Lattice quotients of weak order intervals in subword complexes

Noémie Cartier

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Joint work with :
Nantel Bergeron
Cesar Ceballos
Vincent Pilaud

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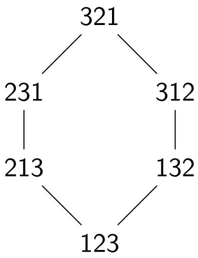

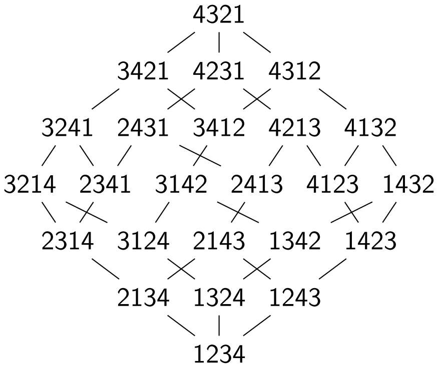

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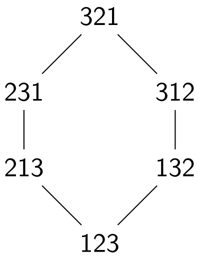

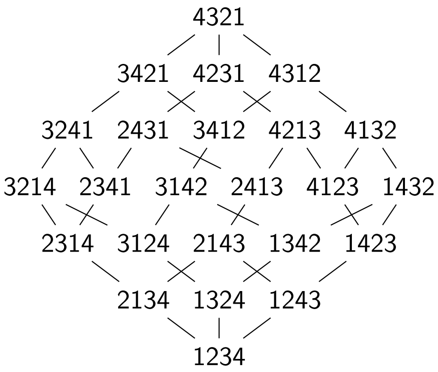

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Theorem

*The weak order on $\mathfrak{S}_n$ is a **lattice**.*

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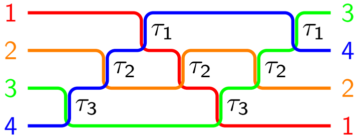
Sorting network $\leftrightarrow$ simple reflections product



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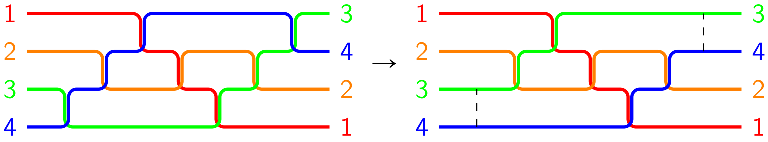
Reduction to minimal length :



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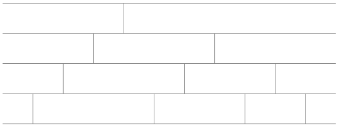

Facet $\{1, 2, 3, 8, 9\}$ of SC$(\tau_4\tau_3\tau_2\tau_1\tau_4\tau_3\tau_2\tau_4\tau_3\tau_4, 25143)$

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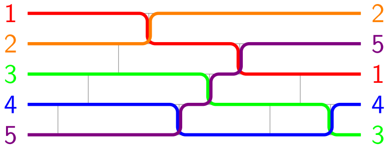

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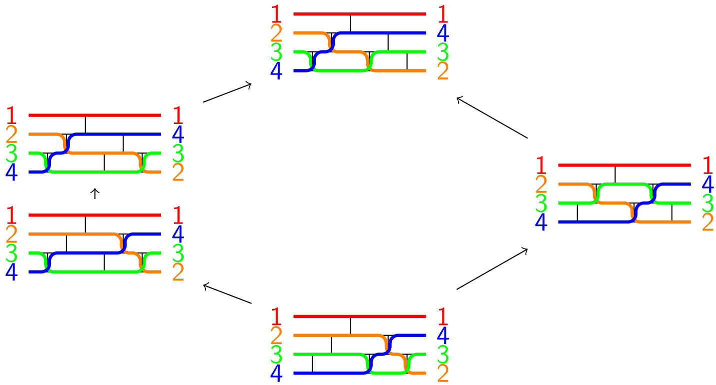
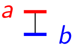
Structure given by **flips** : from one facet to another



