4-cycle AC_i has edges $\{x_i y'_i, y'_i x'_i, x'_i y_i, y_i x_i\}$ and $C(xy) = C(x'_i y'_i) = c_i \in C(M), C(xy'_i) = C(x'_i y) = c'_i \notin C(M)$, where $xy \in E(M)$, and $x'_i \in S_x, y'_i \in S_y$.

Denote

$$X_L = \{x'_1, x'_2, \cdots, x'_l\}, Y_L = \{y'_1, y'_2, \cdots, y'_l\},$$
$$X_{M_l} = \{x_1, x_2, \cdots, x_l\} \subseteq X_M,$$
$$Y_{M_l} = \{y_1, y_2, \cdots, y_l\} \subseteq Y_M,$$

where $\{x_1y_1, x_2y_2, \cdots, x_ly_l\} = E(M_l) \subseteq E(M)$. We abbreviate $C_l = C(M_l) = \{c_1, c_2, \cdots, c_l\}$ and $C_L = \{c_1^{'}, c_2^{'}, \cdots, c_l^{'}\}$, where $c_i^{'} \notin C(M)$ and $c_i^{'} \neq c_j^{'}$ if $i \neq j$, $1 \leq i, j \leq l$.

Then put $S_x^{'} = X - X_M - X_L$ and $S_y^{'} = Y - Y_M - Y_L$. Let $N^c(S_x^{'})$ and $N^c(S_y^{'})$ be a maximum color neighborhood of $S_x^{'}$ and $S_y^{'}$, respectively. Write $N^c(S_x^{'}) = Y_P^{'} \cup Y_Q^{'}(Y_P^{'} \cap Y_Q^{'} = \phi)$, where $C(S_x^{'}, Y_P^{'}) \cap C(M - M_l) = \phi$ and $C(S_x^{'}, Y_Q^{'}) \subseteq C(M - M_l)$. And let $N^c(S_y^{'}) = X_P^{'} \cup X_Q^{'}(X_P^{'} \cap X_Q^{'} = \phi)$, where $C(S_y^{'}, X_P^{'}) \cap C(M - M_l) = \phi$ and $C(S_y^{'}, X_Q^{'}) \subseteq C(M - M_l)$. Clearly $|Y_Q^{'}| \leq t - l$ and $|X_Q^{'}| \leq t - l$.

Claim 3. $Y_P^{'} \in Y_M - Y_{M_l}$.

Proof. By contradiction. Then there exists an edge $e \in (S_x^{'}, Y - (Y_M - Y_{M_l}))$ with $C(e) \notin C(M - M_l)$.

We distinguish the following three cases.

Case 1. $e \in E(S_x^{'}, S_y^{'})$. Let

$$M^1 = \begin{cases} M \cup e & C(e) \notin C_l; \\ M \cup e \cup x_iy_i^{'} - x_iy_i & C(e) \in C_l, \text{w.l.o.g, suppose } C(e) = c_i. \end{cases}$$

Then M^1 is a heterochromatic matching with $|M^1| > |M|$, a contradiction.

Case 2. $e \in E(S_x^{'}, Y_{M_l})$. Without loss of generality, suppose e is adjacent with y_i. Let

$$M^1 = \begin{cases} M \cup e \cup x_iy_i^{'} - x_iy_i & C(e) \notin C_l \cup C_L; \\ M \cup e \cup x_i^{'}y_i^{'} - x_iy_i & C(e) \in C_L; \\ M \cup e \cup x_iy_i^{'} - x_iy_i & C(e) = c_i \in C_l; \\ M \cup e \cup x_iy_i^{'} \cup x_jy_j^{'} - x_iy_i - x_jy_j & C(e) = c_j \in C_l \text{ and } j \neq i. \end{cases}$$

Then M^1 is a heterochromatic matching and $|M^1| > |M|$, a contradiction.

