# Intervals in $m$-Tamari and $m$-cambrian lattices

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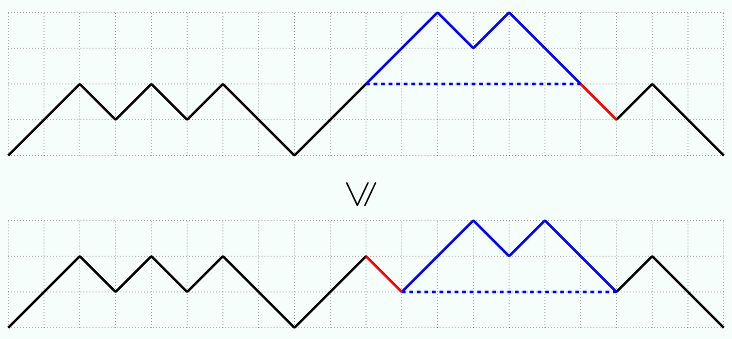
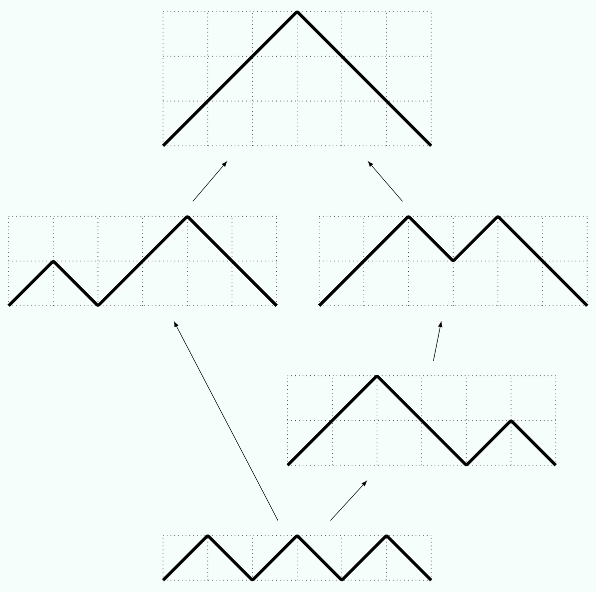
## Tamari lattices

A Dyck path of length (or size) $n$ is a path on $\mathbb{N}^2$ consisting of up steps $(1,1)$ and down steps $(1,-1)$, starting from $(0,0)$ and ending at $(2n,0)$.
We can define an order on Dyck paths of length $n$ where covering relations consist of swapping a down step with the following excursion. We obtain the Tamari lattice of size $n$ [Tamari, 1962], which we denote $\mathcal{T}_n$.



**Example**

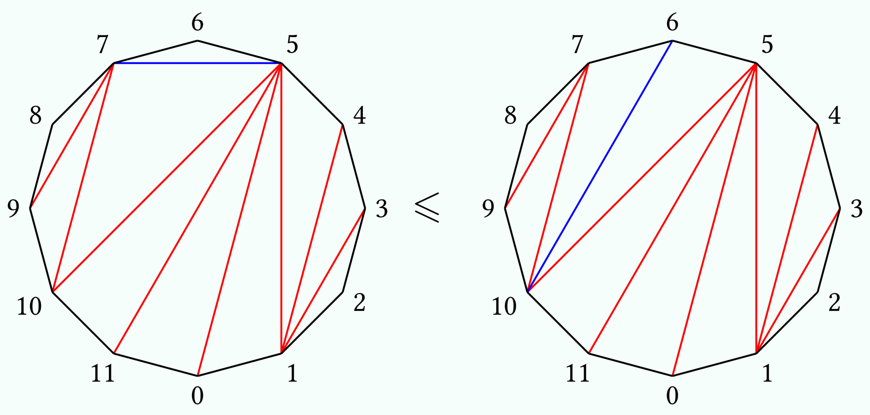
A covering relation of Dyck paths in $\mathcal{T}_{10}$.



