Lambda-calculus and programming language semantics

Thibaut Balabonski @ UPSay
Fall 2023
https://www.lri.fr/~blsk/LambdaCalculus/

Chapter 5: λ-computability

1 Basic data and operations

Functions

Identity function

$$\mathsf{I} \equiv \lambda x.x$$

Function composition

$$g \circ f \equiv \lambda x.g(f(x))$$

Example

$$\begin{array}{rcl} \mathsf{I} \circ \mathsf{I} & \equiv & \lambda x.\mathsf{I}(\mathsf{I}x) \\ & \equiv & \lambda x.(\lambda y.y)\ ((\lambda z.z)\ x) \\ & \longrightarrow_\beta & \lambda x.(\lambda z.z)\ x \\ & \longrightarrow_\beta & \lambda x.x \end{array}$$

Booleans and conditionals

Boolean values

$$\begin{array}{rcl} \mathsf{T} & \equiv & \lambda xy.x \\ \mathsf{F} & \equiv & \lambda xy.y \end{array}$$

Conditional expression

$$\mathsf{if}\ c\ \mathsf{then}\ a\ \mathsf{else}\ b \equiv c\ a\ b$$

Example

$$\begin{array}{rcl} \mathsf{if}\ \mathsf{T}\ \mathsf{then}\ a\ \mathsf{else}\ b & \equiv & \mathsf{T}\ a\ b \\ & \equiv & (\lambda xy.x)\ a\ b \\ & \longrightarrow_\beta & (\lambda y.a)\ b \\ & \longrightarrow_\beta & a \end{array}$$

Exercise: boolean operators

The following λ-term encodes a boolean operator. Which one?

$$\lambda ab.ab\mathsf{F}$$

Write terms for the other common operators.

Pairs and projections

Pair

$$\langle a, b\rangle \equiv \lambda s.s\ a\ b$$

Projections

$$\begin{array}{rcll} \pi_1 & \equiv & \lambda p.p\ (\lambda ab.a) & (\equiv \lambda p.p\ \mathsf{T}) \\ \pi_2 & \equiv & \lambda p.p\ (\lambda ab.b) & (\equiv \lambda p.p\ \mathsf{F}) \end{array}$$

Example

$$\begin{array}{rcl}
\pi_2\ \langle A, B\rangle & \equiv & (\lambda p.p\ (\lambda ab.b))\ \langle A, B\rangle \\
& \longrightarrow_\beta & \langle A, B\rangle\ \lambda ab.b \\
& \equiv & (\lambda s.s\ A\ B)\ \lambda ab.b) \\
& \longrightarrow_\beta & (\lambda ab.b)\ A\ B \\
& \longrightarrow_\beta & (\lambda b.b)\ B \\
& \longrightarrow_\beta & B
\end{array}$$

Algebraic data types and pattern matching

The principle used for representing booleans can be generalized for representing any finite set, by using more parameters (for instance: $\{\lambda abc.a,\ \lambda abc.b,\ \lambda abc.c\}$ for a set of three elements). The principle used for representing pairs can be generalized to arbitrary tuples, by using more arguments (for instance: $\lambda x.xabc$ for a triple (a, b, c)).

Combinations of these can be used to represent any algebraic data type: we have a finite set of constructors, each of which contains a (possibly empty) tuple of parameters.

For instance, here is a definition of binary trees in caml (with integers at the leaves)

```
type tree =
  | L of int
  | N of tree * tree
```

We can encode such a tree following these shapes:

$$\begin{array}{rcll}
\mathsf{L(k)} & \mapsto & \lambda ab.a\ [k] & \text{(k assumed non-negative)} \\
\mathsf{N}(t_1, t_2) & \mapsto & \lambda ab.b\ t_1\ t_2 &
\end{array}$$

Then pattern matching, as was the conditional, is just an application of the encoded term to the terms representing the various branches.

```
match t with
  | L(k)    -> f
  | N(x, y) -> g
```

will be encoded as

$$t\ (\lambda k.f)\ (\lambda xy.g)$$

(where the term f may contain occurrences of the variable k, and the term g may contain occurrences of the variables x and y)

Integers

For each $n \in \mathbb{N}$ we define a λ-term $[n]$

$$\begin{array}{rcl}
[0] & \equiv & \mathsf{I} \\
[n+1] & \equiv & \langle \mathsf{F}, [n]\rangle
\end{array}$$

Some basic operations

$$\begin{array}{rcll}
\mathsf{S} & \equiv & \lambda x.\langle \mathsf{F}, x\rangle & \text{successor} \\
\mathsf{P} & \equiv & \lambda x.x\mathsf{F} & \text{predecessor} \\
\mathsf{isZ} & \equiv & \lambda x.x\mathsf{T} & \text{zero?}
\end{array}$$

