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FROM "METABELIAN $\mathbb{Q}$-VECTOR SPACES" TO NEW ω-STABLE GROUPS

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The aim of this paper is to describe (without proofs) an analogue of the theory of nontrivial torsion-free divisible abelian groups for metabelian groups. We obtain illustrations for "old-fashioned" model theoretic algebra and "new" examples in the theory of stable groups. We begin this paper with general considerations about model theory. In the second section we present our results and we give the structure of the rest of the paper. Most parts of this paper use only basic concepts from model theory and group theory (see [14] and especially Chapters IV, V, VI and VIII for model theory, and see for example [23] and especially Chapters II and V for group theory). However, in Section 5, we need some somewhat elaborate notions from stability theory. One can find the beginnings of this theory in [14], and we refer the reader to [16] or [21] for stability theory and to [22] for stable groups.

§1. Some model theoretic considerations. Denote by $\mathfrak{A}_{(1)}$ the theory of torsion-free abelian groups in the language of groups $\mathcal{L}_{gp}$. A finitely generated group G satisfies $\mathfrak{A}_{(1)}$ iff G is isomorphic to a finite direct power of $\mathbb{Z}$. It follows that $\mathfrak{A}_{(1)}$ axiomatizes the universal theory of free abelian groups and that the theory of nontrivial torsion-free abelian groups is complete for the universal sentences. Denote by $\mathfrak{T}_{(1)}$ the theory of nontrivial divisible torsion-free abelian groups. The models of this theory are the nontrivial $\mathbb{Q}$-vector spaces and $\mathfrak{T}_{(1)}$ is the model completion of $\mathfrak{A}_{(1)}$. Another example of this situation is: the theory ACF_p of algebraically closed fields of characteristic p is the model completion of the theory of commutative domains of characteristic p. A notion which is more general and conceptually better than model completion is model companion. Model companions are intimately

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connected with existentially closed (e.c.) models[1]: an inductive theory T_0 has a model companion T iff the e.c. models of T_0 are the models of T.

The theories $\mathfrak{T}_{(1)}$ and ACF_p are complete and uncountably categorical. Moreover, the theories $\mathfrak{T}_{(1)}$ and ACF_p are ω-stable of Morley rank and Morley degree one, in other words they are strongly minimal (a definable subset of a model is either finite or cofinite). To distinguish the theories $\mathfrak{T}_{(1)}$ and ACF_p we can adopt the point a view of geometrical stability theory. If M is a strongly minimal structure, the algebraic closure *acl* gives rise to a pregeometry[2]. Then, we can define, as for vector spaces, a notion of independence and a dimension (for a model of $\mathfrak{T}_{(1)}$, $acl(X)$ is the subvector space generated by X). One condition on a pregeometry that simplifies its structure is (local) modularity. A pregeometry is modular if for all closed sets X, Y, $dim(X)+dim(Y) = dim(X \cup Y)+dim(X \cap Y)$, it is locally modular if the equation holds whenever $X \cap Y \neq \emptyset$. The pregeometries associated with the models of $\mathfrak{T}_{(1)}$ are modular as those associated with the models of ACF_p are not locally modular. More generally, we can associate a pregeometry with any regular type of a stable theory[3]. Hrushovski [15] gives a fine analysis of locally modular regular type. Again (local) modularity is a geometrical condition that simplifies the analysis of a theory. A stable theory is (locally) modular if all the regular types have a (locally) modular pregeometry. The analysis of [15] implies that a connected superstable locally modular group is solvable and that we can not interpret an infinite field in a superstable locally modular theory.

