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A Second-Order Cone Programming approach for Linear programs with joint probabilistic constraints

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Abstract

This paper deal with a special case of Linear programs with joint probabilistic constraints (LPPC), where the left-hand side of probabilistic constraints is normally distributed stochastic coefficients and the rows of the matrix are assumed independent to each other. Through the piecewise linear approximation and the piecewise tangent approximation, we approximate the stochastic linear programs with two second-order cone programming (SOCP) problems. Furthermore, under weak assumptions, the optimums of the two SOCPs problems are a lower bound and an upper bound of the original problem respectively.

Keywords: Stochastic programming, Joint probabilistic constraints, Second-order cone programming, Piecewise linear approximation

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1. Introductions

In this paper, we focus on the following linear program with joint probabilistic or chance constraints:

$$
\begin{array}{lll}
 & \min & c^T x \\
(LPPC) & s.t. & \boldsymbol{Pr}\{Tx \le D\} \ge 1 - \alpha \\
 & & x \in X
\end{array}
\tag{1}
$$

where $X \subset R^n_+$ is a polyhedron, $c \in R^n$, $D = (D_1, \ldots, D_k) \in R^K$, $T = [T_1, \ldots, T_K]^T$ is a $K \times n$ random matrix, where $T_k, k = 1, \ldots, K$, is a random vector in R^n,

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and α is a confidence parameter chosen by decision maker, typically near zero, e.g., $\alpha = 0.05$. Note that in (1), we only use a single probability constraint on all the rows rather than requiring each row to be satisfied with high probability individually [6]. Such a constraint is known as a *joint probability constraint* [7].

Chance-constrained programming has been studied for a long time and plays an important role in engineering, telecommunication, finance, etc. Charnes, Cooper and Symonds in [4] dealt with individual probabilistic constraints, while joint probabilistic constraints were first consider by Miller and Wager in [8]. A general theory of chance-constrained programming was studied by Prékopa in [10]. In chance-constrained programming, linear programs with joint probabilistic constraints (LPPC) are one of main challenges of stochastic programming because the feasible region generally is not convex [7].

In this paper, we assume that the coefficient matrix T is normally distributed and the rows of T are independent to each other. In [5], Henrion and Strugarek had derived its convexity and proved that there exists an upper bound α^*, for which the linear program is a convex problem. However the α^* is very close to zero. So for most cases, the linear program with normally distributed coefficients is not convex. In [1], they provide a fully explicit way to calculate the gradient of the constraint function and employed other existing algorithms to solve the problem. In this paper, we approximate the stochastic linear programming with normally distributed coefficients with two second-order cone programming (SOCP) problems, which are convex problems. Furthermore, under weak assumptions, the optimums of the two SOCPs problems are a lower bound and an upper bound of the original problem respectively.

2. Normally distributed LPPC

Here, we study a special class of LLPC whereT_k is multivariate normally distributed with mean $\mu_k = (\mu_{k1}, \dots, \mu_{kn})$ and covariance matrix Σ_k. Moreover, T_{k_i} and T_{k_j} are independent of each other when $k_i \neq k_j$.

Since multivariate normally distributed vectors $T_k, k = 1, \dots, K$, are independent of each other, we have a deterministic reformulation of the special case of

LPPC as follows:

$$
\begin{array}{lll}
 & \min & c^T x \\
 & s.t. & \mu_k^T x + F^{-1}(p^{y_k})\|\Sigma_k^{1/2} x\| \leq D_k, k = 1, ..., K \\
(NLPPC) & & \sum_{k=1}^{K} y_k = 1 \\
 & & y_k \geq 0 \\
 & & x \in X
\end{array}
\qquad (2)
$$

where $p = 1 - \alpha$ and $F^{-1}(\cdot)$ is the inverse of the standard normal cumulative distribution function, and y^k is an intermediate variable.

In [5], they have given some results about the convexity of the feasible set of NLPPC as follows.