Contact graph :

- vertices : pipes
- edges : from a to b if 

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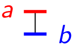
- edges : from *a* to *b* if 

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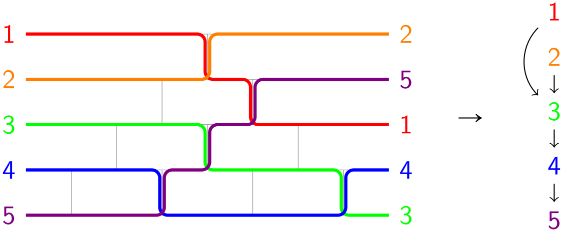

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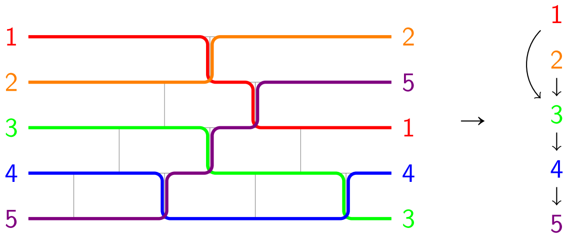

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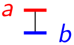
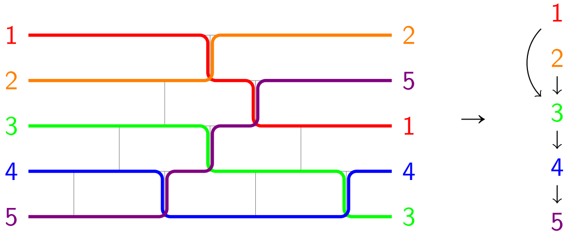
- edges : from *a* to *b* if  appears in the picture



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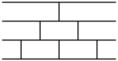
Acyclic contact graph $\iff$ vertex of the **brick polytope**

A very special case

Q : triangular word  and $\omega = 1\; n\; (n-1)\; \dots\; 2$

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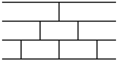
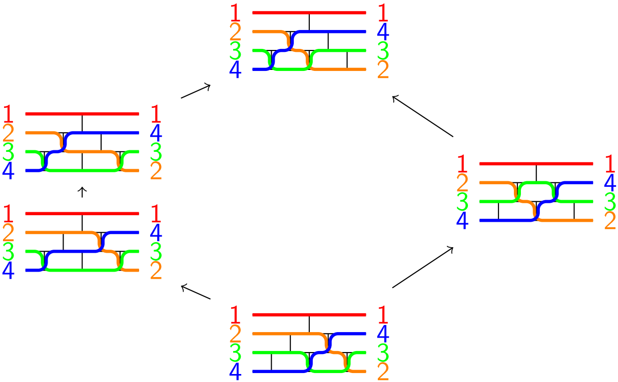
Q : triangular word  and $\omega = 1\ n\ (n-1)\ \ldots\ 2$



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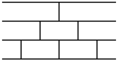
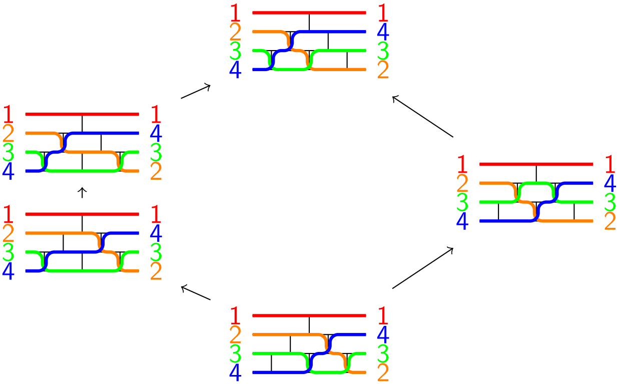
Q : triangular word  and $\omega = 1\, n\, (n-1)\, \ldots\, 2$



$\Rightarrow$ this is the Tamari lattice !