In [10], Cherlin conjectured that an ω-stable simple group is an algebraic group (over an algebraically closed field). Independently, Zil'ber in [29], made the analogous conjecture for $\aleph_1$-categorical simple groups. Related conjectures are the Cherlin-Zil'ber conjecture: an ω-stable simple group of finite Morley rank is algebraic and the Berline conjecture [4]: a simple superstable group is of finite Morley rank. All these conjectures are open. The Cherlin-Zil'ber conjecture gave rise to an extensive theory. In particular, there is an elaborate theory of solvable centerless groups of finite Morley rank due to the importance of maximal solvable algebraic (definable) subgroups of a simple algebraic (finite Morley rank) group. For example, Nesin *et al.* proved that in a metabelian centerless group of finite Morley rank

[1]We recall that T is a model companion of T_0 if T and T_0 have the same universal consequences and if T is model complete. A model M is e.c. for T_0 if $M \models T_0$ and if for every existential sentence φ with parameters in M, if there is a model N of T_0 such that $M \leq N$ and $N \models \varphi$ then $M \models \varphi$. We refer the reader to [14, Chap. VIII].

[2]We recall that $acl(X) = \{a \in M \mid a \text{ is in some finite } X\text{-definable set}\}$. A pregeometry is a set M with a closure operation $cl : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$ which satisfies some natural properties (see [14, Chap. IV]).

[3]A type is regular if a well behaved dimension theory can be developed for the set of its realizations (we refer the reader to [16, Chap. VII] or [21, Chap. XIX]).

we can interpret algebraically closed fields in such a way that the group is interpretably embeddable in a direct product of algebraic groups over the interpretable fields (see [6, Chap. IX]). Here, an obstruction is the problem of the existence of a bad field: a field (K, Γ) of finite Morley rank with a predicate for a nontrivial proper subgroup Γ of K^*. If such a field exists, then an exotic solvable centerless group of finite Morley rank is obtained by considering the natural semidirect product $\Gamma \propto (K, +)$. At the level of infinite Morley rank we have "bad fields"; for example, if K is a differentially closed field with field of constants k, then (K, k^*) is ω-stable since a differentially closed field is. Also, Zil'ber [28] proved the ω-stability of $(\mathbb{C}, \mathbb{U})$ where $\mathbb{U}$ is the multiplicative group of roots of unity and Grünenwald and Haug [11] used this result to construct "small" superstable metabelian centerless groups of infinite rank. Moreover, Pillay in [20] considered structures of the form (K, Γ) where K is an algebraically closed field and Γ is a "finite rank" subgroup of a semiabelian variety over K.

§2. **Presentation of the results.** A group G is called metabelian (or solvable of class ≤ 2) if it has an abelian normal subgroup H such G/H is abelian. The class of metabelian groups is a variety of groups[4], thus it has free objects: for all cardinals r we have a metabelian group (unique up to isomorphism) M_r generated by elements a_i, $i \in r$, such that every function from $\{a_i\}_{i \in r}$ into a metabelian group G can be extended to a homomorphism from M_r into G. In [7] we provide simple explicit axioms for the universal theory of free metabelian groups which we denote by $\mathfrak{A}$ or by $\mathfrak{A}_{(2)}$ and we describe the finitely generated models of this theory, obtaining in this way an analogue of the classification of torsion-free finitely generated abelian groups. We recall the main result of [7] in Section 3. In Section 4, we describe a complete and decidable theory $\mathfrak{T}_{(2)} = \mathfrak{T}$ which plays the same role for $\mathfrak{A}$ as $\mathfrak{T}_{(1)}$ does for $\mathfrak{A}_{(1)}$. $\mathfrak{T}$ is not a model companion of $\mathfrak{A}$ but we are able to describe the e.c. models of $\mathfrak{A}$. Here the situation is similar to Sabbagh's description [25] of e.c. modules over a noncoherent ring and we introduce the notion of quasi-model companion to unify the two situations. In Section 5, we state the stability properties of $\mathfrak{T}$. $\mathfrak{T}$ is ω-stable of Lascar U-rank $\omega + 1$ and we have two classes of regular types for nonorthogonality (i.e., $\mathfrak{T}$ is bidimensional, a notion which generalizes categoricity). Moreover, $\mathfrak{T}$ is modular and it follows that we cannot interpret an infinite field in a model of $\mathfrak{T}$. Note that we have here something that looks like the situation described by Baudisch in [1, 2] for the infinite free nilpotent groups of class c and exponent p^n (p a prime number and $2 \leq c < p$). In the last section of this paper we explain how to generalize

