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## A NOTE ON k-WALKS IN BRIDGELESS GRAPHS

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# A note on $k$-walks in bridgeless graphs

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**Abstract**

We show that every bridgeless graph of maximum degree $\Delta$ has a spanning $\lceil(\Delta+1)/2\rceil$-walk. The bound is optimal.

## 1 Introduction

Following Jackson and Wormald [6], we define a *$k$-walk* in a graph $G$ to be a closed spanning walk visiting each vertex at most $k$ times, where $k \geq 1$ is an integer. Being an interesting variation on the notion of a Hamilton cycle, this concept has received considerable attention (see, e.g., [2, 3, 5]).

Our aim in this note is to determine the least possible $k = k(\Delta)$ such that every graph of maximum degree $\Delta$ admits a $k$-walk. For general graphs, this problem is trivial since a tree of maximum degree $\Delta$ has a $\Delta$-walk [6], and it clearly does not admit any $k$-walk with $k < \Delta$. The situation changes if we restrict ourselves to *bridgeless* (i.e., 2-edge-connected) graphs. We prove the following result:

**Theorem 1.** *Every bridgeless graph of maximum degree $\Delta$ admits a $\lceil(\Delta+1)/2\rceil$-walk.*

Theorem 1 follows directly from a more general statement (Theorem 5) which we prove in Section 2. In Section 3, we complement this result by showing that the bound in Theorem 1 is best possible.

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## 2 The upper bound

All the graphs we consider are finite and loopless, multiple edges are allowed. Throughout this section, $G$ is a graph. Its vertex set and the edge set are denoted by $V(G)$ and $E(G)$, respectively. If $W$ is a walk in $G$, we let $p_W(x)$ denote the number of times a vertex $x \in V(G)$ is visited by $W$. An *edge-cut* in $G$ is an inclusionwise minimal set of edges whose removal disconnects $G$.

Let $v$ be a vertex of $G$ and $e_1, e_2$ be two distinct edges incident with $v$. Let $v_i$ be the endvertex of $e_i$ distinct from $v$. We recall the operation of *splitting* $e_1$ *and* $e_2$ *off* $v$. The resulting graph $G(v, e_1, e_2)$ is defined to be $G$ with an added vertex $v^*$ and the edges $e_1, e_2$ replaced with $e_1^*, e_2^*$, where $e_i^*$ has ends $v^*$ and $v_i$. The following assertion is an easy consequence of Fleischner's Splitting Lemma [4] (see also [9, Theorem A.5.2]):

**Lemma 2.** *Let $v$ be a vertex of degree at least 4 in a bridgeless graph $G$. There exist edges $e_1, e_2$ incident with $v$ such that the graph $G(v, e_1, e_2)$ is bridgeless.*

**Lemma 3.** *Let $v$ be a vertex of a graph $G$, let $e_1, e_2$ be two edges incident with $v$, and $H = G(v, e_1, e_2)$. If $W$ is a spanning closed walk in $H$ such that $p_W(v^*) \leq 2$ (where $v^*$ is defined as above), then $G$ admits a closed walk $\tilde{W}$ such that*

*(i) for all $z \in V(G) \setminus \{v\}$, $p_{\tilde{W}}(z) \leq p_W(z)$, and*

*(ii) $1 \leq p_{\tilde{W}}(v) \leq p_W(v) + 1$.*

*Proof.* Enumerate the vertices visited by $W$ as

$$W = x_0 x_1 \dots x_\ell,$$

where $x_0 = x_\ell$. Any operations on the indices of the vertices in $W$ are performed modulo $\ell$. A *subwalk* of $W$ is a walk of the form

$$[x_i, x_j] = x_i x_{i+1} \dots x_{j-1} x_j.$$

We write $[x_i, x_j]^-$ for the reverse subwalk $x_i x_{i-1} \dots x_{j+1} x_j$.

If $p_W(v^*) = 1$, then we may set $\tilde{W} = W$. Thus, it may be assumed that $p_W(v^*) = 2$. Let the two occurences of $v^*$ in $W$ be $x_i$ and $x_j$, where $i < j$. We use the symbols $v_1$, $v_2$ as introduced in the definition of splitting.