**Example**

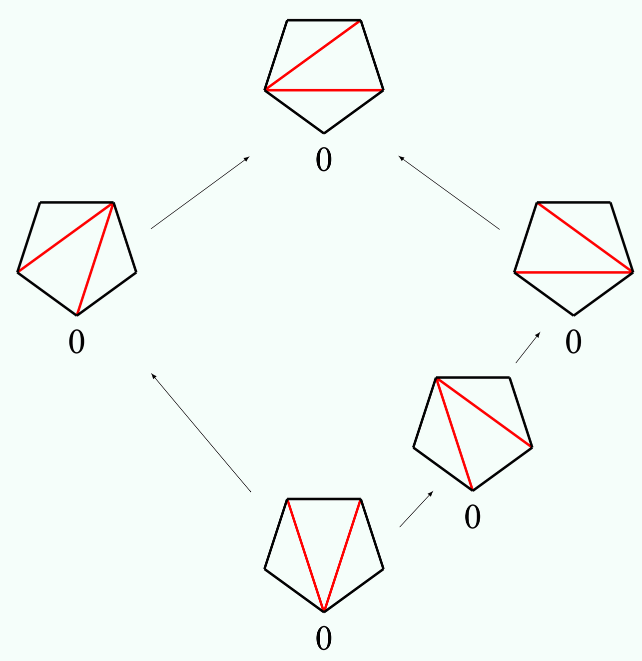
The Tamari lattice $\mathcal{T}_3$ on Dyck paths.

A triangulation of size $n$ is the decomposition of a $(n+2)$-gon into $n$ triangles by non-crossing diagonals.
We get another description of $\mathcal{T}_n$ by setting an order on triangulations of size $n$ where covering relations are increasing flips of a diagonal.



**Example**

A covering relation of triangulations in $\mathcal{T}_{10}$.

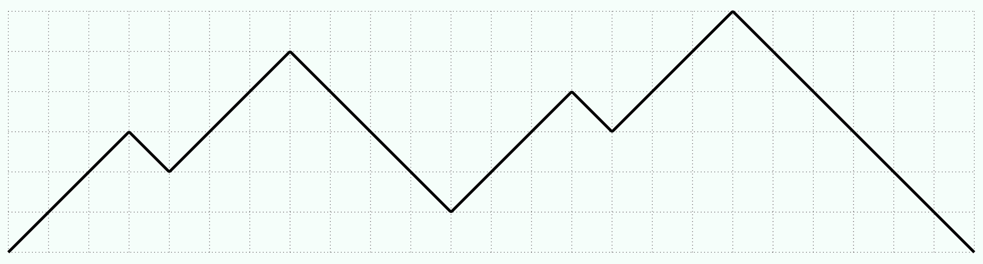


**Example**

The Tamari lattice $\mathcal{T}_3$ on triangulations.

## $m$-Tamari lattices

An $m$-Dyck path of size $n$ is a Dyck path of size $mn$ whose rises lengths are all multiples of $m$.



**Example**

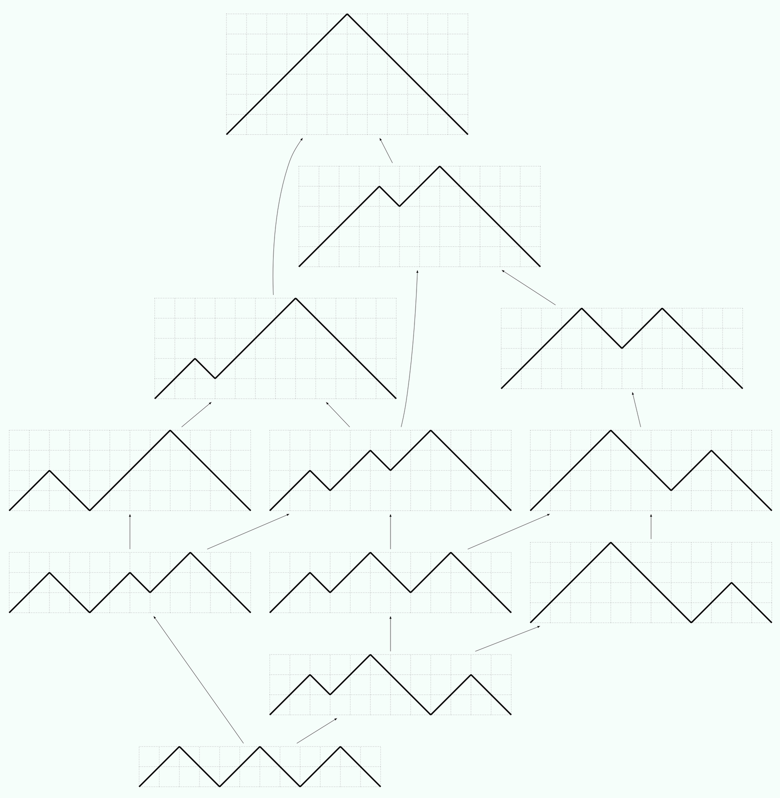
A 3-Dyck path of length 4.