Theorem 2.0.1. *The feasible set of NLPPC is convex when* $p > F(\max\{\sqrt{3}, u^*\})$, *where* $u^* = \max_{i=1,\ldots,K} 4\lambda_{max}^{(k)}[\lambda_{max}^{(k)}]^{-\frac{3}{2}}\|\mu_k\|$. *Correspondingly, NLPPC is a convex problem.*

Remark if the value u^* *is smaller than* $\sqrt{3}$ *in Theorem 2.0.1, then NLPPC is convex when* $p > F(\sqrt{3}) \approx 0.958$.

3. Approximation of NLPPC

The idea to approximate the problem (2) is the following: firstly, we approximate $F^{-1}(p^{y_k})$ with a piecewise tangent function and piecewise linear approximation of y_k respectively. Afterwards, we get two approximations of NLPPC, which are SOCP problems. Secondly, we solve the SOCP problems, whose optimal solutions are the approximated solutions of NLPPC.

3.1. Piecewise tangent approximation of $F^{-1}(p^z)$

We choose N tangent points $z_j, j = 1, \ldots, N$ from interval $(0, 1]$ and denote $F^{-1}(p^{z_j})$ by F_j. Without loss of generality, we assume that $z_1 < z_2 <, \ldots, < z_N$. $F^{-1}(p^z)$ is approximated by using first-order Taylor series expansion around $z = z_j, j = 1, \ldots N$ as follows:

$$\hat{F}_{2_j} = F^{-1}(p^{z_j}) + (F^{-1})'(p^{z_j})p^{z_j} \ln p(z - z_j)$$

where $(F^{-1})'(p^{z_j}) = \frac{1}{F'(F^{-1}(p^{z_j}))} = \frac{1}{f(F^{-1}(p^{z_j}))}$, $\hat{b}_j = (F^{-1})'(p^{z_j})p^{z_j} \ln p$ and $\hat{a}_j = F^{-1}(p^{z_j}) - \hat{b}_j \cdot z_j$.

Then we obtain approximation of $F^{-1}(p^z)$, which is piecewise tangent line approximation of $F^{-1}(p^z)$, and we denote it by $\hat{F}_2$

$$\hat{F}_2 = \max_{j=1,\dots,N}\{\hat{F}_{2_j}\}, \quad z \in (0,1]$$

Theorem 3.1.1. *Let $y_{ki} = y_k x_i, k = 1,\dots,K, i = 1,\dots,n$ and $\tilde{z}_k = (\tilde{z}_{k1},\dots,\tilde{z}_{kn})$. Together with the approximation of $F^{-1}(p^{y_k})$, we have the approximation of NLPPC, correspondingly:*

$$\begin{aligned}
\min \quad & c^T x \\
s.t. \quad & \mu_k^T x + \|\Sigma_k^{1/2}\tilde{z}_k\| \leq D_k, \quad k = 1,\dots,K \\
& \tilde{z}_{ki} \geq \hat{a}_j x_i + \hat{b}_j y_{ki}, \quad j = 0,1,\dots,N-1, i = 1,\dots,n \\
& \sum_{k=1}^{K} y_{ki} = x_i, \quad i = 1,\dots,n \\
& y_{ki} \geq 0, \\
& x \in X
\end{aligned} \tag{3}$$

where $a_0 = 0, b_0 = 0$. Moreover, the optimum of the approximation is an lower bound of NLPPC.

Proof. In this proof, we do not prove how we obtain (3), but only the optimum of (3) is an upper bound. First we prove that the function $F^{-1}(p^z)$ is convex. Since function p^z is convex and $F^{-1}(\cdot)$ is non-decreasing, $F^{-1}(p^z)$ is convex. So for any tangent point $z_j, j \in \{1,\dots,N\}$

$$F^{-1}(p^z) \geq \hat{F_{2_j}} = F^{-1}(p^{z_j}) + F^{'-1}(p^{z_j})p^{z_j}\ln p(z - z_j), \quad z \in [0,1]$$

Thus,

$$F^{-1}(p^z) \geq \hat{F}_2 = \max_{j=1}^{N}\{\hat{F}_{2_j}\}, \quad z \in (0,1]$$