Exercise: integers

Summary of the definitions

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
[0] & \equiv & \mathsf{I} & \mathsf{S} & \equiv & \lambda x.\langle \mathsf{F}, x\rangle & \langle a, b\rangle & \equiv & \lambda c.cab \\
[n+1] & \equiv & \langle \mathsf{F}, [n]\rangle & \mathsf{P} & \equiv & \lambda x.x\mathsf{F} & \mathsf{T} & \equiv & \lambda ab.a \\
 & & & \mathsf{isZ} & \equiv & \lambda x.x\mathsf{T} & \mathsf{F} & \equiv & \lambda ab.b
\end{array}$$

Check the following equalities

$$\begin{aligned} \mathsf{S}\ [n] &=_\beta [n+1] \\ \mathsf{P}\ [n+1] &=_\beta [n] \\ \mathsf{P}\ [0] &=_\beta \mathsf{F} \\ \mathsf{isZ}\ [0] &=_\beta \mathsf{T} \\ \mathsf{isZ}\ [n+1] &=_\beta \mathsf{F} \end{aligned}$$

Define a term add such that

$$\mathsf{add}\ [n]\ [m] \quad = \quad [n+m]$$

Addition

We would like to write a recursive function

$$\mathsf{add}\ n\ m \quad = \quad \mathsf{if\ isZ}\ n\ \mathsf{then}\ m\ \mathsf{else\ add}\ (\mathsf{P}\ n)\ (\mathsf{S}\ m)$$

Problem: finding a λ-term add this way consists in solving an equation

2 Fixpoints

Fixpoints for numeric functions

A fixpoint of a function f is an x such that

$$f(x) = x$$

Finding such a fixpoint f means solving the equation $x = f(x)$

Numeric functions may have various numbers of fixpoints

$x \longmapsto x$	∞
$x \longmapsto x+1$	none
$x \longmapsto x^2$	two (0 and 1)
$f\ :\ [0;1] \longrightarrow [0;1]$	at least one if continuous

Fixpoints for λ-calculus

In the λ-calculus, t is a fixpoint of f if

$$f\ t\ =_\beta\ t$$

Fixpoint theorem

Any λ-term f has a fixpoint

The fixpoint theorem guarantees that, in the λ-calculus, the equation $t =_\beta f\ t$ has always a solution

Church's fixpoint combinator

A term that builds fixpoints

$$Y \quad \equiv \quad \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

First remark that

$$\begin{aligned} Y\,f &\equiv (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))\ f \\ &\longrightarrow_\beta (\lambda x.f(xx))(\lambda x.f(xx)) \end{aligned}$$

The term $(\lambda x.f(xx))(\lambda x.f(xx))$, written Fix_f below, is a fixpoint of f.

Indeed,

$$\begin{aligned} \mathsf{Fix}_f &\equiv (\lambda x.f(xx))(\lambda x.f(xx)) \\ &\longrightarrow_\beta f\ ((\lambda x.f(xx))(\lambda x.f(xx))) \\ &\equiv f\ \mathsf{Fix}_f \end{aligned}$$

For any λ-term f, the term $Y\,f$ builds a fixpoint of f.

Turing's fixpoint combinator

Another term that builds fixpoints, even more directly.

$$\begin{array}{rcl} \Theta & \equiv & A\,A \\ A & \equiv & \lambda xy.y(xxy) \end{array}$$

Checking that $f(\Theta f) =_\beta \Theta f$

$$\begin{array}{rcl} \Theta\, f & \equiv & (\lambda xy.y(xxy))\; A\, f \\ & \longrightarrow_\beta & (\lambda y.y(AAy))\, f \\ & \equiv & (\lambda y.y(\Theta y))\, f \\ & \longrightarrow_\beta & f(\Theta f) \end{array}$$

For any λ-term f, the term $\Theta\ f$ *is* a fixpoint of f

Mutual recursion

Double fixpoint theorem

$$\forall f, g \quad \exists a, b \qquad a =_\beta f\ a\ b \quad \wedge \quad b =_\beta g\ a\ b$$

Proof: define

$$\begin{array}{rcl} d & \equiv & \Theta\ (\lambda x.\langle f\ (\pi_1 x)\ (\pi_2 x), g\ (\pi_1 x)\ (\pi_2 x)\rangle) \\ a & \equiv & \pi_1\ d \\ b & \equiv & \pi_2\ d \end{array}$$

Then

$$\begin{array}{rcl} d & \longrightarrow^* & \langle f\ (\pi_1 d)\ (\pi_2 d), g\ (\pi_1 d)\ (\pi_2 d)\rangle \\ a \equiv \pi_1 d & \longrightarrow^* & f\ (\pi_1 d)\ (\pi_2 d) \ \equiv\ f\ a\ b \\ b \equiv \pi_2 d & \longrightarrow^* & g\ (\pi_1 d)\ (\pi_2 d) \ \equiv\ g\ a\ b \end{array}$$

This can be extended to a n-ary fixpoint, for any n.