Case 3. $e \in E(S_x^{'}, Y_L)$. Without loss of generality, suppose e is adjacent with $y_i^{'}$. Let

$$M^1 = \begin{cases} M \cup e & C(e) \notin C_l; \\ M \cup e \cup x_i^{'}y_i - x_iy_i & C(e) = c_i \in C_l; \\ M \cup e \cup x_jy_j^{'} - x_jy_j & C(e) = c_j \in C_l \text{ and } j \neq i. \end{cases}$$

Then M^1 is a heterochromatic matching and $|M^1| > |M|$, a contradiction.

This completes the proof of the claim. □

An *anti alternating* 4-cycle AAC is a cycle $e_1e_2e_3e_4e_1$ such that $e_1 \in E(M - M_l)$, $e_3 \in E(S_x^{'}, S_y^{'})$ and $C(e_1) = C(e_3)$, $C(e_2) = C(e_4) \in C_l \cup C_L$. Given two anti alternating 4-cycles $AAC = e_1e_2e_3e_4e_1$ and $AAC^{'} = e_1^{'}e_2^{'}e_3^{'}e_4^{'}e_1^{'}$, AAC is $different$ from $AAC^{'}$, we mean that $e_1 \neq e_1^{'}$ and $e_3 \neq e_3^{'}$.

Claim 4. There exists an anti alternating 4-cycle in G.

Proof. Since $|N^c(S_x')| = |Y_P'| + |Y_Q'| \geq |S_x'|$ and $Y_P' \in Y_M - Y_{M_l}$, it follows that $|Y_P'| \geq n - t - l - |Y_Q'| \geq n - t - l - (t - l) \geq n - 2t$. Similarly it holds that $X_P' \in X_M - X_{M_l}$ and hence $|X_P'| \geq n - 2t$.

Since $t < \frac{2n}{5}$ and $l \geq 1$, it holds that

$$|X_P'| + |Y_P'| \geq 2n - 4t \geq t - l + 1 = |X_M - X_{M_l}| + 1 = |Y_M - Y_{M_l}| + 1.$$

Then there exists an edge $\bar{x}\bar{y} \in E(M - M_l)$ such that $\bar{x}$ is adjacent with a vertex $\bar{y}' \in S'_y$ and $\bar{y}$ is adjacent with a vertex $\bar{x}' \in S_x'$. Clearly $C(\bar{x}\bar{y}') \notin C(M - M_l)$ and $C(\bar{x}'\bar{y}) \notin C(M - M_l)$.

Then we conclude that $C(\bar{x}\bar{y}') = C(\bar{x}'\bar{y}) \in C_l$ or $C(\bar{x}'\bar{y}) = C(\bar{x}'\bar{y}) \in C_L$. Otherwise, we distinguish the following cases.

Case 1. $C(\bar{x}\bar{y}') \notin C(M) \cup C_L$ and $C(\bar{x}'\bar{y}) \notin C(M) \cup C_L$, or $C(\bar{x}\bar{y}') \notin C(M) \cup C_L$ and $C(\bar{x}'\bar{y}) \in C_L$, or $C(\bar{x}'\bar{y}) \notin C(M) \cup C_L$ and $C(\bar{x}\bar{y}') \in C_L$.

Let $M^1 = M \cup \bar{x}\bar{y}' \cup \bar{x}'\bar{y} - \bar{x}\bar{y}$, then M^1 is a heterochromatic matching and $|M^1| > |M|$, a contradiction.

Case 2. $C(\bar{x}\bar{y}') \notin C(M) \cup C_L$ and $C(\bar{x}'\bar{y}) \in C_l$, or $C(\bar{x}'\bar{y}) \notin C(M) \cup C_L$ and $C(\bar{x}\bar{y}') \in C_l$.

If $C(\bar{x}\bar{y}') \notin C(M) \cup C_L$ and $C(\bar{x}'\bar{y}) \in C_l$, without less of generality, suppose $C(\bar{x}'\bar{y}) = c_i$. Then let $M^1 = M \cup \bar{x}\bar{y}' \cup \bar{x}'\bar{y} \cup x_i y_i' - \bar{x}\bar{y} - x_i y_i$, hence M^1 is a heterochromatic matching with $|M^1| > |M|$, a contradiction. Similarly, if $C(\bar{x}'\bar{y}) \notin (C(M) \cup C_L)$ and $C(\bar{x}\bar{y}') \in C_l$, we can also obtain a contradiction.