[4]i.e., a class of group closed for the operation of subgroup, homomorphic image and cartesian product or equivalently the models of an equational theory of $\mathcal{L}_{gp}$.

the previous situation to some theories $\mathfrak{T}_{(m)}$ of solvable groups of class m for arbitrary m. The theories $\mathfrak{T}_{(m)}$ are related to recent work of Delon and Simonetta [27, Chap. IV] which had helped the author, in particular for the theories $\mathfrak{T}_{(m)}$ when $m \geq 3$, but our point of view is different. We obtain with the theories $\mathfrak{T}_{(m)}$, when $m \geq 3$, ω-stable connected centerless solvable groups which are far from being linear and nilpotent-by-abelian (but, for $m \geq 3$, $\mathfrak{T}_{(m)}$ does not give a completion for the universal theory of free solvable groups of class m). Notice that an ω-stable connected centerless solvable groups of U-rank $< \omega$ interprets an infinite field and is nilpotent-by-abelian. So, ω-stable centerless solvable groups of infinite rank are really different from those of finite rank. Moreover, the groups of the theories $\mathfrak{T}_{(m)}$ give natural and purely group theoretic examples for Berline-Lascar's theory [5] of superstable groups and for Hrushovski's theory [15] of locally modular types.

§3. **The universal theory of free metabelian groups.** In this section we recall the main result of [7]. Let G be group. We denote by $\mathit{Fit}(G)$ the Fitting subgroup of G. We recall that $\mathit{Fit}(G)$ is the subgroup of G generated by all the nilpotent normal subgroups of G (see [23, Chap. V]). Let H be an abelian normal subgroup of G and set $\bar{G} = G/H$. Then H is a module over the integral group ring $\mathbb{Z}[\bar{G}]$, where the action of $\mathbb{Z}[\bar{G}]$ on H is (well) defined as follows: if $g \in G$ and $v \in H$, $v.\bar{g} = g^{-1}vg$. So, adopting a multiplicative notation, if $\sum n_{\bar{g}}\bar{g} \in \mathbb{Z}[\bar{G}]$ and if $v \in H$ then $v.\sum n_{\bar{g}}\bar{g} = v^{\sum n_{\bar{g}}g} = \prod g^{-1}v^{n_{\bar{g}}}g$.

The theory $\mathfrak{A}$ is a (recursive) universal theory of $\mathcal{L}_{gp}$ which says the following of a model G: (a.1) G is a metabelian torsion-free group; (a.2) the relation "commute" is an equivalence relation on $G \setminus \{1\}$ (it follows that $\mathit{Fit}(G)$ is abelian and that if G is nonabelian, G is centerless); (a.3) the quotient $G/\mathit{Fit}(G)$ is torsion-free (and abelian since G is metabelian); (a.4) $\mathit{Fit}(G)$ is torsion-free as a $\mathbb{Z}[\bar{G}]$-module where $\bar{G} = G/\mathit{Fit}(G)$ (this makes sense since $\mathit{Fit}(G)$ is normal and abelian). Notice that in a group which satisfies (a.2) the Fitting subgroup is definable by a universal formula and this allows one to express (a.3) and (a.4) by universal sentences.