Back on the addition

$$\begin{array}{rcl} \mathrm{add}\ n\ m & = & \mathrm{if\ \ isZ}\ n\ \ \mathrm{then}\ \ m\ \ \mathrm{else\ \ add}\ (\mathrm{P}\ n)\ (\mathrm{S}\ m) \\ \mathrm{add} & = & \lambda nm.\mathrm{if\ \ isZ}\ n\ \ \mathrm{then}\ \ m\ \ \mathrm{else\ \ add}\ (\mathrm{P}\ n)\ (\mathrm{S}\ m) \\ \mathrm{add} & = & (\lambda f nm.\mathrm{if\ \ isZ}\ n\ \ \mathrm{then}\ \ m\ \ \mathrm{else\ \ f}\ (\mathrm{P}\ n)\ (\mathrm{S}\ m))\ \ \mathrm{add} \end{array}$$

We define add as a fixpoint with

$$\mathrm{add} \ \equiv\ \Theta\ (\lambda f nm.\mathrm{if\ \ isZ}\ n\ \ \mathrm{then}\ \ m\ \ \mathrm{else\ \ f}\ (\mathrm{P}\ n)\ (\mathrm{S}\ m))$$

Exercise: Fibonacci sequence

Define a λ-term representing the Fibonacci function, defined by

$$\begin{array}{rcl} f(0) & = & 0 \\ f(1) & = & 1 \\ f(n+2) & = & f(n+1) + f(n) \end{array}$$

Exercise: paradoxical fixpoint?

We said that:

- $f\ :\ x \mapsto x + 1$ is function with zero fixpoint
- $F = \lambda x.\mathrm{S}\ x$ is a λ-term, and therefore it has a fixpoint

How can these two facts both be true?

Exercise: Church integers (iterators)

Alternative representation for $[n]$

$$[n] \equiv \lambda f x. f^n\ x$$

Idea: $[n]$ *takes as argument of function* f *and returns a function that iterates n times f*

Show that $\lambda n f x. f(n f x)$ represents the successor function

Find terms representing addition, multiplication, and predecessor

Exercise: Curry's Y-combinator

Another fixpoint combinator

$$\mathsf{Y} \quad \equiv \quad \lambda f.(\lambda x. f(xx))(\lambda x. f(xx))$$

Check that for any term t we have

$$\mathsf{Y}\ t\ =_\beta\ t\ (\mathsf{Y}\ t)$$

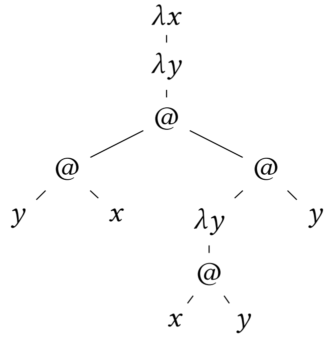
Do we also have $\quad \mathsf{Y}\ t\ \longrightarrow_\beta^*\ t\ (\mathsf{Y}\ t)\quad$?

3 Decidability

New version presented live, with λ-terms encoded by their AST.

de Bruijn notation: use numbers instead of variable names

$$\lambda x. \lambda y.(y\ x\ ((\lambda y. xy)\ y))$$



Replace each variable occurrence with the number of λ between the occurrence and its binder

$$\lambda.\lambda.0\ 1\ ((\lambda.20)\ 0)$$

What we gain: the need for variable renamings disappears. Also, the syntax of terms will be easier to represent as a λ-encoded data structure

Translations between named and nameless variables

For any named closed term t, write $\llbracket t \rrbracket$ its nameless version. Generalization to term with free variables: let ℓ be a list of variable names that contains all the free variables of t, define $\llbracket t \rrbracket_\ell$ the translation where each free variable x of t is associated to the index at which x appears in t.

$$\begin{array}{rcl} \llbracket x \rrbracket_\ell & = & \mathsf{index\_of}(x, \ell) \\ \llbracket t\ u \rrbracket_\ell & = & \llbracket t \rrbracket_\ell\ \llbracket u \rrbracket_\ell \\ \llbracket \lambda x. t \rrbracket_\ell & = & \lambda. \llbracket t \rrbracket_{x:\ell} \end{array}$$

(assume $\mathsf{index\_of}$ is a function that returns the index at which the name x appears in the list ℓ).

Reverse: for any nameless closed term t, write $\llparenthesis t \rrparenthesis$ its named version. Generalization to term with free variables: let ℓ be a list of variable names that is long enough to account for every indices in t, define $\llparenthesis t \rrparenthesis_\ell$ the translation where each free index of t is associated to the element at corresponding index of ℓ.

$$\begin{array}{rcll} \llparenthesis k \rrparenthesis_\ell & = & \mathsf{nth}(k, \ell) & \\ \llparenthesis t\ u \rrparenthesis_\ell & = & \llparenthesis t \rrparenthesis_\ell\ \llparenthesis u \rrparenthesis_\ell & \\ \llparenthesis \lambda. t \rrparenthesis_\ell & = & \lambda x. \llparenthesis t \rrparenthesis_{x:\ell} & \text{for } x \text{ a fresh variable name} \end{array}$$

(assume nth is a function that returns the element at index k in the list ℓ).