Case 3. $C(\bar{x}\bar{y}') \in C_L$ and $C(\bar{x}'\bar{y}) \in C_l$, or $C(\bar{x}'\bar{y}) \in C_L$ and $C(\bar{x}\bar{y}') \in C_l$.

If $C(\bar{x}\bar{y}') \in C_L$ and $C(\bar{x}'\bar{y}) \in C_l$, without loss of generality, suppose $C(\bar{x}\bar{y}') = c_j'$ and $C(\bar{x}'\bar{y}) = c_i$. If $i \neq j$, let $M^1 = M \cup \bar{x}\bar{y}' \cup \bar{x}'\bar{y} \cup x_i y_i' - \bar{x}\bar{y} - x_i y_i$, then M^1 is a heterochromatic matching with $|M^1| > |M|$, a contradiction. If $i = j$, let $M^1 = M \cup \bar{x}\bar{y}' \cup \bar{x}'\bar{y} - \bar{x}\bar{y} - x_i y_i$, then M^1 is a heterochromatic matching with $|M^1| = |M|$ and $b_{M_1} > b_M$, a contradiction. Similarly, if $C(\bar{x}'\bar{y}) \in C_L$ and $C(\bar{x}\bar{y}') \in C(M_l)$, we can also get a contradiction.

Case 4. $C(\bar{x}\bar{y}'), C(\bar{x}'\bar{y}) \in C_l$ and $C(\bar{x}\bar{y}') \neq C(\bar{x}'\bar{y})$.

Suppose $C(\bar{x}\bar{y}') = c_j$, $C(\bar{x}'\bar{y}) = c_i$ and $i \neq j$. Let $M^1 = M \cup \bar{x}\bar{y}' \cup \bar{x}'\bar{y} \cup x_i y_i' \cup x_j y_j' - \bar{x}\bar{y} - x_i y_i - x_j y_j$, then M^1 is a heterochromatic matching with $|M^1| > |M|$, a contradiction.

Case 5. $C(\bar{x}'\bar{y}), C(\bar{x}'\bar{y}) \in C_L$ and $C(\bar{x}'\bar{y}) \neq C(\bar{x}\bar{y}')$.

Suppose $C(\bar{x}\bar{y}') = c_j'$, $C(\bar{x}'\bar{y}) = c_i'$ and $i \neq j$. Let $M^1 = M \cup \bar{x}\bar{y}' \cup \bar{x}'\bar{y} - \bar{x}\bar{y}$, then M^1 is a heterochromatic matching with $|M^1| > |M|$, a contradiction.

So $C(\bar{x}\bar{y}') = C(\bar{x}'\bar{y}) \in C_l$ or $C(\bar{x}'\bar{y}) = C(\bar{x}'\bar{y}) \in C_L$.

Then we claim that there exists an edge $e \in E(G - V(M))$ such that $C(e) = C(\bar{x}\bar{y})$. Otherwise let

$$M^1 = \begin{cases} M \cup x_i y_i^{'} \cup \bar{x}^{'}\bar{y} - \bar{x}\bar{y} - x_i y_i & C(\bar{x}\bar{y}^{'}) \in C_l, \text{w.l.o.g, let } C(\bar{x}\bar{y}^{'}) = c_i; \\ M \cup \bar{x}^{'}\bar{y} - \bar{x}\bar{y} & C(\bar{x}\bar{y}^{'}) \in C_L. \end{cases}$$

Then $|M_1| = |M|$ and $b_{M_1} > b_M$, a contradiction. And if $e \neq \bar{x}\bar{y}$, w.l.o.g, suppose $\bar{x}^{'}$ is not incident with e and if $C(\bar{x}\bar{y}^{'}) = c_i (1 \leq i \leq l)$, $x_i^{'}$ is not incident with e. Let

$$M^1 = \begin{cases} M \cup e \cup x_i^{'} y_i \cup \bar{x}^{'}\bar{y} - \bar{x}\bar{y} - x_i y_i & C(\bar{x}\bar{y}^{'}) \in C_l, \text{w.l.o.g, let } C(\bar{x}\bar{y}^{'}) = c_i; \\ M \cup e \cup \bar{x}^{'}\bar{y} - \bar{x}\bar{y} & C(\bar{x}\bar{y}^{'}) \in C_L. \end{cases}$$

