Counting linear intervals in the Tamari and Dyck lattices

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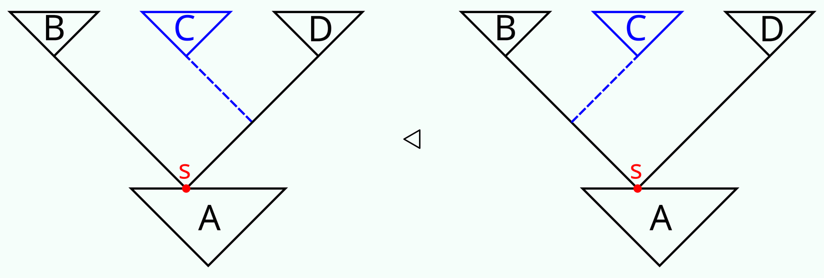
In a poset, when two elements P and Q are comparable, the interval $[P, Q]$ is the subset of elements R that satisfy $P \leqslant R \leqslant Q$. The simplest intervals are those which are totally ordered. They are called linear intervals. In this work, we are interested in the linear intervals in two classical posets on Catalan objects, namely the Tamari and the Dyck lattices. We prove that both lattices have the same number of linear intervals of any height and we count them.

The Tamari lattices

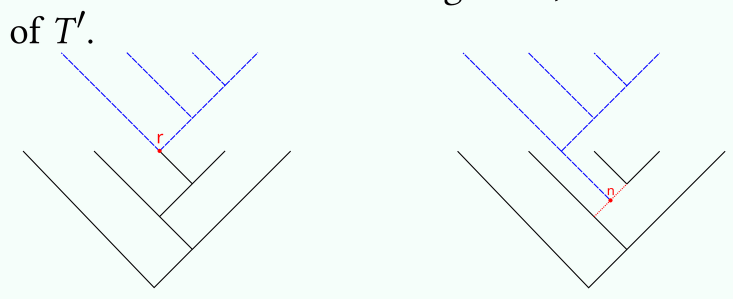
A **planar rooted binary tree** is a finite connected acyclic planar graph with n vertices of degree 3, and other vertices of degree 1, one of which is marked and called the root. We will simply call them trees.
The **Tamari lattice** Tam_n of size n is an order on trees of size n where covering relations consist of left rotations.



Example

A covering relation in the Tamari lattices.

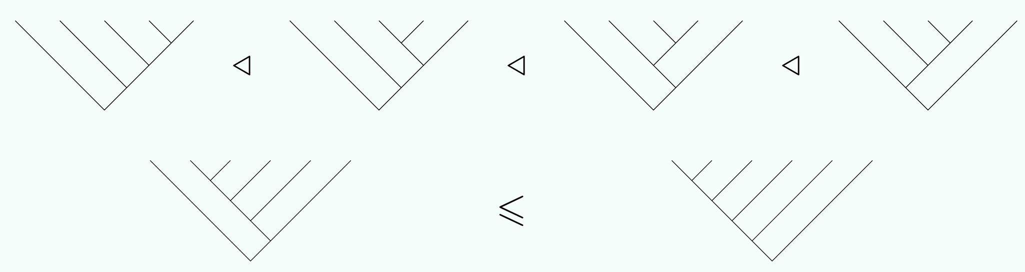
We can **graft** a tree T' on a chosen leaf of another tree T by deleting the root of T' and identifying its root node r with the chosen leaf of T.
We can graft an interval $I' = [P', Q']$ on the k-th leaf of another interval $I = [P, Q]$ by grafting P' (resp. Q') on the k-th leaf of P (resp. Q) as bottom (resp. top) element.
We can also **plug** a tree T' into a chosen edge of another tree T. To do so, we create a new node n on the selected edge of T, and we identify this node with the root of T'.



Example

Grafting (left) and plugging (right) a tree on another one.

We define left comb ℓ_n and right comb r_n recursively as follows:
$\ell_1 = r_1 =$: Y is the only tree of size 1 and for $n \geqslant 2$, ℓ_n (resp. r_n) is obtained by grafting ℓ_{n-1} (resp. r_{n-1}) on the left (resp. right) leaf of Y.
We define two particular intervals L_n and R_n in Tam_{n+1}, for $n \geqslant 2$. The interval R_n has r_{n+1} as bottom element and the grafting of r_n on the left leaf of Y as top element. The interval L_n is the mirrored version of R_n.
The intervals L_n and R_n are linear of height n.



Example

The intervals R_3 (with all 4 elements) and L_4.