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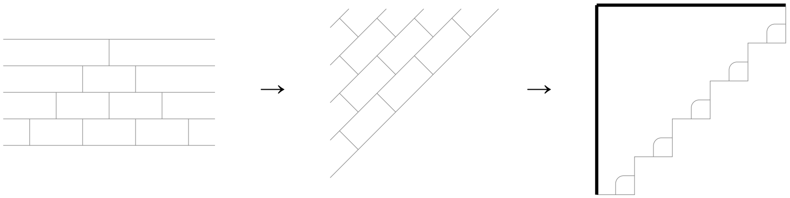

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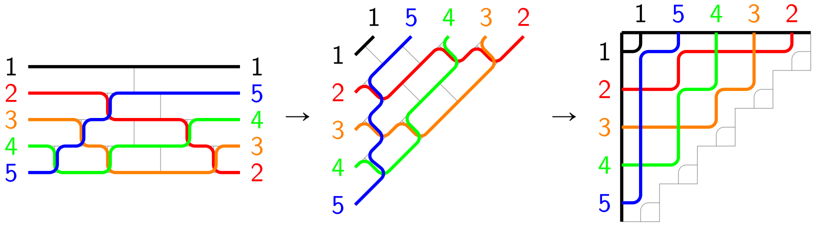

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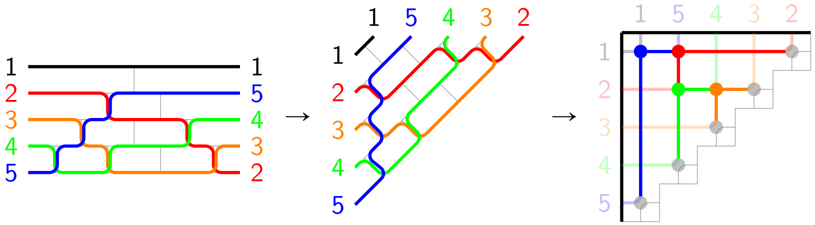

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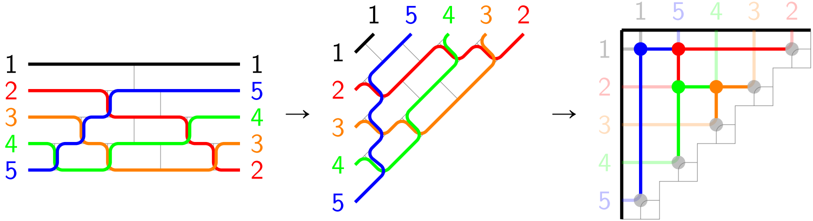

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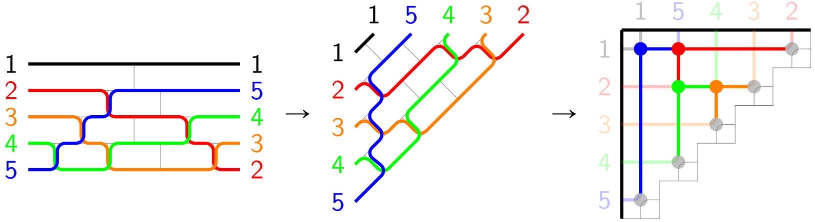

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Tree rotations ≡ flips → lattice isomorphism (Woo, 2004)

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Can we find other lattice quotients of parts of the weak order with pipe dreams ?

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Theorem (Pilaud)

*For any $\omega \in \mathfrak{S}_n$, the set $\Pi(\omega)$ of **acyclic pipe dreams** of exit permutation ω, ordered by ascending flips, is a **lattice quotient** of the **weak order interval** $[\mathrm{id}, \omega]$.*

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Two algorithms to compute the morphism :

- insertion algorithm (pipe by pipe)
- sweeping algorithm (cell by cell)

→ name of the morphism : Ins_ω

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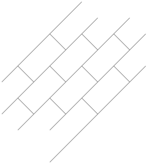

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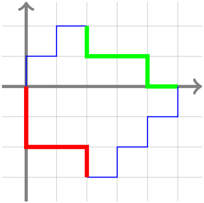

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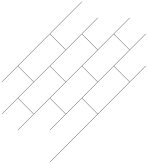