This leads to a first generalization of $\mathcal{T}_n$ by considering the poset on $m$-Dyck paths of length $n$ whose covering relations are the same as before, which we call $m$-Tamari lattice and denote $\mathcal{T}_n^{(m)}$.

**Theorem** [Bousquet-Mélou, Fusy, Préville-Ratelle, 2011]

The number of intervals, i.e. pairs of comparable elements, of $\mathcal{T}_n^{(m)}$ is

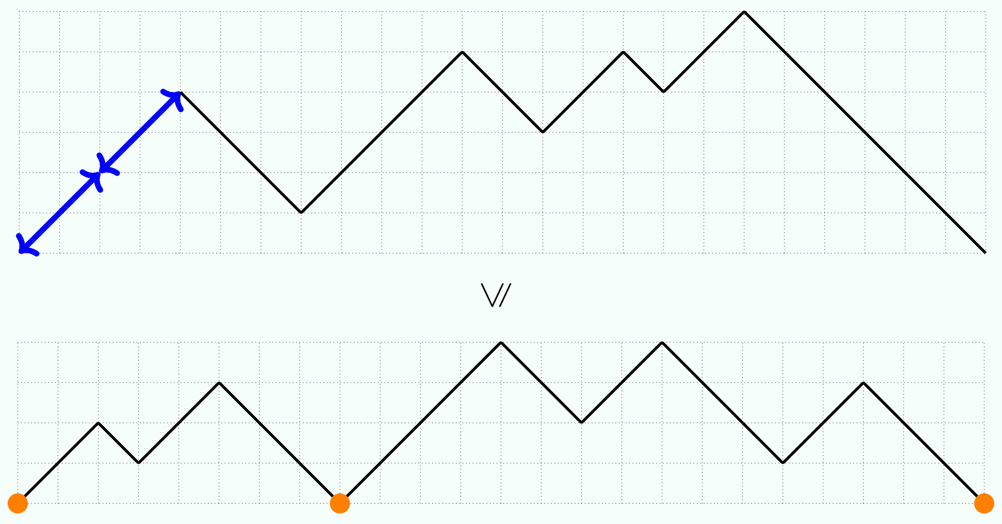
$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}.$$



**Example**

The 2-Tamari lattice for $n=3$, $\mathcal{T}_3^{(2)}$.

We define the contacts of a $(m$-$)$Dyck path $P$ as the vertices of $P$ at height 0 and denote their number as $c(P)$ and the initial rise $r(P)$ of $P$ as the number of up steps (divided by $m$) at the beginning of the path.
The contacts of an interval $[P,Q]$ are contacts of its lower path $P$ and its initial rise is the one of its upper path $Q$.