Then $|M_1| > |M|$, a contradiction. This completes the proof of Claim 4. □

Suppose that the maximum number of the pairwise different anti alternating 4-cycles in G is k, clearly $1 \leq k \leq t - l$. Assume that the anti alternating 4-cycle AAC_i has edges $\{\bar{x}_i\bar{y}_i, \bar{x}_i^{'}\bar{y}_i, \bar{x}_i{'}\bar{y}_i{'}, \bar{x}_i\bar{y}_i^{'}\}$. where $C(\bar{x}_i\bar{y}_i) = C(\bar{x}_i{'}\bar{y}_i{'})$, $C(\bar{x}_i^{'}\bar{y}_i) = C(\bar{x}_i\bar{y}_i^{'}) \in (C_L \cup C_l)$ and $\bar{x}_i\bar{y}_i \in E(M - M_l)$, $\bar{x}_i{'}\bar{y}_i{'} \in E(S_x^{'}, S_y^{'})$.

Denote

$$\begin{aligned} &X_{M_{\bar{k}}} = \{\bar{x}_1^{'}, \bar{x}_2^{'}, \cdots, \bar{x}_l^{'}\}, Y_{M_{\bar{k}}} = \{\bar{y}_1^{'}, \bar{y}_2^{'}, \cdots, \bar{y}_l^{'}\}, \\ &X_{K_1} = \{x \mid x \in (S_x^{'} \cap V(AAC_i)), 1 \leq i \leq k\}, k_1 = |X_{K_1}|, \\ &Y_{K_2} = \{y \mid y \in (S_y^{'} \cap V(AAC_i)), 1 \leq i \leq k\}, k_2 = |Y_{K_2}|, \\ &C_k = \{C(\bar{x}_1\bar{y}_1), C(\bar{x}_2\bar{y}_2), \cdots, C(\bar{x}_k\bar{y}_k)\}, \end{aligned}$$

where $\{\bar{x}_1\bar{y}_1, \bar{x}_2\bar{y}_2, \cdots, \bar{x}_k\bar{y}_k\} = E(M_{\bar{k}}) \subseteq E(M - M_l)$. Clearly, $1 \leq k_1, k_2 \leq k$.

Then put $S_x^{''} = X - X_M - X_L - X_{k_1}$ and $S_y^{''} = Y - Y_M - Y_L - Y_{k_2}$. Note that $S_x^{''} \neq \phi, S_y^{''} \neq \phi$. Otherwise, it holds that $n \leq 2t$, a contradiction. Let $N^c(S_x^{''})$ and $N^c(S_y^{''})$ be a maximum color neighborhood of $S_x^{''}$ and $S_y^{''}$, respectively. Set $N^c(S_x^{''}) = Y_P^{''} \cup Y_Q^{''} (Y_P^{''} \cap Y_Q^{''} = \phi)$ where $C(S_x^{''}, Y_P^{''}) \cap (C(M) - C_l - C_k) = \phi$, $C(S_x^{''}, Y_P^{''}) \in (C(M) - C_l - C_k)$ and let $N^c(S_y^{''}) = X_P^{''} \cup X_Q^{''} (X_P^{''} \cap X_Q^{''} = \phi)$ where $C(S_y^{''}, X_P^{''}) \cap C(M) - C_l - C_k) = \phi$, $C(S_y^{''}, X_Q^{''}) \subseteq C(M) - C_l - C_k)$. Clearly, $|Y_Q^{''}| \leq t - l - k$ and $|X_Q^{''}| \leq t - l - k$.

Claim 6. $Y_P^{''} \subseteq (Y_M - Y_{M_l} - Y_{M_{\bar{k}}})$.