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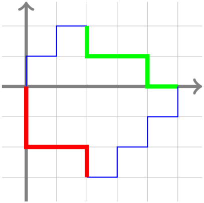

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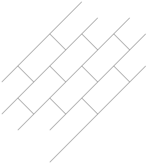

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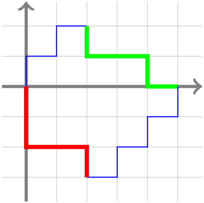

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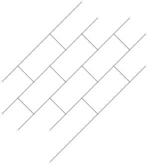
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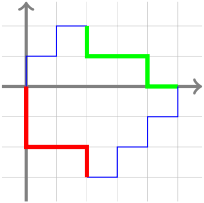

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Restrictions :

- only consider **strongly acyclic** pipe dreams
- order on pipe dreams : **acyclic order** (weaker than flip order)

Theorem

For any n-shape F and $\omega \in \mathfrak{S}_n$ sortable on F, the map $\mathrm{Ins}_{F,\omega}$ is a ***lattice morphism*** *from the* ***weak order interval*** $[\mathrm{id}, \omega]$ *to the* ***strongly acyclic*** *pipe dreams ordered by the acyclic order.*

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Theorem

*If the maximal permutation $\omega_0 = n\,(n-1)\,\dots 2\,1$ is sortable on F, then any linear extension of a pipe dream on F with exit permutation ω is in $[\mathrm{id}, \omega]$, and **all acyclic pipe dreams are strongly acyclic**.*

Further generalization : Coxeter groups

symmetric group $\mathfrak{S}_n$	Coxeter group W
transpositions $(i, i+1)$	simple reflections
reduced pipe dreams	subword complex
pair of pipes	root in Φ
$P^{\#}$ acyclic	root cone is pointed
$\pi \in \mathrm{lin}(P)$	root configuration $\subseteq \pi(\Phi^+)$

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Theorem (Jahn & Stump 2022)

If the Demazure product of Q is w_0, then for any $w \in W$ the application $\mathrm{Ins}_Q(w, \cdot)$ is ***surjective on acyclic facets*** *of $\mathrm{SC}(Q, w)$.*

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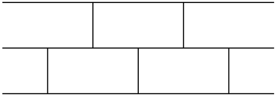
Conjecture

If Q is an alternating word on S and $w \in W$ is sortable on Q, then the application $\mathrm{Ins}_{Q,w} : [e, w] \mapsto \mathrm{SC}(Q, w)$ is a **lattice morphism** from the left weak order on $[e, w]$ to its image.

Thank you for your attention !

Q a word on S seen as a sorting network, here $\omega = \omega_0 = n(n-1)\dots1$



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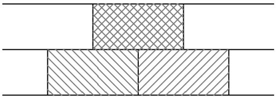

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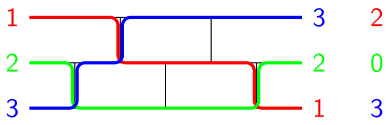

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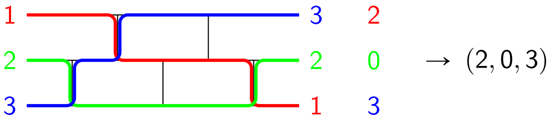

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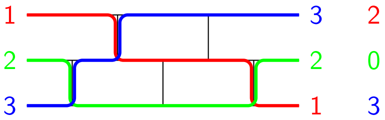
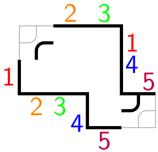


$\rightarrow (2,0,3)$

- **brick polytope** of $\mathrm{SC}(Q,\omega)$: convex hull of brick vectors of facets

Sweeping algorithm for $\omega = 23145$ and $\pi = 21345$



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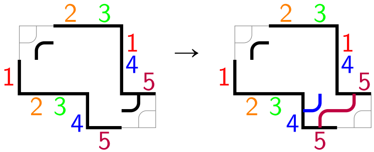

