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Universal Theory of Certain Solvable Groups and Bounded Ore Group Rings

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INTRODUCTION

A free group has a decidable universal theory (see [13]). Moreover, it is known (see [22]) that a noncyclic free nilpotent group of class 2 has a decidable universal theory if and only if Hilbert's tenth problem has a positive answer for the field of the rationals (this problem is still open); and it has been proved recently that if Hilbert's tenth problem has a negative answer for the field of the rationals, then a noncyclic free solvable group of class ≥ 3 has an undecidable universal theory (see [4]). This paper was motivated by the following question: Is the universal theory of a noncyclic free metabelian group decidable? We will prove that this question has an affirmative answer and we will also prove that some other solvable groups (left-iterated restricted wreath products of torsion-free abelian groups) have decidable universal theories. This kind of problem led us to compare universal theories and to study some properties of group rings which are Ore domains. We can note that our decidability results contrast with the large number of undecidability results that exist for non abelian-by-finite solvable groups (see Section 4).

This paper is organized as follows. In the first section, we fix notation and we recall a certain number of definitions and elementary facts about logic and group theory. In the second section, we compare the universal theories of certain solvable groups. We prove, in particular, that a non-

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cyclic free metabelian group has the same universal theory as the wreath product of two nontrivial torsion-free abelian groups; this result is essential for the study of the universal theory of free metabelian groups. In the third section, we study the following notions: Let G be a group; we define a norm on the integral group ring $\mathbb{Z}G$ of G by $\|\Sigma n_g g\| = \Sigma |n_g|$, and we say that G is a (recursively) bounded Ore group if $\mathbb{Z}G$ is an Ore domain and if there exists a (recursive) function f of $\mathbb{N}^*$ into $\mathbb{N}$ such that for all nonzero elements a and b of $\mathbb{Z}G$ there exist nonzero elements c and d of $\mathbb{Z}G$ such that $ac = bd$ and such that $\|c\| \leq f(\max(\|a\|, \|b\|))$ and $\|d\| \leq f(\max(\|a\|, \|b\|))$. We give some "nontrivial" examples and we study systems of equations and inequations in such group rings. In the last section, we use the notions introduced in the third to give a general criterion to prove that certain wreath products have decidable universal theories. With this criterion, we obtain our decidability results. Note that the proof of the decidability of the universal theory of a free metabelian group given in this paper does not give an explicit description of this theory; we give such a description in [5].

1. NOTATIONS AND PRELIMINARIES

Almost all the notions and notations of group theory and logic used in this paper are classical. We refer the reader to [21] and [17] for group theory, to [20] for group rings, and to [3] and [8] for logic ([17] especially for varieties of groups and [8] especially for decidability and undecidability problems). But, for the convenience of the reader and to be accessible to both algebraists and logicians, we are going to recall a certain number of definitions and elementary facts.

Let L be a first-order language, M a model of L, and P a property pertaining to the sentences of L. The P-theory of M is the set of all the sentences of L with the property P which are true in M. We say that M has a decidable P-theory if there exists an algorithm which decides if a sentence with the property P is in the P-theory of M. We say that M is decidable if M has a decidable theory. Any sentence of L is equivalent to a sentence φ of the form

$$Q_1 x_1 Q_2 x_2 \cdots Q_n x_n \; \phi(x_1, \ldots, x_n),$$

where $\phi(x_1, \ldots, x_n)$ is a quantifier-free formula and where the Q_i are quantifier symbols $\exists$ or $\forall$. If all the Q_i are $\exists$, then we say that φ is an existential sentence; if all the Q_i are $\forall$, then we say that φ is a universal sentence. Clearly, two models of L have the same universal theory iff they have the same existential theory; and M has a decidable universal theory iff it has a decidable existential theory. We say that φ is a $\forall\exists$ sentence if φ is existential or if there exists $p \in \{1, \dots, n\}$ such that Q_i is $\forall$ for $1 \le i \le p$ and such that Q_i is $\exists$ for $i > p$. If N is a submodel of M, then we say that N is existentially closed in M if any existential sentence with parameters in N true in M is true in N. If u is an ultrafilter we denote by M^u the ultrapower of M by u.