Encoding the abstract syntax of nameless λ-terms.

Nameless terms can be represented with the following three constructors.

```
type term =
  | Var of int
  | App of term * term
  | Abs of term
```

Representation of such a data structure using λ-terms:

$$\begin{array}{rcl} [k] & = & \lambda abc.a\,[k] \\ [t\ u] & = & \lambda abc.b\,[t]\,[u] \\ [\lambda.t] & = & \lambda abc.c\,[t] \end{array}$$

(note: $[k]$ on the left of the first equation is the encoding of a λ-term made of the de Bruijn index k, defined by the equation, whereas $[k]$ on the right of the same equation is the encoding of the naturel number k, as proposed at the beginning of the chapter)

Encoding the abstract syntax of named λ-terms.

One obtains an encoding of usual, named λ-terms by composing the translation to nameless representation with the previous translation. Here is a set of combined equations:

$$\begin{array}{rcl} [x]_\ell & = & \lambda abc.a\,[\mathsf{index\_of}(x,\ell)] \\ [t\ u]_\ell & = & \lambda abc.b\,[t]_\ell\,[u]_\ell \\ [\lambda x.t]_\ell & = & \lambda abc.c\,[t]_{x:\ell} \end{array}$$

(again, $[\mathsf{index\_of}(x,\ell)]$ is the encoding of a natural number as defined at the beginning of the chapter)

Self-interpreter

Using the previous term representation, one can define an interpreter of the λ-calculus, in the λ-calculus. Such a function can be called a *self-interpreter*, and also corresponds to the concept of *universal machine* that you will hear of again in the computability course. This interpreter is a term e such that for any term t and any list ℓ we have

$$e\,[t]_\ell\ \ell =_\beta t$$

(this assumes that the list ℓ can also encoded as a λ-term, which is left as an exercise)

For such an interpreter, we want the following equations:

$$\begin{array}{rclcl} e\,[x]_\ell\ \ell & = & e\,(\lambda abc.a\,[k])\ \ell & = & \mathsf{nth}(k,\ell) \\ e\,[t\ u]_\ell\ \ell & = & e\,(\lambda abc.bb\,[t]_\ell\,[u]_\ell)\ \ell & = & (e\,[t]_\ell\ \ell)\,(e\,[t]_\ell\ \ell) \\ e\,[\lambda x.t]_\ell\ \ell & = & e\,(\lambda abc.c\,[t]_{x:\ell})\ \ell & = & \lambda x.(e\,[t]_{:\ell}\ x : \ell) \end{array}$$

Thus we propose the following term:

$$\begin{array}{rcl} e & = & Y\,(\lambda e.\lambda t.\lambda \ell.\ t\,(\lambda k.\mathsf{nth}(k,\ell)) \\ & & \qquad\qquad\qquad (\lambda tu.(e\ t\ \ell)\,(e\ u\ \ell)) \\ & & \qquad\qquad\qquad (\lambda t.\lambda x.e\ t\,(x : \ell))) \end{array}$$

Correctness of the self-interpreter

Assuming that lists of names ℓ can be encoded as λ-terms as well as the two functions $\mathsf{index\_of}$ and nth, we prove that for any term t and any list ℓ containing (at least) the free variables of t :

$$e\,[t]_\ell\ \ell =_\beta t$$

Write $e = Y\ e'$. We have in one step

$$e \;=\; Y\ e' \;\longrightarrow\; (\lambda x.e'(xx))(\lambda x.e'(xx)) \;=\; e''$$

where the obtained term e'' is the fixpoint of e' produced by Y.

Since all encodings share a common structure, first remark that

$$\begin{array}{rcl} e\,[t]_\ell\;\ell & = & Y\;e'\;[t]_\ell\;\ell \\ & \longrightarrow & (\lambda x.e'(xx))(\lambda x.e'(xx))\;[t]_\ell\;\ell \\ & = & e''\;[t]_\ell\;\ell \\ & \longrightarrow & e'\;e''\;[t]_\ell\;\ell \\ & \longrightarrow^3 & [t]_\ell\;e_1\;e_2\;e_3 \end{array}$$

where

$$\begin{array}{rcl} e_1 & = & \lambda k.\mathsf{nth}(k,\ell) \\ e_2 & = & \lambda tu.(e''\;t\;\ell)\;(e''\;u\;\ell) \\ e_3 & = & \lambda t.\lambda x.e''\;t\;(x\;:\;\ell) \end{array}$$

Now prove the result by induction on t:

- Case of a variable x (assumed in ℓ):

  $$\begin{array}{rcl} e\,[x]_\ell\;\ell & \longrightarrow & [x]_\ell\;e_1\;e_2\;e_3 \\ & = & (\lambda abc.a\;[\mathsf{index\_of}(x,\ell)])\;e_1\;e_2\;e_3 \\ & \longrightarrow^3 & e_1\;[\mathsf{index\_of}(x,\ell)] \\ & = & (\lambda k.\mathsf{nth}(k,\ell))\;[\mathsf{index\_of}(x,\ell)] \\ & = & \mathsf{nth}([\mathsf{index\_of}(x,\ell)],\ell) \end{array}$$

  The specifications of nth and $\mathsf{index\_of}$ indeed require that $\mathsf{nth}([\mathsf{index\_of}(x,\ell)],\ell)$ is equal to x (when x is in ℓ).