If A is a group and if L is a right A-module (i.e., a $\mathbb{Z}[A]$-module), we denote by $M(A, L)$ the set of matrices $\begin{pmatrix} a & 0 \\ v & 1 \end{pmatrix}$ where $a \in A$ and $v \in L$. $M(A, L)$ is a group under matrix multiplication, and this group is, of course, the usual semidirect product $A \propto L$ of L by A. In particular, if A is a free abelian group on $\{X_i\}_{i \in r}$ and if L is the free module of rank k over the ring of Laurent polynomials $\mathbb{Z}[X_i^{\pm 1}]_{i \in r}$ then $M(A, L)$ is isomorphic to the standard restricted wreath product $\mathbb{Z}^{(r)} wr \mathbb{Z}^{(k)}$. In [7] we prove the following

THEOREM 3.1. *Let G be a group. The following properties are equivalent:* (1) *G satisfies the universal theory of a noncyclic free metabelian group;* (2) *G satisfies $\mathfrak{A}$;* (3)*G is a subgroup of a group $M(A, L)$ where A is a torsion-free abelian group and L is a torsion-free $\mathbb{Z}[A]$-module;* (4) *for all $g_1, \dots, g_n \in G$ there exist $k, r \in \mathbb{N}$ such that the group $\langle g_1, \dots, g_n \rangle$ can be embedded in $\mathbb{Z}^{(k)} wr \mathbb{Z}^{(r)}$. Moreover, if G is a nonabelian group satisfying one of the properties above, then G has the same universal theory as a noncyclic free metabelian group.*

As a corollary we see that the theory obtained by adding to $\mathfrak{A}$ the axiom which says that G is not abelian is complete for universal sentences and that the universal theory of a free metabelian group is decidable. Note that a noncyclic free metabelian group is unstable, has an undecidable $\forall\exists$ theory and an undecidable universal theory if we allow constants in the language (see [17, 8, 24]).

§**4. Metabelian $\mathbb{Q}$-vector spaces.** If (r_1, r_2) is a pair of cardinals, then we denote by $E(r_1, r_2)$ the group $M(D, V)$ where D is a multiplicatively noted $\mathbb{Q}$-vector space of dimension r_1 and where V is a vector space of dimension r_2 over the field of fractions of the commutative domain $\mathbb{Z}[D]$ (V is a D-module via the multiplication of the ring $\mathbb{Z}[D]$). The groups $E(r_1, r_2)$ arise naturally: whenever A is a torsion-free abelian group and L is a torsion-free $\mathbb{Z}[A]$-module there exists a couple of cardinals (r_1, r_2) such that $M(A, L)$ can be embedded in $E(r_1, r_2)$.

The theory $\mathfrak{T}$ is a $\forall\exists$ theory consisting of $\mathfrak{A}$ and sentences which say: (a.5) G is nonabelian; (a.6) $G/Fit(G)$ is divisible; (a.7) $Fit(G)$ is $\mathbb{Z}[\bar{G}]$-divisible where $\bar{G} = G/Fit(G)$; (a.8) for all $g \in G \setminus Fit(G)$, $G \simeq C_G(g) \propto Fit(G)$, where $C_G(g)$ is the centralizator of g in G. Then, we have the suggestive

LEMMA 4.1. *$G \models \mathfrak{T}$ iff there exists cardinals $r_1, r_2 > 0$ such that $G \simeq E(r_1, r_2)$.*

Unfortunately, $\mathfrak{T}$ is not model complete, the reason for this is that we have bad embeddings between models of $\mathfrak{T}$, for example, if r is an infinite cardinal we can embed $E(r, r)$ in $E(r, 1)$. So, we have to work in a new language. We consider the language $\mathcal{L}_{mc}$ consisting of $\mathcal{L}_{gp}$ and for all $m \geq 1$ and all $n \geq 2$ a new relation $R_{m,n}(x_1, \dots, x_n)$. Then, we consider a set of sentences $\mathfrak{D}$ consisting of all sentences of the form $\forall x_1 \dots x_n\ R_{m,n}(\bar{x}) \leftrightarrow \delta_{n,m}(\bar{x})$, where the formulae $\delta_{n,m}(\bar{x})$ are *universal* formulae of $\mathcal{L}_{gp}$ which are satisfied by elements $g_1, \dots, g_n$ of a model of $\mathfrak{A}$ iff $g_1, \dots, g_n$ are elements of $Fit(G)$ which are linearly independent for the elements of $\mathbb{Z}[\bar{G}]$ of norm $\leq m$ (if $\sum n_g g \in \mathbb{Z}[X]$ then $\| \sum n_g g \| = \sum |n_g|$). $\mathfrak{D}$ is a universal expansion by definition of $\mathcal{L}_{gp}$. Then, we have