Proof. By contradiction. Otherwise Then there exists an edge $e \in (S_x^{''}, Y - (Y_M - Y_{M_l} - Y_{M_{\bar{k}}}))$ with $C(e) \notin (C(M) - C_l - C_k)$.

We distinguish the following three cases.

Case 1. $e \in E(S_x^{''}, Y - Y_M - Y_L)$. By the proof of Case 1 in Claim 3, we only consider the case when $C(e) \in C_k$. Without loss of generality, suppose that $C(e) = C(\bar{x}_p\bar{y}_p)(1 \leq p \leq k)$. Let

$$M^1 = \begin{cases} M \cup e \cup \bar{x}_p\bar{y}_p^{'} - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) \in C_L; \\ M \cup e \cup \bar{x}_p\bar{y}_p^{'} \cup x_i y_i^{'} - \bar{x}_p\bar{y}_p - x_i y_i & C(\bar{x}_p\bar{y}_p^{'}) \in C_l, \text{w.l.o.g, let } C(\bar{x}_p\bar{y}_p^{'}) = c_i. \end{cases}$$

Then M^1 is a heterochromatic matching and $|M^1| > |M|$, a contradiction.

Case 2. $e \in E(S_x^{''}, Y_{M_l})$. Without loss of generality, suppose e is adjacent with y_i. By the proof of Case 2 in Claim 3, we only consider the case when $C(e) \in C_k$. Without loss of generality, suppose that $C(e) = C(\bar{x}_p\bar{y}_p)(1 \le p \le k)$. Let

$$M^1 = \begin{cases} M \cup e \cup x_i^{'}y_i^{'} \cup \bar{x}_p\bar{y}_p^{'} - x_iy_i - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) \in C_L; \\ M \cup e \cup x_i^{'}y_i^{'} \cup \bar{x}_p\bar{y}_p^{'} \cup x_jy_j^{'} - x_iy_i - \bar{x}_p\bar{y}_p - x_jy_j & C(\bar{x}_p\bar{y}_p^{'}) = c_j \in C_l \text{ and } j \ne i; \\ M \cup e \cup x_i^{'}y_i^{'} - x_iy_i - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) = c_i \in C_l. \end{cases}$$

Then M^1 is a heterochromatic matching such that either $|M^1| > |M|$ or $|M^1| = |M|$ and $b_{M_1} > b_M$, a contradiction.

Case 3. $e \in E(S_x^{''}, Y_L)$. Without loss of generality, suppose e is adjacent with $y_i^{'}$. By the proof of Case 3 in Claim 3, we only consider the case when $C(e) \in C_k$. Suppose that $C(e) = C(\bar{x}_p\bar{y}_p)(1 \le p \le k)$. Since $e \in E(G - V(M))$, by the proof of Claim 4, we conclude that $e = \bar{x}_p^{'}\bar{y}_p^{'}$, a contradiction.

Case 4. $e \in E(S_x^{''}, Y_{M_{\bar{k}}})$. Without loss of generality, suppose e is adjacent with $\bar{y}_p$ $(1 \le p \le k)$.

If $C(e) \notin C(M) \cup C_L$, then let

$$M^1 = \begin{cases} M \cup e \cup \bar{x}_p\bar{y}_p^{'} \cup x_iy_i^{'} - x_iy_i - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) \in C_l, \text{w.l.o.g, let } C(\bar{x}_p\bar{y}_p^{'}) = c_i; \\ M \cup e \cup \bar{x}_p\bar{y}_p^{'} - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) \in C_L, \text{w.l.o.g, let } C(\bar{x}_p\bar{y}_p^{'}) = c_i^{'}. \end{cases}$$