In this paper we will work most often with the language of groups: $L = \{1, {}^{-1}, .\}$ where 1 is a constant symbol for the identity, ${}^{-1}$ is a unary function symbol for the inverse, and . is a binary function symbol for the multiplication. In this language any quantifier-free formula is (effectively) equivalent (modulo the theory of groups) to a formula of the form

$$\bigvee_{i=1}^{m} \left(\bigwedge_{j=1}^{t_i} w_{j,i}(x_1, \dots, x_n) = 1 \wedge \bigwedge_{j=t_i+1}^{s_i} w_{j,i}(x_1, \dots, x_n) \neq 1 \right),$$

where the $w_{i,j}$ are elements of the free group on $x_1, \dots, x_n$. Hence, any existential sentence is (effectively) equivalent to a sentence of the form

$$\bigvee_{i=1}^{m} \left(\exists x_1 \cdots \exists x_n \left(\bigwedge_{j=1}^{t_i} w_{j,i}(x_1, \dots, x_n) = 1 \wedge \bigwedge_{j=t_i+1}^{s_i} w_{j,i}(x_1, \dots, x_n) \neq 1 \right) \right),$$

and thus we see that a group G has a decidable universal theory iff there is an algorithm which decides if a system of equations and inequations has a solution in G or not.

Let G be a group. For every set X we denote by $G^{(X)}$ the group of functions f of X into G such that $\{x \in X \mid f(x) \neq 1\}$ is finite. If $g_1, \dots, g_n \in G$, then we denote by $\langle g_1, \dots, g_n \rangle$ the subgroup of G generated by $g_1, \dots, g_n$. If $g, h \in G$, then we put $[g, h] = g^{-1}h^{-1}gh$ and $g^h = h^{-1}gh$. For any group H we denote by $G * H$ the free product of G by H. If n is a positive integer, we denote by $\delta_n G$ the nth term of the derived series ($\delta_1 G = G' = \langle [g,h] \mid g, h \in G \rangle$, $\delta_{n+1} G = \langle [g,h] \mid g, h \in \delta_n G \rangle$, and G is called solvable of class $\le n$ if $\delta_n G = 1$). We say that a group is metabelian if it is solvable of class ≤ 2. We denote by $\gamma_n G$ the nth term of the lower central series ($\gamma_2 G = G'$, $\gamma_{n+1} G = \langle [g,h] \mid g \in \gamma_n G,\ h \in G \rangle$). If R is a commutative ring, we denote by RG the group ring of G and we recall that the augmentation ideal of RG is the kernel of the ring homomorphism ε of RG into R defined by $\varepsilon(\Sigma a_g \cdot g) = \Sigma a_g$. If R is a ring, then we denote by R^* the set of nonzero elements of R and, if X is a set, we denote by $R^{(X)}$ the R-module of functions f of X into R such that $\{x \in X \mid f(x) \neq 0\}$ is finite. We say that R is a domain if R is without divisors of zero (even if

R is not commutative), and we say that R is an Ore ring if for all $a, b \in R^*$ there exist $c_1, d_1, c_2, d_2 \in R^*$ such that $ac_1 = bd_1$ and $c_2 a = d_2 b$.

Let V be a variety of groups. If r is a cardinal, then we denote by $F_r(V)$ the free group of V of rank r; if V is the variety of solvable groups of class $\leq n$, we denote this group by $F_r(n)$. If r is a cardinal and if n is a positive integer, we denote by $M_r(n)$ the group of 2×2 matrices $(\alpha_{i,j})$ such that $\alpha_{1,1} \in F_r(n-1)$, $\alpha_{1,2} = 0$, $\alpha_{2,1} \in \Omega$, and $\alpha_{2,2} = 1$, where Ω is the free $\mathbb{Z}F_r(n-1)$-module on $(\lambda_i \mid i \in r)$; we call this group the Magnus group of class n and rank r. If we denote by a_i $(i \in r)$ free generators of $F_r(n)$ and by $\bar{a}_i$ free generators of $F_r(n-1)$, then we define a homomorphism f of $F_r(n)$ into $M_r(n)$ by

$$f(a_i) = \begin{pmatrix} \bar{a}_i & 0 \\ \lambda_i & 1 \end{pmatrix}.$$

By a theorem of Magnus, f is a monomorphism (see [10] for a modern exposition of this Magnus embedding).