. Together with Correspondingly $\|\Sigma_k^{1/2}x\| \geq 0$, we have

$$\begin{aligned}
& \{x : \mu_k^T x + F^{-1}(p^{y_k})\|\Sigma_k^{1/2}x\| \leq D_k, k = 1,\dots,K\} \\
\subset \quad & \{x : \mu_k^T x + \hat{F}_2\|\Sigma_k^{1/2}x\| \leq D_k, k = 1,\dots,K\}
\end{aligned}$$

□

3.2. Piecewise linear approximation of $F^{-1}(p^z)$

We choose N interpolation points $z_j, j = 1, \ldots, N$ from interval $(0, 1]$ and denote $F^{-1}(p^{z_j})$ by F_j. Without loss of generality, we assume that $z_1 < z_2 <, \ldots, < z_N$. Let $\hat{F}_1$ be the corresponding piecewise linear approximation of $F^{-1}(p^z)$. We have:

$$\hat{F}_{1_j} = F_j + \frac{z - z_j}{z_{j+1} - z_j}(F_{j+1} - F_j) = a_j + b_j \cdot z, \quad z \in [z_j, z_{j+1}], j = 1, \ldots, N-1$$

Where $a_j = \frac{z_{j+1}F_j - z_j F_{j+1}}{z_{j+1} - z_j}$ and $b_j = \frac{F_{j+1} - F_j}{z_{j+1} - z_j}$.

Lemma 3.1.

$$\hat{F}_1 = \max_{j=1,\ldots,N-1}\{a_j + b_j \cdot z\}, \quad z \in (0, 1]$$

Proof. It is easy to prove, as $F^{-1}(p^z)$ is convex. □

Theorem 3.2.1. *Let $y_{ki} = y_k x_i, k = 1, \ldots, K, i = 1, \ldots, n$ and $\tilde{z}_k = (\tilde{z}_{k1}, \ldots, \tilde{z}_{kn})$. Together with the approximation of $F^{-1}(p^{y_k})$, we have the approximation of NLPPC, correspondingly:*

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & \mu_k^T x + \|\Sigma_k^{1/2}\tilde{z}_k\| \leq D_k, \quad k = 1, \ldots, K \\ & \tilde{z}_{ki} \geq a_j x_i + b_j y_{ki}, \quad j = 0, 1, \ldots, N-1, i = 1, \ldots, n \\ & \sum_{k=1}^{K} y_{ki} = x_i, \quad i = 1, \ldots, n \\ & y_{ki} \geq 0, \\ & x \in X \end{aligned} \tag{4}$$

where $a_0 = 0, b_0 = 0$. Moreover, if $z_N = 1$ and the feasible set of $y_k, k = 1, \ldots, K$ of NLPPC is bounded by $[z_1, 1]^K$, then the optimum of the approximation is an upper bound of NLPPC.

Proof. As in Theorem 3.1.1, we do not prove how we obtain (4), but only the optimum of (4) is a lower bound. In the proof of Theorem 3.1.1, we proved that the function $F^{-1}(p^z)$ is convex. So for any interval $[z_j, z_{j+1}], j \in \{1, \ldots, N-1\}$,

$$F^{-1}(p^z) \leq \hat{F}_{1_j} = F_j + \frac{z - z_j}{z_{j+1} - z_j}(F_{j+1} - F_j), \quad z \in [z_j, z_{j+1}]$$

Thus,

$$F^{-1}(p^z) \leq \hat{F}_1 = \max_{j=1}^{N}\{\hat{F}_{1_j}\}, \quad z \in [z_1, z_N]$$

Together with $\|\Sigma_k^{1/2}x\| \geq 0$, when $y_k \in [z_1, z_N], k = 1, \ldots, K$, we have:

$$\begin{aligned} & \{x : \mu_k^T x + \hat{F}_1\|\Sigma_k^{1/2}x\| \leq D_k, k = 1, ..., K\} \\ \subset \; & \{x : \mu_k^T x + F^{-1}(p^{y_k})\|\Sigma_k^{1/2}x\| \leq D_k, k = 1, ..., K\} \end{aligned}$$

Under the assumption that the feasible set of $y_k, k = 1, \ldots, K$, is bounded by $[z_1, 1]^K$ and $z_N = 1$, the feasible set of the approximation is a subset of the feasible set of NLPPC. So the optimum of is an upper bound of NLPPC. □

4. Numerical study

NLPPC can be used widely in stochastic combinatorial optimization where Branch and Bound (*B&B*) is a general algorithm for finding optimal solutions. As we know, bounding plays a very important role in the *B&B*. For many stochastic combinatorial optimization problems with joint probabilistic constraints, their linear relaxation are LPPC problems, such as the resource constrained shortest path problem (RCSP), knapsack problem etc.