Linear intervals in the Tamari lattices

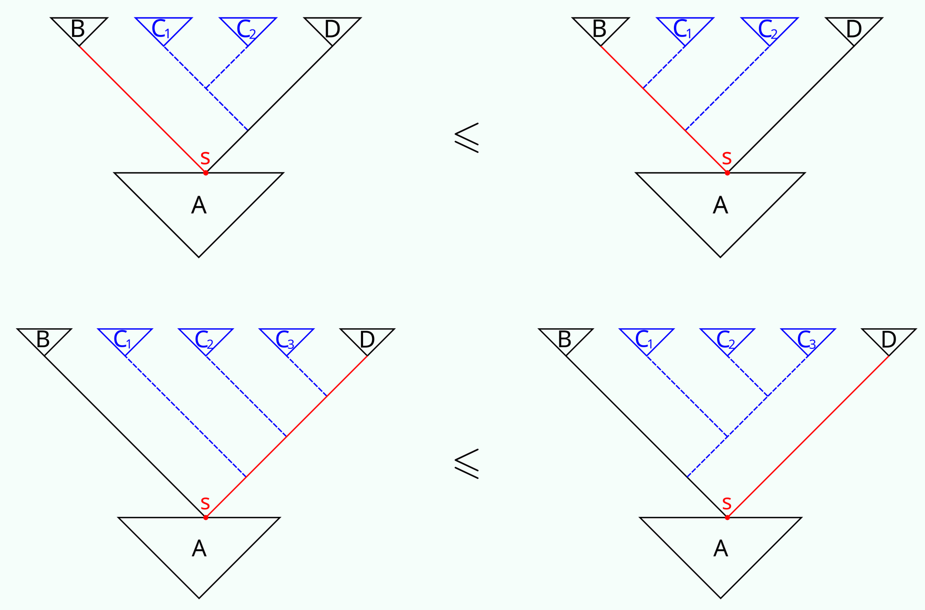
The **height** of an interval $[P, Q]$ is the length k of a longest maximal chain $P = P_0 \lessdot \cdots \lessdot P_k = Q$ from P to Q.
In any poset, an interval of height 0 is a trivial interval $[P, P]$ and an interval of height 1 is a covering relation $P \lessdot Q$.
We define left and right intervals in the Tamari lattices Tam_n as follows: for $k \geqslant 2$, a left (resp. right) interval of height k is obtained by grafting the interval L_k (resp. R_k) on some leaf of a trivial interval, and then trivial intervals on its leaves. All left and right intervals are linear.

Proposition

All linear intervals of height $k \geqslant 2$ in the Tamari lattices are either left or right intervals.

Corollary

There are no linear intervals of height $k \geqslant n$ in Tam_n.



Example

A left interval of height 2 (top) and a right interval of height 3 (bottom).

Let $A(t)$ be the generating series of trees. For $k \geqslant 0$, let $S_k(t)$ be the generating series of linear intervals of height k in the Tamari lattices.
A linear interval of height 0 is a trivial interval $[P, P]$ for P a non trivial tree so $S_0 = A - 1$.
A linear interval of height 1 (resp. $k \geqslant 2$) can be understood as a tree with a marked node s (resp. and a direction "left" or "right") and another tree (resp. k other trees) plugged on the edges out of this node. So we have $S_1 = (tA')tA$ and $S_k = 2(tA')(tA)^k$ for $k \geqslant 2$.

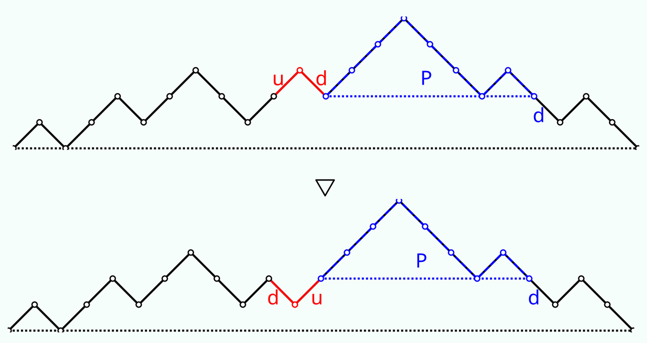
Theorem

In the Tamari lattice Tam_n of size n, there are:

- $\frac{1}{n+1}\binom{2n}{n}$ linear intervals of height 0,
- $\binom{2n-1}{n-2}$ linear intervals of height 1,
- $2\binom{2n-k}{n-k-1}$ linear intervals of height k, for $2 \leqslant k < n$.