THEOREM 4.2. *The theory $\mathfrak{T} \cup \mathfrak{D}$ is model complete (in $\mathcal{L}_{mc}$). Thus, every formula $\varphi(\bar{x})$ of $\mathcal{L}_{gp}$ is equivalent modulo $\mathfrak{T}$ to a $\forall\exists$ formula of $\mathcal{L}_{gp}$ and to a $\exists\forall$ formula of $\mathcal{L}_{gp}$.*

COROLLARY 4.3. (1) *$E(1,1)$ is an elementary prime model for $\mathfrak{T}$ and thus $\mathfrak{T}$ is complete and decidable.* (2) *The e.c. models of $\mathfrak{A}$ are exactly the groups $E(r,1)$ where r is a nonzero cardinal. Thus, two e.c. models of $\mathfrak{A}$ are elementary equivalent and $\mathfrak{A}$ has no model companion.*

The second part of the corollary implies that for every cardinal $r \geq \aleph_1$ we have a unique e.c. model of $\mathfrak{A}$ of cardinal r and that we have $\aleph_0$ countable e.c. models of $\mathfrak{A}$, one for each nonzero cardinal $\leq \aleph_0$ which represents its "dimension". So, the class of e.c. models of $\mathfrak{A}$ looks like the class of models of an $\aleph_1$-categorical theory (we lose a dimension in the sense of Section 5).

The situation above and the description of e.c. modules over a noncoherent ring led us to the following definition. Let T_0 be an inductive theory in $\mathcal{L}$. We say that T_0 has a quasi-model companion if there exists a theory T of $\mathcal{L}$ such that: (i) T_0 and T have the same universal consequences; (ii) every e.c. model of T_0 is a model of T; (iii) there exists a universal (or existential) expansion by definition Δ such that $T \cup \Delta$ is model complete[5]. Then, we say that (T, Δ) is a quasi-model companion of T_0. $(\mathfrak{T}, \mathfrak{D})$ is a quasi-model companion of $\mathfrak{A}$ and it follows from [25] that if R is a ring, then the theory of R-modules has a quasi-model companion (with Δ as the pp-formulae).

PROPOSITION 4.4. *We suppose that T_0 has a quasi-model companion (T, Δ).* (1) *If T_0^{ec} is the theory of the e.c. models of T_0, then (T_0^{ec}, Δ) is a quasi-model companion of T_0 and $T \subseteq T_0^{ec}$;* (2) *the class of e.c. models of T_0, the class of infinitely generic models of T_0 and the class of finitely generic models of T_0 coincide;* (3) *the following properties are equivalent:* (i) *T_0 has the J.E.P. in $\mathcal{L}$,* (ii) *T_0 has a complete quasi-model companion.*

We refer the reader to [12] for finitely and infinitely generic models. We can use the proposition above and known results to show that some theories do not have a quasi-model companion. For example, the following universal theories do not have a quasi-model companion: the theory of groups, the theory of (torsion-free) nilpotent groups of class m, for any $m \geq 2$ and the theory of solvable groups of class 2 (all these theories have the J.E.P and have non-elementary equivalent e.c. models; one may consult [13] for references and proofs). In contrast, there are a lot of model complete metabelian groups of the form $(K^*, .) \propto (K, +)$, where K is a field (see [27, Chap. I]).

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[5] I do not know if T must be unique. One can define various notions of minimality for Δ but, at the present time, I have no substantial results.