We performed computational tests on a stochastic version of the resource constrained shortest path problem (RCSP). When the resources consumed by traversing the arc are random and independently normally distributed each other, the relaxation of the RCSP is a NLPPC problem. The RCSP consists of finding the shortest path between two nodes s and t in a network, with the constraint that traversing an arc of the network implies the consumption of certain limited resources [3]. The RCSP can be mathematically formulated as below;

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Tx \leq D \\ & Mx = b \\ & x \in \{0, 1\}^n \end{aligned} \tag{5}$$

where $c \in R^n$, $M \in \mathbb{R}^{m \times n}$, which is the *node-arc incidence matrix* [2] and $b \in \mathbb{R}^m$, where all elements are 0 except the *s-th* and the *t-th* which are 1 and -1 respectively. T is a non negative $K \times n$ matrix, D is a positive vector of K elements.

4.1. Relaxation of the stochastic RCSP

Here, we assume that, for each arc of the network, the resources consumed by traversing the arc are random and independently normally distributed each other. So the corresponding linear relaxation of the stochastic RCSP is as follows:

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & \boldsymbol{Pr}\{Tx \leq D\} \geq 1 - \alpha \\ & Mx = b \\ & x \geq 0 \end{aligned}$$

which is a NLPPC problem.

4.2. Computational results

The algorithm was implemented in Matlab and all tests ran on a Pentium(R)D @ 3.00 GHz with 2.0 GB RAM. The tests consisted of 5 instances drawn from the OR-library [9], which are deterministic RCSPs and all of which have 10 different deterministic resources consumed for each arch. In the network instances considered, the graph sizes are $(100, 990)$, $(100, 990)$, $(100, 999)$, $(200, 1960)$ and $(200, 1960)$, respectively. The costs of all the arcs are set to the same as the deterministic RCSPs. For *k-th* resource, we choose the fixed resources consumed as their means and their variance are generated on the interval $[0, \sigma^2(k)]$, where$\sigma^2(k)$ is the variance of all the *k-th* resources. For the networks with 100 nodes, the recource threshold D is set to the same as the threshold of the deterministic RCSP, while for the networks with 200 nodes, D is 1.5 times of the one of the deterministic RCSP. For the piecewise linear approximation, we choose three interpolation points and $z_1 = e^{-6}, z_2 = 0.15$ and $z_3 = 1$, while we choose two tangent points $z_1 = 0.15$ and $z_2 = 0.45$ for the piecewise tangent approximation. Here, we set the confidence parameter $\alpha = 0.1$. The results are shown in Table 1, where we list the upper bound and the lower bound of the relaxed stochastic RCSP together with their CPU time. We also give the gap between the upper bounds and the low bounds computed by

$$\text{Gap} = \frac{UB - LB}{LB}$$

where UB denotes the upper bound of the relaxed problem while the LB denotes the lower bound.

Instances	(Nodes, Arcs)	Lower bound	CPU time (s)	Upper bound	CPU time (s)	Gap(%)
RCSP1	(100,990)	100.20	18.72	104.20	15.74	3.84
RCSP2	(100,990)	97.45	17.35	105.44	16.98	7.58
RCSP3	(100,999)	6.16	17.54	6.61	17.91	6.81
RCSP4	(200,1960)	5.00	24.84	5.00	28.36	0.00
RCSP5	(200,1960)	5.40	39.79	5.65	35.95	4.42

Table 1: Computational results

From Table 1, on the one hand, we observe that for all instances, the CPU time is less than 70 seconds and there is not significant difference between the piecewise tangent approximation and the piecewise linear approximation from CPU time point of view. On the other hand, the gap between the upper bound and the lower bound doesn't exceed 8%, which shows that both two approximated optimum values are not far from the optimum value.

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