Linear intervals in the Dyck lattices

A **Dyck path** of length (or size) n is a path on $\mathbb{N}^2$ consisting of up steps $(1, 1)$ and down steps $(1, -1)$, starting from $(0, 0)$ and ending at $(2n, 0)$. We define a **valley** as a down step followed by an up step and a **peak** as an up step followed by a down step.
The **Dyck lattice** Dyck_n of size n is an order on Dyck paths of size n where covering relations consist of changing a peak into a valley. A path P is less than or equal to a path Q if and only if P is always below Q.



Example

A covering relation in the Dyck lattice of size 11. In the bottom element, the excursion following d is uPd.

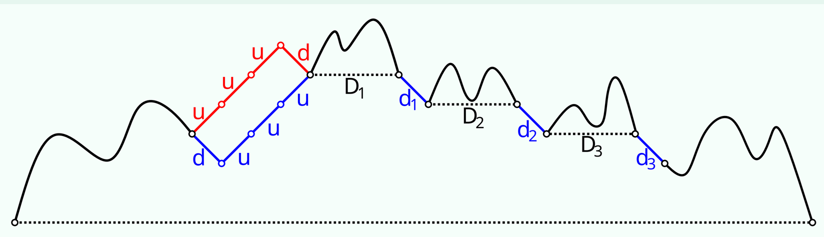
Remark

We can define another order on Dyck paths where covering relations consist of exchanging the down step of a valley with the excursion that follows it. We get a poset that is isomorphic to the Tamari lattice.

We can also define left and right intervals in the Dyck lattice Dyck_n. For $k \geqslant 2$, a left (resp. right) interval of height k is an interval $[P, Q]$ where Q is obtained from P by changing $d^k u$ into ud^k (resp du^k into $u^k d$).

Proposition

In the Dyck lattice, all left and right intervals are linear and all linear intervals of height $k \geqslant 2$ are either left or right intervals.



Example

A right interval of height 3 in the Dyck lattices.

For $k \geqslant 0$, let $T_k(t)$ be the generating series of linear intervals of height k in the Dyck lattices.

There is a bijection between Dyck paths and binary trees so $A(t)$ is the generating series of Dyck paths. As previously, we have $T_0 = A - 1$.

A covering relation can be understood as a Dyck path with a marked down step d, preceded by duP for bottom element and udP for top element, with P any Dyck path. Hence, we have $T_1 = (tA')tA$.

For $k \geqslant 2$, a right interval of height k can be understood as a Dyck path with a marked down step d_k, before which we insert $du^k D_1 d \ldots dD_k$ for the bottom element and $u^k dD_1 d \ldots dD_k$ for the top element, with k Dyck paths $D_1, \ldots, D_k$.
Similarily, a left interval of height k can be understood as a Dyck path with a marked up step and a list of k Dyck paths $D_1, \ldots, D_k$.
Thus, we have $T_k = 2(tA')(tA)^k$.

Theorem ⚠

For all $n \geqslant 1$ and $k \geqslant 0$, there are as many linear intervals of height k in the Tamari lattice Tam_n and in the Dyck lattice Dyck_n.

Other prospects

We can define a family of posets Tam_n^δ (where $\delta\colon \{2, \ldots, n\} \to \{0, 1\}$) on Dyck paths of size n which contains both the Tamari and the Dyck lattices.

Conjecture

For all $n \geqslant 1$ and $k \geqslant 0$, all posets Tam_n^δ have the same number of linear intervals of height k.

To further investigate this, we could look at the **Cambrian lattices** of type A and the posets of **tilting modules** that all generalize the Tamari lattice, and we conjecture they all have the same number of linear intervals.
We could as well investigate if these techniques give some results in the case of **m-Tamari** and **m-Cambrian** lattices, or in other types.
The poset of the **weak order** on the symmetric group seems to have nice numbers of linear intervals as well as the **Pallo's comb** poset.

In some posets, there are **no linear intervals** of height at least 2. It is in fact equivalent to have no linear intervals of height 2. Such posets are said to be 2-thick and for lattices of finite length, this is equivalent to be relatively complemented.

In particular, the **Boolean lattices**, the **set partitions** lattices and the **non crossing partitions** lattices with coarsening order (fusing blocks) only have trivial intervals and covering relations as linear intervals.

Indeed, in the Boolean lattice all intervals of height 2 are isomorphic to the Boolean lattice B_2, which is not totally ordered.
In the set partition and the non crossing partition lattices, all intervals of height 2 are isomorphic to the Boolean lattice B_2 or to the set partition lattice P_2, which are not totally ordered.

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