If $C(e) \in C_l$, w.l.o.g, suppose that $C(e) = c_i$, then let

$$M^1 = \begin{cases} M \cup e \cup \bar{x}_p\bar{y}_p^{'} \cup x_iy_i^{'} \cup x_jy_j^{'} - x_iy_i - \bar{x}_p\bar{y}_p - x_jy_j & C(\bar{x}_p\bar{y}_p^{'}) = c_j \in C_l \text{ and } j \ne i; \\ M \cup e \cup \bar{x}_p^{'}\bar{y}_p^{'} \cup x_iy_i^{'} - x_iy_i - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) = c_i \in C_l; \\ M \cup e \cup \bar{x}_p\bar{y}_p^{'} \cup x_iy_i^{'} - x_iy_i - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) = c_j^{'} \in C_L \text{ and } j \ne i; \\ M \cup e \cup \bar{x}_p\bar{y}_p^{'} - x_iy_i - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) = c_i^{'} \in C_L. \end{cases}$$

If $C(e) \in C_L$, w.l.o.g, suppose that $C(e) = c_i^{'}$, then let

$$M^1 = \begin{cases} M \cup e \cup \bar{x}_p\bar{y}_p^{'} \cup x_jy_j^{'} - x_jy_j - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) = c_j \in C_l \text{ and } j \ne i; \\ M \cup e \cup \bar{x}_p\bar{y}_p^{'} - x_iy_i - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) = c_i \in C_l; \\ M \cup e \cup \bar{x}_p\bar{y}_p^{'} - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) = c_j^{'} \in C_L \text{ and } j \ne i; \\ M \cup e \cup \bar{x}_p^{'}\bar{y}_p^{'} - x_iy_i - \bar{x}_p\bar{y}_p & C(\bar{x}_p\bar{y}_p^{'}) = c_i^{'} \in C_L. \end{cases}$$

If $C(e) \in C_k$ and $C(e) = C(\bar{x}_q\bar{y}_q)(q \ne p)$, then let

$$M^1 = \begin{cases} M \cup e \cup \bar{x}_p^{'}\bar{y}_p^{'} \cup \bar{x}_q\bar{y}_q^{'} - \bar{x}_q\bar{y}_q - \bar{x}_p\bar{y}_p & C(\bar{x}_q\bar{y}_q^{'}) \in C_L, \text{w.l.o.g, let } C(\bar{x}_q\bar{y}_q^{'}) = c_i^{'}; \\ M \cup e \cup \bar{x}_p^{'}\bar{y}_p^{'} \cup \bar{x}_q\bar{y}_q^{'} \cup x_iy_i^{'} - \bar{x}_p\bar{y}_p - \bar{x}_q\bar{y}_q - x_iy_i & C(\bar{x}_q\bar{y}_q^{'}) \in C_l, \text{w.l.o.g, let } C(\bar{x}_q\bar{y}_q^{'}) = c_i. \end{cases}$$

If $C(e) \in C_k$ and $C(e) = C(\bar{x}_p\bar{y}_p)$, then let $M^1 = M \cup \bar{x}_p^{'}\bar{y}_p^{'} - \bar{x}_p\bar{y}_p$.

In any cases, M^1 is a heterochromatic matching such that either $|M^1| > |M|$ or $|M^1| = |M|$ and $b_{M^1} > b_M$, which gives a contradiction.

$\square$.

Similarly it holds that $X_P^{''} \subseteq (X_M - X_{M_l} - X_{M_{\bar{k}}})$.

Then it follows that $|X_P^{''}| + |Y_P^{''}| - |X_M - X_{M_l} - X_{M_{\bar{k}}}|$

$$
\begin{aligned}
&\geq (n - t - l - k_1) + (n - t - l - k_2) - 3(t - l - k) \\
&= 2n - 5t + l + k + (2k - k_1 - k_2) \\
&\geq l + k.
\end{aligned}
$$

So there exists an edge $x_0y_0 \in E(M - M_l - M_{\bar{k}})$, where x_0 is adjacent with a vertex $y_0^{'} \in S_y^{''}$ and y_0 is adjacent with a vertex $x_0^{'} \in S_x^{''}$ such that at least one of $C(x_0y_0^{'}), C(x_0^{'}y_0)$ is not in C_k. Suppose $C(x_0y_0^{'}) \notin C_k$. Clearly, $C(x_0^{'}y_0) \notin (C(M) - C_l - C_k)$.

We distinguish the following cases.