Suppose first that both neighbors of $x_i$ on $W$ coincide with $v_1$, i.e., $[x_{i-1}, x_{i+1}] = v_1 v^* v_1$. Then we may set

$$\tilde{W} = [x_0, x_{i-1}]\,[x_{i+2}, x_{j-1}]\,v\,[x_{j+1}, x_\ell]$$

(see Figure 1a). Note that we may indeed concatenate the subwalks $[x_0, x_{i-1}]$ and $[x_{i+2}, x_{j-1}]$ since $x_{i+2}$ is a neighbor of $x_{i-1} = x_{i+1}$. It is easy to check that $\tilde{W}$ satisfies the conditions (i)–(ii). By symmetry, we may assume that the neighbors of $x_i$ on $W$ are $v_1$ and $v_2$, and the same holds for $x_j$. We now distinguish two cases.

*Case 1:* $[x_{i-1}, x_{i+1}] = [x_{j-1}, x_{j+1}] = v_1 v^* v_2$. We set

$$\tilde{W} = [x_0, x_{i-1}]\,[x_{j-2}, x_{i+2}]^-\,[x_{j+1}, x_\ell]$$

(see Figure 1b). Note that the conditions (i)–(ii) are satisfied. By symmetry, this case also covers the possibility that $[x_{i-1}, x_{i+1}]$ and $[x_{j-1}, x_{j+1}]$ equal $v_2 v^* v_1$.

*Case 2:* $[x_{i-1}, x_{i+1}] = [x_{j-1}, x_{j+1}]^- = v_1 v^* v_2$. Since $W$ is spanning, there is $k$ such that $x_k = v$. We may assume that $i < k < j$ since the other possibility ($k < i$ or $k > j$) is symmetric. The walk

$$\tilde{W} = [x_0, x_{i-1}]\,v\,[x_{k-1}, x_{i+2}]^-\,[x_{j-1}, x_{k+1}]^-\,v\,[x_{j+1}, x_\ell]$$

(see Figure 1c for an illustration) meets the requirements.

Since we have covered, up to symmetry, all the possibilities, the proof is complete. □

Spanning closed walks correspond to edge weight functions in the following straightforward way. Let $w$ be a function assigning to each edge $e \in E(G)$ a non-negative integer $w(e)$. For any set $X \subset E(G)$, we define

$$w(X) = \sum_{e \in X} w(e).$$

The function $w$ is an *Eulerian weight* if for each edge-cut $C$ in $G$, the value $w(C)$ is positive and even. Note that if $w$ is an Eulerian weight, then each vertex $v$ must be incident with an edge of nonzero weight, since the set

$$\partial v = \{e : e \text{ is incident with } v\}$$

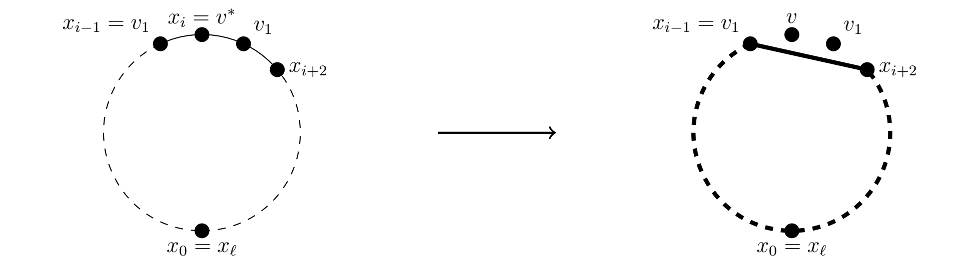
contains an edge-cut.

**Lemma 4.** *Let $G$ be a graph and $k \geq 1$ a positive integer. The graph $G$ has a $k$-walk if and only if it admits an Eulerian weight $w$ such that for each $v \in V(G)$,*