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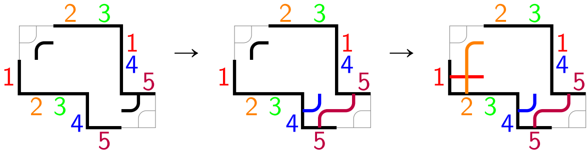

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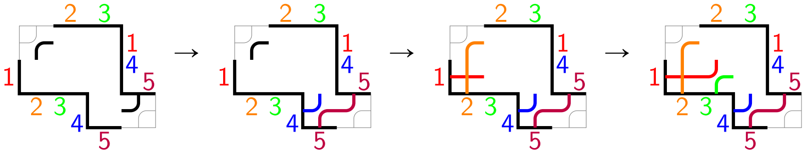

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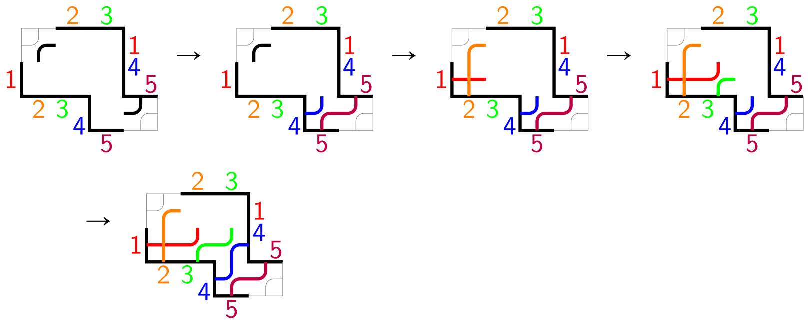

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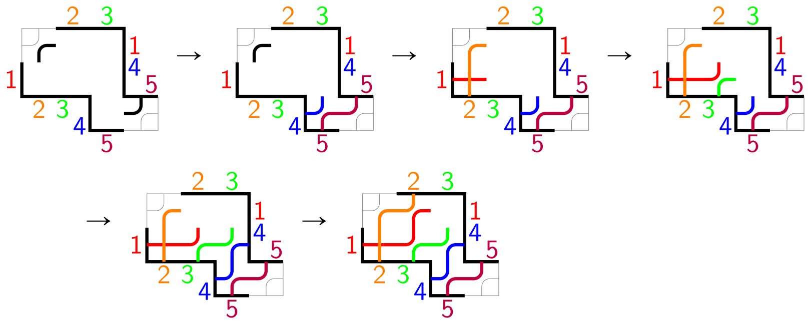

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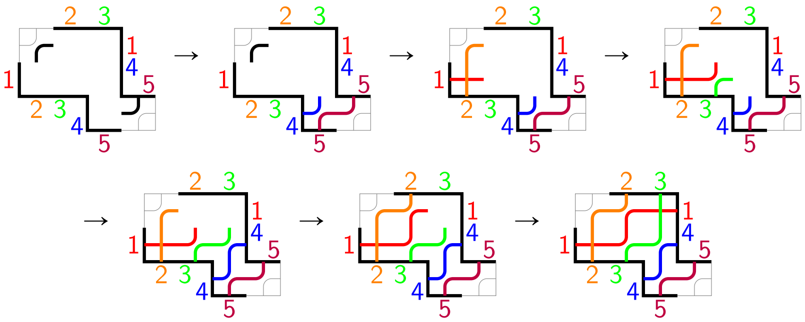

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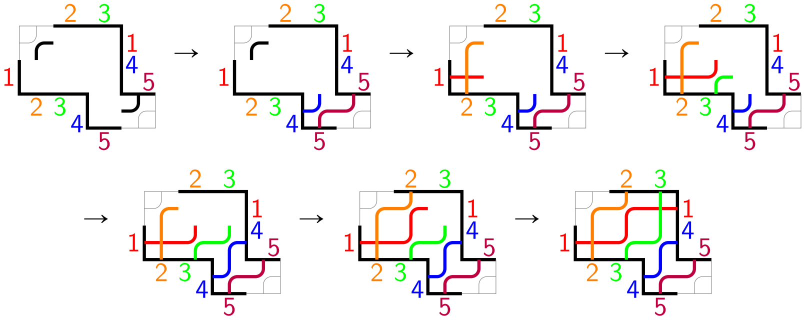