Case 1. $C(x_0^{'}y_0) \notin C_k$. By the proof in Claim 4, it holds that $C(x_0y_0^{'}) = C(x_0^{'}y_0) \in C_l \cup C_L$ and $C(x_0^{'}y_0^{'}) = C(x_0^{'}y_0^{'})$. Then there is an anti alternating 4-cycle different from any AAC_i, $1 \leq i \leq k$, contradicting with the maximum number of the anti alternating 4-cycle of G.

Case 2. $C(x_0^{'}y_0) \in C_k$. Without loss of generality, let $C(x_0^{'}y_0) = C(\bar{x}_p\bar{y}_p)$ $(1 \leq p \leq k)$.

In this case, we have the following claim.

Claim 7. If there is an edge $e \in E(G - V(M))$ such that $C(e) = C(x_0y_0)$, then $e = x_0^{'}y_0^{'}$.

Proof. Otherwise, we have the the following two cases.

Case 1. $x_0^{'}$ is not incident with e.

If $C(\bar{x}_p\bar{y}_p^{'}) \in C_l$, w.l.o.g, let $C(\bar{x}_p\bar{y}_p^{'}) = c_i$ and suppose $\bar{x}_p^{'}$ and $x_i^{'}$ are not incident with e. Let $M^1 = M \cup e \cup x_0^{'}y_0 \cup \bar{x}_p^{'}\bar{y}_p \cup x_i^{'}y_i - x_0y_0 - \bar{x}_p\bar{y}_p - x_iy_i$, then M^1 is a heterochromatic matching of G with $|M^1| > |M|$, a contradiction.

If $C(\bar{x}_p\bar{y}_p^{'}) \in C_L$, w.l.o.g, let $C(\bar{x}_p\bar{y}_p^{'}) = c_i^{'}$ and suppose $\bar{x}_p^{'}$ is not incident with e. Let $M^1 = M \cup e \cup x_0^{'}y_0 \cup \bar{x}_p^{'}\bar{y}_p - x_0y_0 - \bar{x}_p\bar{y}_p$, then M^1 is a heterochromatic matching of G with $|M^1| > |M|$, a contradiction.

Case 2. $y_0^{'}$ is not incident with e.

If $C(x_0y_0^{'}) \in C_l$, w.l.o.g, let $C(x_0y_0^{'}) = c_i$ and suppose $x_i^{'}$ is not incident with e. Let $M^1 = M \cup e \cup x_0y_0^{'} \cup x_i^{'}y_i - x_0y_0 - x_iy_i$, then M^1 is a heterochromatic matching with $|M^1| > |M|$, a contradiction.

If $C(x_0y_0^{'}) \notin C_l$, let $M^1 = M \cup e \cup x_0y_0^{'} - x_0y_0$, then M^1 is a heterochromatic matching with $|M^1| > |M|$, a contradiction.

This completes the proof of Claim 7. □

In the following, we end the proof of Case 2, then the proof of Theorem 6 is complete.

If $C(\bar{x}_p^{'}\bar{y}_p) \in C_l$, w.l.o.g, let $C(\bar{x}_p^{'}\bar{y}_p) = c_i$. Then let

$$M^1 = \begin{cases} M \cup x_0y_0^{'} \cup x_0^{'}y_0 \cup x_iy_i^{'} - x_iy_i - \bar{x}_p\bar{y}_p - x_0y_0 & C(x_0y_0^{'}) = c_i; \\ M \cup x_0y_0^{'} \cup x_0^{'}y_0 \cup \bar{x}_p^{'}\bar{y}_p \cup x_iy_i^{'} \cup x_jy_j^{'} - x_iy_i - x_jy_j - \bar{x}_p\bar{y}_p - x_0y_0 & C(x_0y_0^{'}) = c_j (j \neq i); \\ M \cup x_0y_0^{'} \cup x_0^{'}y_0 \cup \bar{x}_p^{'}\bar{y}_p - x_iy_i - \bar{x}_p\bar{y}_p - x_0y_0 & C(x_0y_0^{'}) = c_i^{'}; \\ M \cup x_0y_0^{'} \cup x_0^{'}y_0 \cup \bar{x}_p^{'}\bar{y}_p \cup x_iy_i^{'} - x_iy_i - \bar{x}_p\bar{y}_p - x_0y_0 & C(x_0y_0^{'}) = c_j^{'} (j \neq i); \\ M \cup x_0y_0^{'} \cup x_0^{'}y_0 \cup \bar{x}_p^{'}\bar{y}_p \cup x_iy_i^{'} - x_iy_i - \bar{x}_p\bar{y}_p - x_0y_0 & C(x_0y_0^{'}) \notin C_L \cup C_l. \end{cases}$$