- Case of an application $t\;u$:

  $$\begin{array}{rcll} e\,[t\;u]_\ell\;\ell & \longrightarrow & [t\;u]_\ell\;e_1\;e_2\;e_3 & \\ & = & (\lambda abc.b\;[t]_\ell\;[u]_\ell)\;e_1\;e_2\;e_3 & \\ & \longrightarrow^3 & e_2\;[t]_\ell\;[u]_\ell & \\ & = & (\lambda tu.(e''\;t\;\ell)\;(e''\;u\;\ell))\;[t]_\ell\;[u]_\ell & \\ & \longrightarrow^2 & (e''\;[t]_\ell\;\ell)\;(e''\;[u]_\ell\;\ell) & \\ & =_\beta & t\;u & \text{by induction hypotheses} \end{array}$$

- Case of an abstraction $\lambda x.t$:

  $$\begin{array}{rcll} e\,[\lambda x.t]_\ell\;\ell & \longrightarrow & [\lambda x.t]_\ell\;e_1\;e_2\;e_3 & \\ & = & (\lambda abc.c\;[t]_{x:\ell})\;e_1\;e_2\;e_3 & \\ & \longrightarrow^3 & e_3\;[t]_{x:\ell} & \\ & = & (\lambda t.\lambda x.e''\;t\;(x\;:\;\ell))\;[t]_{x:\ell} & \\ & \longrightarrow & \lambda x.e''\;[t]_{x:\ell}\;(x\;:\;\ell) & \text{(note: } x \notin \mathsf{fv}([t]_{x:\ell})\text{)} \\ & =_\beta & \lambda x.t & \text{by induction hypothesis} \end{array}$$

Second fixpoint theorem

$$\forall f\;\exists t\quad f\;[t] =_\beta t$$

Proof of the second fixpoint theorem

First remark that one could write two terms A and N such that

$$\begin{array}{rcl} \mathsf{A}\;[t]\;[u] & =_\beta & [t\;u] \\ \mathsf{N}\;[t] & =_\beta & [[t]] \end{array}$$

(A is simply $\lambda tu.\lambda abc.b\;t\;u$, whereas N is defined as the fixpoint of a function defined by pattern matching on the representation $[t]$ of t)

Then define

$$\begin{array}{rcl} w & \equiv & \lambda x.f\ (\mathsf{A}\ x\ (\mathsf{N}\ x)) \\ z & \equiv & w\ [w] \end{array}$$

Then z is a fixpoint for f.

$$\begin{array}{rlll} z \equiv w\ [w] & =_\beta & f\ (\mathsf{A}\ [w]\ (\mathsf{N}\ [w])) & \\ & =_\beta & f\ (\mathsf{A}\ [w]\ [[w]]) & \\ & =_\beta & f\ [w\ [w]] & \equiv\ f\ [z] \end{array}$$

Scott's undecidability theorem

Theorem

1. any two non-empty sets $A, B \subseteq \Lambda$ closed by β-equality are not effectively separable
2. no non-trivial set $A \subseteq \Lambda$ closed by β-equality can be effectively characterized

Definitions

- E is closed by β-equality if $\forall x, y \in \Lambda\ x \in E \wedge x =_\beta y \implies y \in E$
- E is non-trivial if there are $x \in E$ and $y \notin E$
- A and B are effectively separable if there is an effectively characterized set C such that $t \in A \implies t \in C$ and $t \in B \implies t \notin C$
- C is effectively characterized if there is a λ-term f such that $f\ t =_\beta \mathsf{T}$ for any $t \in C$ and $f\ t =_\beta \mathsf{F}$ for any $t \notin C$

(note: in the definition of "effectively characterized" it is of critical importance that the application of the λ-term f to *any* λ-term t is normalizable)

Proof of Scott's theorem

Any two non-empty sets $A, B \subseteq \Lambda$ closed by β-equality are not effectively separable

Assume there is a separating set C such that $A \subseteq C$ and $B \cap C = \emptyset$, characterized by a λ-term f such that

$$\begin{array}{rcl} t \in C & \implies & f\ [t] =_\beta \mathsf{T} \\ t \notin C & \implies & f\ [t] =_\beta \mathsf{F} \end{array}$$

Since A and B are not empty, we can find two terms $a \in A$ and $b \in B$. Define

$$g \quad \equiv \quad \lambda x.\mathsf{if}\, f\ x\ \mathsf{then}\ a\ \mathsf{else}\ b$$

Then

$$\begin{array}{rcl} t \in C & \implies & g\ [t] =_\beta b \\ t \notin C & \implies & g\ [t] =_\beta a \end{array}$$

From the second fixpoint theorem, there is z such that $g\ [z] = z$

$$\begin{array}{rcccl} z \in C & \implies & z =_\beta g\ [z] =_\beta b \in B & \implies & z \notin C \\ z \notin C & \implies & z =_\beta g\ [z] =_\beta a \in A & \implies & z \in C \end{array}$$

Contradiction!