1. if $\omega^{-1}(i) < \omega^{-1}(j)$, add an elbow
2. if $\omega^{-1}(i) > \omega^{-1}(j)$ and $\pi^{-1}(i) > \pi^{-1}(j)$, add a cross $+$
3. if i, j inversion of ω and non-inversion of π, add an elbow if you can still make the pipes end in order ω that way (3a), and a cross $+$ otherwise (3b)

Insertion algorithm for $\omega = 3241$ and $\pi = 2134$

The idea : keep track of cells containing only half of an elbow, and complete as many of those cells as possible when inserting a new pipe.

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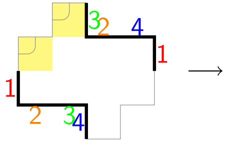

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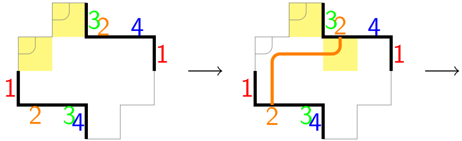

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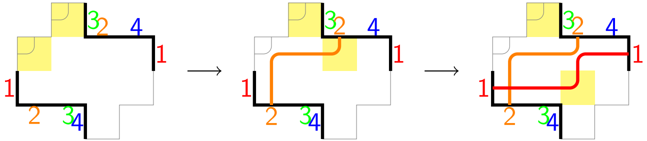

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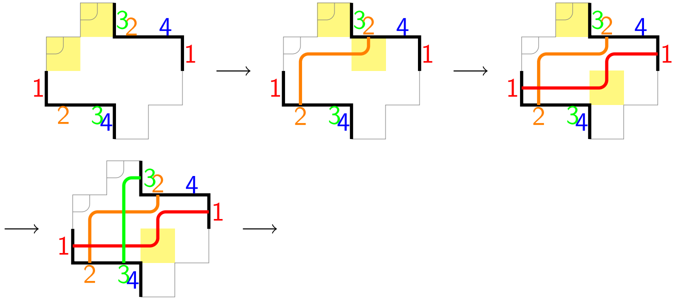

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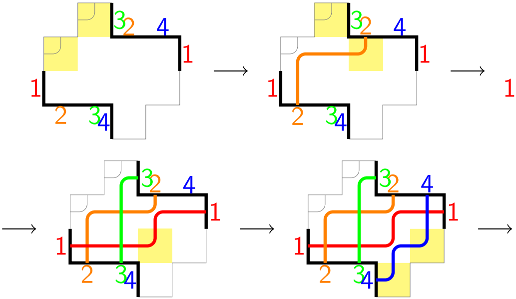

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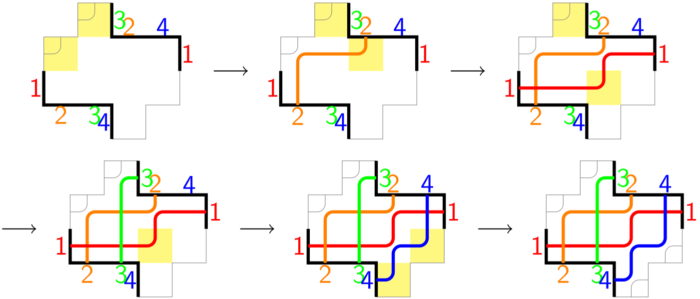
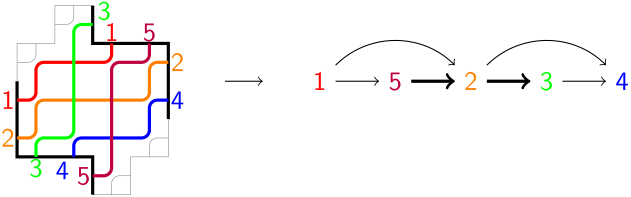


An acyclic but not strongly acyclic facet :



One linear extension : $15234 \nless 31524$.