§5. Stability properties of $\mathfrak{T}$. In a countable language a theory T is ω-stable (superstable) iff the Morley rank (the U-rank) of any type for T is an ordinal. We use Lascar U-rank[6]. This is a good rank to consider in the study of ω-stable (superstable and finite Morley rank) groups because, for example, this rank has good additivity properties. The U-rank does not rank formulae but types; however, we can define the U-rank of a superstable group G (or of a subgroup or of a quotient with all the structure which comes from G) as the U-rank of one of its generic types (i.e., a 1-type of maximal U-rank pertaining to the group, subgroup or quotient). A group G is connected if it has no proper definable subgroup of finite index (or equivalently for an ω-stable G if it has a unique generic type). The regular types are the building blocks for the classification of the models of a "classifiable" theory. In our context, a regular type is strongly regular and two regular types p_1 and p_2 over a model M are nonorthogonal if there are RK-equivalent; that is, if $\bar{a}$ realizes p_i, then p_j is realized in the prime model over $M^\frown\bar{a}$ for $(i, j) = (1, 2)$ and $(2, 1)$.

THEOREM 5.1. *Let G be a model of $\mathfrak{T}$. Then G is a connected ω-stable group of Lascar U-rank $\omega + 1$, $G/Fit(G)$ is strongly minimal and $Fit(G)$ is connected of U-rank ω. Furthermore, $\mathfrak{T}$ is bidimensional: we have two classes of regular types (for nonorthogonality), one represented by the generic type of $G/Fit(G)$, the other represented by the generic type of $Fit(G)$.*

We can use this analysis to prove results on the groups $E(r_1, r_2)$. For example, the groups $E(r_1, r_2) = E$ are "d^2-simple": we have a unique definable normal subgroup H (the Fitting subgroup) such that H and E/H are definably simple as structures interpreted in E (it follows that the other definable subgroups are the centralizers of an element not in the Fitting subgroup). In another direction, G is an ω-stable model of $\mathfrak{A}$ iff G is a model of $\mathfrak{T}$. Moreover, the proof of Theorem 5.1 gives the following.

COROLLARY 5.2. *The theory $\mathfrak{T}$ is modular. It follows that we can interpret neither an infinite field nor an infinite simple group in a model of $\mathfrak{T}$.*

Let K be an algebraically closed field of characteristic 0 and Γ a subgroup of K^*. If Γ is finitely generated as a $\mathbb{Q}$-group, then it follows from [20] and results of number theory that the expanded field structure (K, Γ) is stable and so is the semidirect product $\Gamma \propto K$. In the case where Γ is $\mathbb{Q}$-generated by algebraically independent elements over $\mathbb{Q}$, we obtain a group of the form $E(r_1, r_2)$. So, the result above shows that, in general, the structures (K, Γ) and $\Gamma \propto K$ in $\mathcal{L}_{gp}$ are different.

[6]The U-rank is always $\leq$ the Morley rank. Note that if a group G is ω-stable of finite U-rank then the Morley rank of G is equal to its U-rank.

§6. Generalizations. We define groups $E(r_1, ..., r_m)$ by induction on m where $r_1, ..., r_m$ are nonzero cardinals. $E(r_1)$ is the $\mathbb{Q}$-vector space of dimension r_1, this group is an abelian orderable group. Assume that $E(r_1, ..., r_m) \stackrel{\text{def}}{=} E_m$ has been defined and that it is an orderable solvable group of class m. Then, by [19, Chap. XIII], E_m is an Ore group and thus $\mathbb{Z}[E_m]$ has a division ring K of fractions. Then, if r_{m+1} is a nonzero cardinal we define $E(r_1, ..., r_m, r_{m+1}) \stackrel{\text{def}}{=} E_{m+1}$ to be the natural semidirect product $E_m \propto V$ where V is a vector space of dimension r_{m+1} over K. Then, E_{m+1} is of class exactly $m + 1$ and E_{m+1} is orderable.