Undecidability of β-equality

No algorithm can decide whether two arbitrary λ-terms are β-equal

Assume f is a λ-term such that, for any a and b, $f\ [a]\ [b]$ equals to $[1]$ if $a =_\beta b$ and to $[0]$ otherwise

Define $A = \{x \mid x =_\beta a\}$

- by definition, A is closed by β-equality
- A is not empty, since it contains a
- $\Lambda \setminus A$ is not empty, because:
  - if a has a normal form, then $\Omega \notin A$
  - if a has no normal form, then $\lambda x.x \notin A$

By Scott's theorem, the set A is not recursive

On the other hand, $f\ [a]$ computes the characteristic function of A *Contradiction.*

Exercise: halting problem for the λ-calculus

No algorithm can decide whether an arbitrary λ-term has a normal form

Undecidability of the optimal strategy

Strategy: function $F : \Lambda \longrightarrow \Lambda$ such that

$$\forall t \in \Lambda \quad t \longrightarrow_\beta F(t)$$

Optimal strategy: strategy that always picks a shortest path to the normal form (if there is a normal form)

There is no computable optimal strategy

Undecidability of the optimal strategy: idea

Consider the set

$$t_n \equiv (\lambda x.x\mathsf{E}x)\ (\lambda y.y[n](\mathsf{I}\mathsf{I}))$$

of λ-terms, where E enumerates λ-terms with at most one free variable a

Assuming E is already in normal form, for each n we have to choose between:

- reducing $t_n \longrightarrow_\beta (\lambda y.y[n](\mathsf{I}\mathsf{I}))\ \mathsf{E}\ (\lambda y.y[n](\mathsf{I}\mathsf{I}))$
- reducing $t_n \longrightarrow_\beta (\lambda x.x\mathsf{E}x)\ (\lambda y.y[n]\mathsf{I})$

However, the best choice differs depending on the normal form of $\mathsf{E}\ [n]$

Optimal strategy: first case

If $\mathsf{E}\ [n] \longrightarrow^*_\beta \lambda xyz.z$ in k steps then

$$\begin{aligned} (\lambda y.y[n](\mathsf{I}\mathsf{I}))\ \mathsf{E}\ (\lambda y.y[n](\mathsf{I}\mathsf{I})) &\longrightarrow_\beta \mathsf{E}\ [n]\ (\mathsf{I}\mathsf{I})\ (\lambda y.y[n](\mathsf{I}\mathsf{I})) \\ &\longrightarrow^*_\beta (\lambda xyz.z)\ (\mathsf{I}\mathsf{I})\ (\lambda y.y[n](\mathsf{I}\mathsf{I})) \\ &\longrightarrow^2_\beta \lambda z.z \end{aligned}$$

optimally in $k+3$ steps and

$$\begin{aligned} (\lambda x.x\mathsf{E}x)\ (\lambda y.y[n]\mathsf{I}) &\longrightarrow_\beta (\lambda y.y[n]\mathsf{I})\ \mathsf{E}\ (\lambda y.y[n]\mathsf{I}) \\ &\longrightarrow_\beta \mathsf{E}\ [n]\ \mathsf{I}\ (\lambda y.y[n]\mathsf{I}) \\ &\longrightarrow^*_\beta (\lambda xyz.z)\ \mathsf{I}\ (\lambda y.y[n]\mathsf{I}) \\ &\longrightarrow^2_\beta \lambda z.z \end{aligned}$$

optimally in $k+4$ steps

Optimal strategy: second case

If $\mathsf{E}\ [n] \to_\beta^* a$ in k steps then

$$\begin{array}{rcl}(\lambda y.y[n](\mathsf{II}))\ \mathsf{E}\ (\lambda y.y[n](\mathsf{II})) & \to_\beta & \mathsf{E}\ [n]\ (\mathsf{II})\ (\lambda y.y[n](\mathsf{II})) \\ & \to_\beta^* & a\ (\mathsf{II})\ (\lambda y.y[n](\mathsf{II})) \\ & \to_\beta^2 & a\ \mathsf{I}\ (\lambda y.y[n]\mathsf{I})\end{array}$$