If $C(\bar{x}_p^{'}\bar{y}_p) \in C_L$, w.l.o.g, suppose $C(\bar{x}_p^{'}\bar{y}_p) = c_i^{'}$. Then let

$$M^1 = \begin{cases} M \cup x_0y_0^{'} \cup x_0^{'}y_0 \cup \bar{x}_p^{'}\bar{y}_p - x_iy_i - \bar{x}_p\bar{y}_p - x_0y_0 & C(x_0y_0^{'}) = c_i; \\ M \cup x_0y_0^{'} \cup x_0^{'}y_0 \cup \bar{x}_p^{'}\bar{y}_p \cup x_jy_j^{'} - x_iy_i - x_jy_j - \bar{x}_p\bar{y}_p - x_0y_0 & C(x_0y_0^{'}) = c_j (j \neq i); \\ M \cup x_0y_0^{'} \cup x_0^{'}y_0 - \bar{x}_p\bar{y}_p - x_0y_0 & C(x_0y_0^{'}) = c_i^{'}; \\ M \cup x_0y_0^{'} \cup x_0^{'}y_0 - x_0y_0 & \text{Otherwise.} \end{cases}$$

In any cases, M^1 is a heterochromatic matching such that either $|M^1| > |M|$ or $|M^1| = |M|$ and $b_{M_1} > b_M$, contradicting with the choice of M.

This completes the proof of Theorem 6. □

References

[1] M. Albert, A. Frieze and B. Reed, Multicolored Hamilton cycles, Electronic J.Combin. 2(1995), Research Paper R10.

[2] N. Alon, T. Jiang, Z. Miller and D. Pritikin, Properly colored subgraphs and rainbow subgraphs in edge-colored graphs with local constraints, Random Struct. Algorithms 23(2003), No.4,409-433.

[3] J.A. Bondy and U.S.R. Murty. Graph Theory with Applications, Macmillan Press[M]. New York, 1976.

[4] H.J. Broersma, X. Li, G. Woegingerr and S. Zhang, Paths and cycles in colored graphs, Australian J.combin. 31(2005), 297-309.

[5] H. Chen and X. Li, Long heterochromatic paths in edge-colored graphs, The Electronic J. Combin. 12(1)(2005), Research Paper R33.

[6] P. Erdös and Zs. Tuza, Rainbow subgraphs in edge-colorings of complete graphs, Ann. Discrete Math. 55(1993), 81-83.

[7] A.M. Frieze and B.A. Reed, Polychromatic Hamilton cycles, Discrete Math. 118 (1993), 69-74.

[8] M.R. Garey and D.S. Johnson, Comuters and Intractability, Freeman, New York, 1979, Pages 203. GT55: Multiple Choice Matching Problem.

[9] G. Hahn and C. Thomassen, Path and cycle sub-Ramsey numbers and edge-coloring conjecture, Discrete Math. 62(1)(1986), 29-33.

[10] P. Hall, On representatives of subsets, J.London Math. Soc., 10(1935),26-30.

[11] L. Hu and X. Li, Sufficient conditions for the existence of perfect heterochromatic matcings in colored graphs, arXiv:math.Co/051160v1 24Nov 2005.

[12] E.L. Lawler, Combinatorial Optimization: Networks and Matroids, Holt, Rinehart and Winston, New York, 1976.

[13] H. Li, X. Li, G. Liu,, and G. Wang, The heterochromatic matchings in edge-colored bipartite graphs, preprint 2005.