Here one can apply the general machinery of Delon and Simonetta to prove that the groups $E(r_1, ..., r_m)$ have a decidable theory. More precisely, Delon and Simonetta give an Ax-Kochen-Ershov principle for specific structures of the form $((A, +, 0), (B, ., 1, \leq), *, v)$ where A is an abelian group, B is an ordered group, $*$ is an action of B on A and where v is a valuation and obtain results for groups of the form $B \propto_* A$. In particular, they obtain new decidable groups (see [27, Chap. IV]). Using our techniques and [27, Chap. IV] it is not very difficult to show that all the groups $E(r_1, ..., r_m)$ are connected and ω-stable; also Simonetta put this in a more general setting in [26]. If $m \geq 3$, then the groups $E(r_1, ..., r_m)$ are far from being nilpotent-by-abelian: they generate the variety of solvable groups of class m and are typical nonlinear groups (if the $r_i \geq 2$, they contain a noncyclic free solvable group of class m). This contrast with the fact that a connected solvable group of finite Morley rank is nilpotent-by-abelian (see [22, Sect. 3.e], and [18] and [3] for generalisations). Moreover, the existence of such groups answer the main question of [3].

We can define by induction $\forall\exists$ theories $\mathfrak{T}_{(m)}$ of $\mathcal{L}_{gp}$ with the following properties (some lemmas of [27, Sect. IV.7] helped the author to construct the theories $\mathfrak{T}_{(m)}$ for $m \geq 3$): (1) $G \models \mathfrak{T}_{(m)}$ *iff there exists a m-tuple of nonzero cardinals* $(r_1, ..., r_m)$ *such that* $G \simeq E(r_1, ..., r_2)$*;* (2) $\mathfrak{T}_{(m)}$ *has a universal expansion by definition which is model complete;* (3) $E(1, ..., 1)$ *is an elementary prime model for* $\mathfrak{T}_{(m)}$*, thus* $\mathfrak{T}_{(m)}$ *is complete and decidable;* (4) $\mathfrak{T}_{(m)}$ *is* ω*-stable. If* $G \models \mathfrak{T}_{(m)}$*, then* G *is connected of* U*-rank* $\omega^{m-1} + ... + \omega + 1$*,* $Fit(G)$ *is connected of* U*-rank* ω^{m-1} *and* $\mathfrak{T}_{(m)}$ *is m-dimensional;* (5) $\mathfrak{T}_{(m)}$ *is modular.*

Now we set $\mathfrak{A}_{(m)} = Th_{\forall}(wr_{i=1}^{m}\mathbb{Z})$ and $\mathfrak{A}^{+}{}_{(m)} = \mathfrak{A}_{(m)} \cup Th_{\exists}(wr_{i=1}^{m}\mathbb{Z})$ where $wr_{i=1}^{m}\mathbb{Z}$ is the left-iterated restricted wreath product of m copies of $\mathbb{Z}$. Then, $\mathfrak{A}_{(m)}$ is the set of universal consequences of $\mathfrak{T}_{(m)}$ and we can prove that $\mathfrak{T}_{(m)}$ is a quasi-model companion of $\mathfrak{A}_{(m)}$ and that the e.c. models of $\mathfrak{A}_{(m)}$ are the groups $E(r, 1, ..., 1)$.

Assume that $m \geq 3$. A direct proof of the decidability of $\mathfrak{A}^{+}{}_{(m)}$ can be found in [9]. But, at the present time, I do not have an elegant description of this theory. Notice that $\mathfrak{A}_{(m)}$ does not axiomatize the universal theory of a noncyclic free solvable group of class m (it is known [8] that the decidability of the universal theory of a noncyclic free solvable group of class $m \geq 3$ implies a positive answer to Hilbert's tenth problem for the field of the rationals: a difficult open problem).

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