optimally in $k + 3$ steps and

$$\begin{array}{rcl}(\lambda x.x\mathsf{E}x)\ (\lambda y.y[n]\mathsf{I}) & \to_\beta & (\lambda y.y[n]\mathsf{I})\ \mathsf{E}\ (\lambda y.y[n]\mathsf{I}) \\ & \to_\beta & \mathsf{E}\ [n]\ \mathsf{I}\ (\lambda y.y[n]\mathsf{I}) \\ & \to_\beta^* & a\ \mathsf{I}\ (\lambda y.y[n]\mathsf{I})\end{array}$$

optimally in $k + 2$ steps

Optimal strategy: conclusion

$$t_n \equiv (\lambda x.x\mathsf{E}x)\ (\lambda y.y[n](\mathsf{II}))$$

If F is an optimal strategy, then

- if $\mathsf{E}\ [n] \to_\beta^* \lambda xyz.z$ then $F(t_n) = (\lambda y.y[n](\mathsf{II}))\ \mathsf{E}\ (\lambda y.y[n](\mathsf{II}))$, and
- if $\mathsf{E}\ [n] \to_\beta^* a$ then $F(t_n) = (\lambda x.x\mathsf{E}x)\ (\lambda y.y[n]\mathsf{I})$

An optimal strategy thus separates

$$\{n \mid \mathsf{E}\ [n] \to_\beta^* \lambda xyz.z\} \qquad \text{and} \qquad \{n \mid \mathsf{E}\ [n] \to_\beta^* a\}$$

However, these two sets are not recursively separable, since by Scott's theorem

$$\{t \mid t \to_\beta^* \lambda xyz.z\} \qquad \text{and} \qquad \{t \mid t \to_\beta^* a\}$$

are not recursively separable.

4 The λ-calculus is a model of computable functions

Bonus section, encoding general recursive function into λ-calculus.

Definability

A mathematical function $\varphi : \mathbb{N}^p \to \mathbb{N}$ is *λ-definable* if there is a λ-term $f \in \Lambda$ such that

$$\forall n_1, \dots, n_p \in \mathbb{N}, \quad f[n_1] \dots [n_p] =_\beta [\varphi(n_1, \dots, n_p)]$$

By Church-Rosser property, we could also have given the condition

$$\forall n_1, \dots, n_p \in \mathbb{N}, \quad f[n_1] \dots [n_p] \to_\beta^* [\varphi(n_1, \dots, n_p)]$$

Property: the λ-definable functions are exactly the recursive functions

Initial recursiuve functions

Zero $Z(n) = 0$

- $\mathsf{Z} = \lambda x.[0]$

Successor $S(n) = n + 1$

- $\mathsf{S} = \lambda x.\langle \mathsf{F}, x\rangle$

Projection $U_i^p(n_0, \dots, n_p) = n_i$ with $0 \leqslant i \leqslant p$

- $\mathsf{U}_i^p = \lambda x_0 \dots x_p.x_i$

Composition of recursive functions

If F, G_1, ..., G_m are recursive then the function H defined by

$$H(\vec{n}) = F(G_1(\vec{n}), \ldots, G_m(\vec{n}))$$

is recursive

Assume F, G_1, ..., G_m are defined by f, g_1, ..., g_m then H can be defined by

$$h \equiv \lambda \vec{x}.F\ (G_1\ \vec{x}) \ldots (G_m\ \vec{x})$$

Primitive recursion

If F and G are recursive then the function H defined by

$$\begin{array}{rcl} H(0, \vec{n}) & = & F(\vec{n}) \\ H(k+1, \vec{n}) & = & G(H(k, \vec{n}), k, \vec{n}) \end{array}$$

is recursive

Assume F and G are defined by f and g, we are looking for an h such that

$$h \quad \equiv \quad \lambda x \vec{y}.\mathsf{if\ isZ}\ x\ \mathsf{then}\ f\ \vec{y}\ \mathsf{else}\ g\ (h\ (\mathsf{P}x)\ \vec{y})\ (\mathsf{P}x)\ \vec{y}$$

Fixpoint theorem: such a term h exists

Minimisation

If F is recursive and is such that

$$\forall \vec{n}\ \exists m\ F(\vec{n}, m) = 0$$

then the function M defined by

$$M(\vec{n}) \quad = \quad \text{the smallest } m \in \mathbb{N} \text{ such that } F(\vec{n}, m) = 0$$

is recursive

Assume F is defined by f, then define

$$m \quad \equiv \quad \lambda \vec{x}.(\Theta\ (\lambda h y.\mathsf{if\ isZ}\ (f \vec{x} y)\ \mathsf{then}\ y\ \mathsf{else}\ h(\mathsf{S}y))\ [0])$$

Summary

We encoded in the λ-calculus:

- the initial functions Z, S and U_i^p
- function composition
- primitive recursion
- minimisation

Therefore, any recursive function is λ-definable
The λ-calculus is Turing-complete

5 Decidability, traditional presentation

The historical path, encoding λ-terms as numbers.