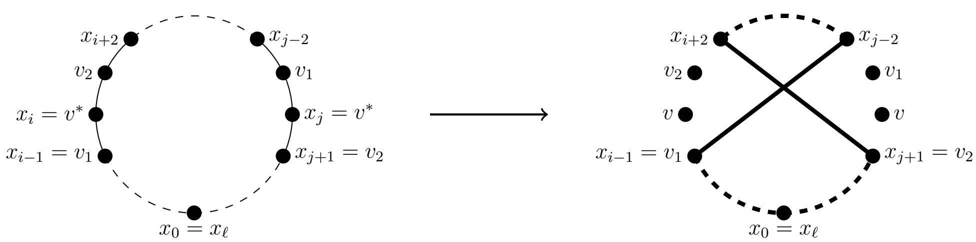
$$w(\partial v) \leq 2k. \tag{1}$$

*Proof.* If $G$ has a $k$-walk $W$, then the function assigning each edge the number of times it is traversed by $W$ (in any direction) is clearly an Eulerian weight satisfying (1). Conversely, let $w$ be such an Eulerian weight. Replacing each edge $e$ by $w(e)$ parallel edges (or deleting it if $w(e) = 0$), we obtain a (connected) Eulerian graph of maximum degree at most $2k$. Any Euler trail in the new graph determines a $k$-walk in $G$. □

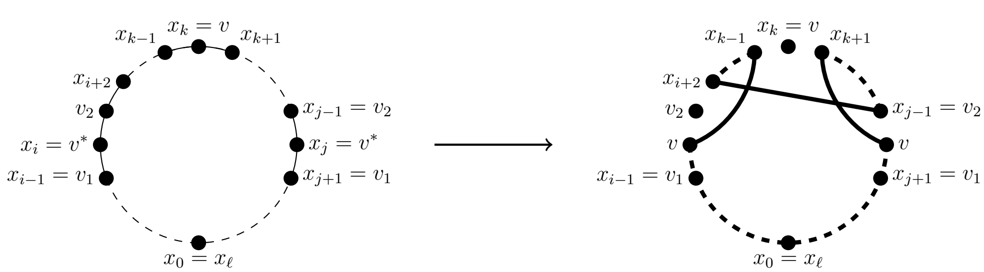
We now proceed to prove the main result of this paper.




(a) $W$ contains a subwalk $v_1 v^* v_1$.




(b) $W$ contains two subwalks $v_1 v^* v_2$.




(c) $W$ contains subwalks $v_1 v^* v_2$ and $v_2 v^* v_1$.

Figure 1: The possibilities considered in the proof of Lemma 3. Dashed lines represent walks, edges are drawn solid. In each case, thick lines give the resulting walk $\tilde{W}$ in $G$.

**Theorem 5.** *Every bridgeless graph admits a closed spanning walk $W$ such that for each vertex $x$,*

$$p_W(x) \le \left\lceil \frac{\deg(x)+1}{2} \right\rceil. \qquad (2)$$

*Proof.* By induction. We first establish the assertion for graphs with maximum degree $\Delta \le 3$. Then, we prove that if $\Delta(G) \ge 4$, the assertion holds for $G$ whenever it holds for all bridgeless graphs that are smaller than $G$ in a certain sense.

Assume first that $\Delta(G) \le 3$. Since the minimum degree is at least 2 and the claim is clearly true if $G$ is a circuit, we may assume that $G$ is a subdivision of a cubic bridgeless graph $H$. By the well-known Petersen theorem (see, e.g., [1, Corollary 2.2.2]), $H$ has a 1-factor $F$. Let $w : E(G) \to \{1, 2\}$ be a function whose value $w(e)$ is 2 if the edge of $H$ corresponding to $e$ belongs to $F$, and 1 otherwise. It is easy to see that $w$ is an Eulerian weight in $G$. By Lemma 4, $G$ admits a 2-walk.

Next, assume that $\Delta(G) \ge 4$ and (2) holds for all graphs $G'$ such that either $\Delta(G') < \Delta(G)$, or $\Delta(G') = \Delta(G)$ and $G'$ has fewer vertices of maximum degree. We show that the assertion holds for $G$.

Let $v$ be any vertex of degree $\Delta(G)$. Lemma 2 ensures that there are two edges $e_1$, $e_2$ such that $G(v, e_1, e_2)$ is bridgeless. Since the resulting graph has fewer vertices of degree $\Delta(G)$, the induction hypothesis implies that $G(v, e_1, e_2)$ admits a closed spanning walk $W_0$ satisfying (2). Using Lemma 3, we find a closed spanning walk $\tilde{W}_0$ in $G$ such that for each vertex $x \in V(G) \backslash \{v\}$, $p_{\tilde{W}_0}(x) \le p_{W_0}(x)$, and

$$1 \le p_{\tilde{W}_0}(v) \le p_{W_0}(v) + 1.$$

Clearly, the closed walk $\tilde{W}_0$ in $G$ is spanning, satisfies (2) at all vertices $x \ne v$, and

$$p_{\tilde{W}_0}(v) \le p_{W_0}(v) + 1 \le \left\lceil \frac{(\deg(v)-2)+1}{2} \right\rceil + 1 = \left\lceil \frac{\deg(v)+1}{2} \right\rceil.$$