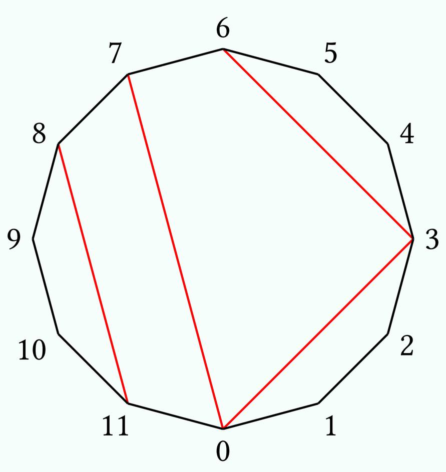


**Example**

An interval of 2-Dyck paths of length 6 with 3 contacts and an initial rise of 2.

## $m$-cambrian lattices

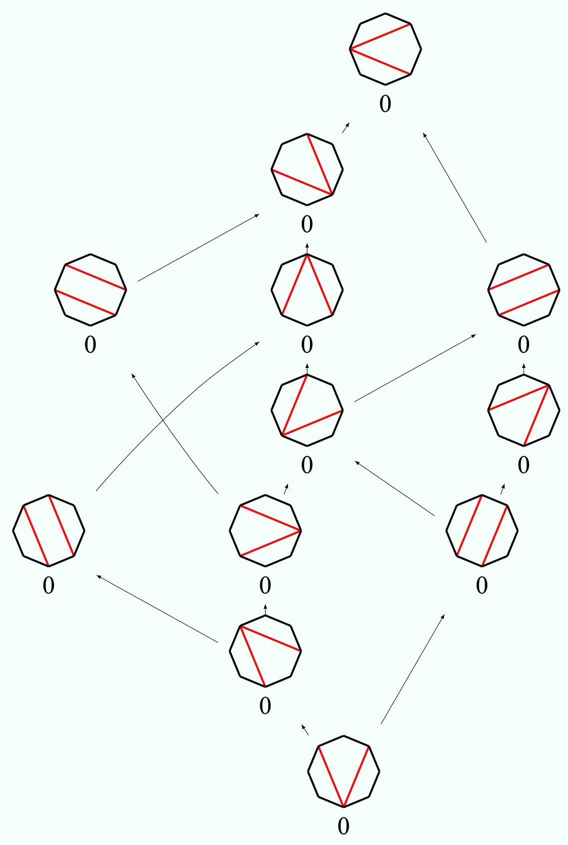
An $(m+2)$-angulation of size $n$ is the decomposition of an $(mn+2)$-gon into $(m+2)$-gons by non-crossing diagonals.



**Example**

A 4-angulation (or quadrangulation) of size 5.

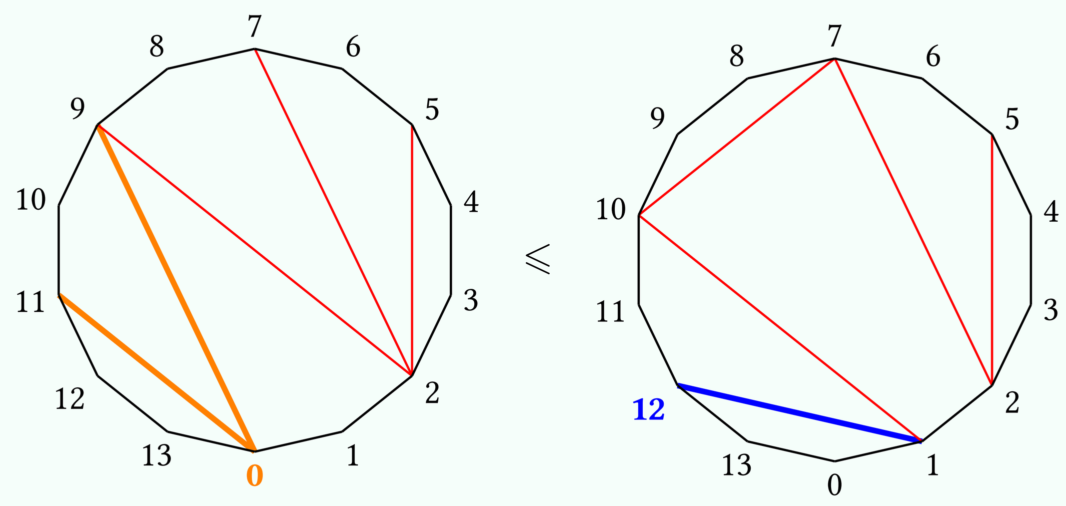
We can again define covering relations on $(m+2)$-angulations of size $n$ as admissible flips of diagonals, which leads to a second generalization of $\mathcal{T}_n$, called the (linear) $m$-cambrian lattice (of type $A$), which we denote $\mathrm{Camb}_n^{(m)}$.
The lattices $\mathcal{T}_n^{(m)}$ and $\mathrm{Camb}_n^{(m)}$ are in general not isomorphic but we conjecture that they have the same number of intervals [Stump et al., 2015].