Encoding λ-terms using numbers

Assume a (computable and) injective function $\varphi : \mathbb{N}^2 \longrightarrow \mathbb{N}$, for instance $\varphi(x, y) \equiv 2^x(2y+1) - 1$

Assign numbers to all variables: $\{x_0, x_1, x_2, \ldots\}$

We deduce a function $\sharp : \Lambda \longrightarrow \mathbb{N}$ assigning a unique number to each λ-term

$$\begin{array}{rcl} \sharp x_i & = & \varphi(0,\ i) \\ \sharp(t\ u) & = & \varphi(1,\ \varphi(\sharp t,\ \sharp u)) \\ \sharp(\lambda x_i.t) & = & \varphi(2,\ \varphi(i,\ \sharp t)) \end{array}$$

Encoding of a λ-term t: the λ-term t' representing the number n representing the encoded λ-term t

$$[t] \equiv [\sharp t]$$

Remark: this is a new encoding, thus all encoding-dependent theorems have to be proved again.

Enumeration theorem (admitted)

There is a λ-term E such that for any closed λ-term t, $\mathsf{E}\ [t] \longrightarrow_\beta^* t$

This is the equivalent of the self-interpreter in the previous presentation. The proof however is far more technical.

Proof of the second fixpoint theorem

The functions φ_A and φ_N defined by

$$\begin{array}{rcl} \varphi_A(\sharp t, \sharp u) & = & \sharp(t\ u) \\ \varphi_N(\sharp t) & = & \sharp[t] \end{array}$$

are recursive. They are thus defined by λ-terms A and N such that

$$\begin{array}{rcl} \mathsf{A}\ [t]\ [u] & =_\beta & [t\ u] \\ \mathsf{N}\ [t] & =_\beta & [[t]] \end{array}$$

Define

$$\begin{array}{rcl} w & \equiv & \lambda x.f\ (\mathsf{A}\ x\ (\mathsf{N}\ x)) \\ z & \equiv & w\ [w] \end{array}$$

Then z is a fixpoint for f.

$$\begin{array}{rcll} z \equiv w\ [w] & =_\beta & f\ (\mathsf{A}\ [w]\ (\mathsf{N}\ [w])) & \\ & =_\beta & f\ (\mathsf{A}\ [w]\ [[w]]) & \\ & =_\beta & f\ [w\ [w]] & \equiv\ f\ [z] \end{array}$$

Scott's undecidability theorem (stated using general vocabulary of recursive functions)

Theorem

1. any two non-empty sets $A, B \subseteq \Lambda$ closed by β-equality are not recursively separable
2. any non-trivial set $A \subseteq \Lambda$ closed by β-equality is not recursive

Definitions

- E is closed by β-equality if $\forall x, y \in \Lambda\ x \in E \wedge x =_\beta y \implies y \in E$
- E is non-trivial if there are $x \in E$ and $y \notin E$
- A and B are recursively separable if there is a recursive set C such that $A \subseteq C$ and $B \cap C = \emptyset$
- C is recursive if its characteristic function is recursive

Proof of Scott's theorem

Any two non-empty sets $A, B \subseteq \Lambda$ closed by β-equality are not recursively separable

Assume there is a recursive set C such that $A \subseteq C$ and $B \cap C = \emptyset$ Its characteristic function is realized by a λ-term f such that

$$\begin{array}{lcl} t \in C & \Longrightarrow & f\ [t] =_\beta [1] \\ t \notin C & \Longrightarrow & f\ [t] =_\beta [0] \end{array}$$

Since A and B are not empty, we can find two terms $a \in A$ and $b \in B$. Define

$$g \quad \equiv \quad \lambda x.\mathsf{if}\ \mathsf{isZ}\ (f\ x)\ \mathsf{then}\ a\ \mathsf{else}\ b$$

Then

$$\begin{array}{lcl} t \in C & \Longrightarrow & g\ [t] =_\beta b \\ t \notin C & \Longrightarrow & g\ [t] =_\beta a \end{array}$$

From the second fixpoint theorem, there is z such that $g\ [z] = z$

$$\begin{array}{lcccl} z \in C & \Longrightarrow & z =_\beta g\ [z] =_\beta b \in B & \Longrightarrow & z \notin C \\ z \notin C & \Longrightarrow & z =_\beta g\ [z] =_\beta a \in A & \Longrightarrow & z \in C \end{array}$$

Contradiction!

Undecidability results...

are proved exactly as in the previous section, now that Scott's theorem is established for this other representation of λ-terms.

Homework

1. Prove that there exists no λ-term h such that $h\ [t] = \mathsf{T}$ for any $t \in \Lambda$ with a normal form and $h\ [t] = \mathsf{F}$ for any $t \in \Lambda$ with no normal form.
2. Using the encoding of algebraic datatypes, and one of the already defined encodings of numbers, propose an encoding of lists, and of the nth function.
3. In your encoding, prove that $\mathsf{nth}\ k\ \ell = \mathsf{nth}\ (k+1)\ (t : \ell)$.