It follows that $W = \tilde{W}_0$ satisfies (2) at all vertices of $G$. □

# 3 The lower bound

**Theorem 6.** *For every even $\Delta \ge 4$, there is a 2-connected graph $G$ with $\Delta(G) = \Delta$ and no $(\Delta/2)$-walk.*

*Proof.* Let $k = \Delta - 1$. For $i \in \{1, \dots, 9\}$, take a copy $H_i$ of the complete bipartite graph $K_{2,k}$, with the degree $k$ vertices denoted by $a_i$ and $b_i$.

The graph $G$ is obtained from the disjoint union of the graphs $H_1, \dots, H_9$ by adding new vertices $a$ and $b$, together with edges

$$\{aa_i : i \in \{1, 4, 7\}\} \cup \{bb_i : i \in \{3, 6, 9\}\} \cup \{b_i a_{i+1} : i \in \{1, 2, 4, 5, 7, 8\}\}$$

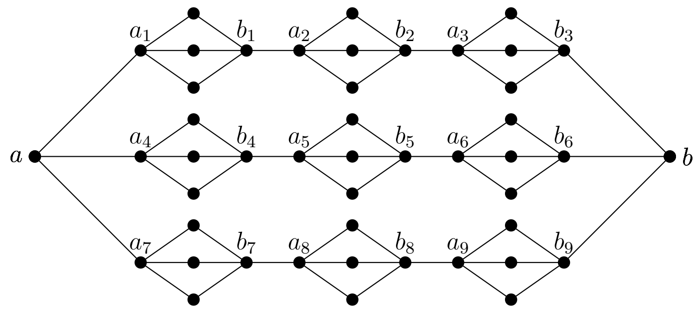



Figure 2: A 2-connected graph with maximum degree 4 and no 2-walk.

(see Figure 2 for an illustration with $k = 3$). Note that the maximum degree of $G$ is $k + 1 = \Delta$.

We now show that $G$ has no $(\Delta/2)$-walk. Assume the contrary. By Lemma 4, there is an Eulerian weight $w$ satisfying

$$w(\partial v) \leq \Delta \tag{3}$$

for each vertex $v$.

Since $w(\partial a)$ is even, there is an edge incident with $a$ that receives an even value. We may assume that $w(aa_1)$ is even. Since each pair of edges from the set

$$C = \{aa_1, b_1a_2, b_2a_3, b_3b\}$$

forms an edge-cut, at most one edge $e \in C$ has $w(e) = 0$. Consequently, for some $i \in \{1, 2, 3\}$, both edges in $C$ that are incident with either $a_i$ or $b_i$ are assigned a positive even value by $w$. Let $C_i$ be the set consisting of these two edges. We have

$$w(E(H_i)) = w(\partial a_i) + w(\partial b_i) - w(C_i) \leq 2\Delta - 4 \tag{4}$$

by (3).

For each vertex $d$ of degree 2 in $H_i$, $\partial d$ is an edge-cut, whence $w(\partial d) \geq 2$. It follows that

$$w(E(H_i)) \geq 2k = 2\Delta - 2,$$

contradicting (4). It follows that $G$ does not admit any $(\Delta/2)$-walk. □

Recall that a *trail* in a graph is a walk using each edge at most once. By a well-known result of Jaeger [7, 8], every 4-edge-connected graph $G$ admits a spanning closed trail. It is easy to see that if the maximum degree of $G$ is $\Delta$, then such a trail gives rise to a $\lceil \Delta/2 \rceil$-walk in $G$. For even $\Delta$, this improves on the bound of Theorem 1 by one. Since the tightness example constructed in the proof of Theorem 6 makes a heavy use of edge-cuts of size 2, one may wonder whether such an improvement is possible even for 3-edge-connected graphs $G$. We leave this as an open problem:

**Problem 7.** *Does every 3-edge-connected graph of maximum degree* $\Delta$ *admit a* $\lceil \Delta/2 \rceil$*-walk?*

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