**Example**

The 2-cambrian lattice for $n=3$, $\mathrm{Camb}_3^{(2)}$. Its Hasse diagram is not planar.

We can remark that the smallest element of $\mathrm{Camb}_n^{(m)}$ has all diagonals starting at vertex 0, which we call initial diagonals and its biggest element has all diagonals ending at vertex $m(n-1)+2$, which we call final diagonals.
The initial diagonals of an interval $[P,Q]$ are those of its lower element $P$ and its final diagonals are those of its upper element $Q$. Let $i(P)$ and $f(Q)$ be their numbers.



**Example**

An interval of quadrangulations of size 6 with 2 initial diagonals and 1 final diagonal.

## Generating series

Let $t$, $x$ and $y$ be three indeterminates and $\mathbb{Z}[x,y][[t]]$ be the ring of formal power series in $t$ with coefficients in $\mathbb{Z}[x,y]$.
Let $\Delta$ be the operator on $\mathbb{Z}[x,y][[t]]$ defined by

$$\Delta S(t;x,y)=\frac{S(t;x,y)-S(t;1,y)}{x-1}.$$

**Theorem** [Bousquet-Mélou, Fusy, Préville-Ratelle, 2011]

The equation

$$F(t;x,y)=x+xyt(F(t;x,1)\cdot\Delta)^{(m)}(F(t;x,y))$$

has a unique solution $F^{(m)}$ in $\mathbb{Z}[x,y][[t]]$.
Furthermore, we have

$$F^{(m)}(t;1,1)=\sum_{n\in\mathbb{N}}\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}t^n.$$

For $m\geqslant 1$, let $T^{(m)}(t;x,y)\in\mathbb{Z}[x,y][[t]]$ be the generating function of $m$-Tamari intervals, where $t$ counts the length of the paths, and $x$ and $y$ keep tracks of the contacts and initial rise of the intervals, that is to say

$$T^{(m)}(t;x,y)=\sum_{[P,Q]\in\mathcal{T}_n^{(m)}}x^{c(P)}y^{r(Q)}t^n.$$

In particular, $T^{(m)}(t;1,1)$ counts the number of intervals in $\mathcal{T}_n^{(m)}$.

**Theorem** [Bousquet-Mélou, Fusy, Préville-Ratelle, 2011]

The generating function $T^{(m)}(t;x,y)$ satisfies the equation

$$F(t;x,y)=x+xyt(F(t;x,1)\cdot\Delta)^{(m)}(F(t;x,y)).$$

Thus, we have $T^{(m)}(t;x,y)=F^{(m)}(t;x,y)$.

Similarily, for $m\geqslant 1$, let $C^{(m)}(t;x,y)\in\mathbb{Z}[x,y][[t]]$ be the generating function of $m$-cambrian intervals, where $t$ counts the size of the angulations, and $x$ and $y$ keep tracks of the initial and final diagonals of the intervals, that is to say

$$C^{(m)}(t;x,y)=\sum_{[P,Q]\in\mathrm{Camb}_n^{(m)}}x^{i(P)}y^{f(Q)}t^n.$$

**Conjecture** [Stump, Thomas, Williams, 2015]

We have $C^{(m)}(t;1,1)=T^{(m)}(t;1,1)$.

**Conjecture** [C., 2021]

We have $x^2yC^{(m)}(t;x,y)=T^{(m)}(t;x,y)$.