Let A and B be two groups; we denote by $A \operatorname{wr} B$ the (restricted) wreath product of A by B. We recall that $A \operatorname{wr} B$ is the semi-direct product of $A^{(B)}$ by B, in which the automorphism of $A^{(B)}$ produced by an element $b \in B$ is given by $f^b(a) = f(ab^{-1})$ for $f \in A^{(B)}$. Suppose that $A = \mathbb{Z}^{(r)}$ where r is a cardinal. If B is given by a presentation $(b_i,\ i \in \kappa \mid R)$ then it is easy (using simple computations) to see that $A \operatorname{wr} B$ has a presentation

$$\left(a_i,\ i \in r; b_i,\ i \in \kappa \mid R; \left[a_i^x, a_j^y\right],\ x, y \in B,\ i, j \in r\right)$$

and that any element of $A \operatorname{wr} B$ can be written in the form

$$b \prod_{i \in r} a_i^{f_i}$$

where $b \in B$, $f_i \in \mathbb{Z}B$, and we adopt the notation

$$a_i^{\sum_{b \in B} n_b b} = \prod_{b \in B} \left(b^{-1} a_i b\right)^{n_b}.$$

In other words, $A \operatorname{wr} B$ is isomorphic to the group of 2×2 matrices $(\alpha_{i,j})$ where $\alpha_{1,1} \in B$, $\alpha_{1,2} = 0$, $\alpha_{2,1} \in (\mathbb{Z}B)^{(r)}$, and $\alpha_{2,2} = 1$. In particular, we see that $M_r(n)$ is isomorphic to $\mathbb{Z}^{(r)} \operatorname{wr} F_r(n-1)$. If $A_1, \ldots, A_n$ is a finite sequence of groups, then we denote by $\operatorname{wr}_{i=1}^n A_i$ the left-iterated wreath product $A_1 \operatorname{wr} (A_2 \operatorname{wr}(\ldots (A_{n-1} \operatorname{wr} A_n) \ldots))$. In general, $A_1 \operatorname{wr} (A_2 \operatorname{wr} A_3) \neq (A_2 \operatorname{wr} A_2) \operatorname{wr} A_3$.

2. COMPARING UNIVERSAL THEORIES

We begin by giving a manageable criterion to prove that two groups have the same universal theory. For this, we generalize the notions of discrimination and strong discrimination. We recall that if V is a variety of groups and if r is a positive integer, $F_r(V)$ discriminates V if for any finite set of words $w_i(x_1, \ldots, x_s)$, $i = 1, \ldots, n$, that are not laws in V there exist $b_1, \ldots, b_s \in F_r(V)$ such that $w_i(b_1, \ldots, b_s) \neq 1$ for $i = 1, \ldots, n$; and $F_r(V)$ strongly discriminates V if for any finite set of words $w_i(x_1, \ldots, x_s, y_1, \ldots, y_r)$, $i = 1, \ldots, n$, that are not laws in V there exist $b_1, \ldots, b_s \in F_r(V)$ such that $w_i(b_1, \ldots, b_s, a_1, \ldots, a_r) \neq 1$ for $i = 1, \ldots, n$, where $a_1, \ldots, a_r$ are free generators of $F_r(V)$ (see [17, Sect. 17] and [9]).

DEFINITION 2.1. (1) Let G and H be two groups and $(X \mid R)$ a presentation of G. We say that H discriminates G if for any finite set of words on X, $w_i(x_1, \ldots, x_s)$, $i = 1, \ldots, n$, that are not 1 in G and any finite set of relations in R, $r_i(x_1, \ldots, x_s)$, $i = 1, \ldots, m$, there exist $h_1, \ldots, h_s \in H$ such that $w_i(h_1, \ldots, h_s) \neq 1$ for $i = 1, \ldots, n$ and $r_i(h_1, \ldots, h_s) = 1$ for $i = 1, \ldots, m$.

(2) Let G be a group, H a subgroup of G generated by Y, and $(X \cup Y \mid R)$ a presentation of G with $X \cap Y = \emptyset$. We say that H strongly discriminates G if for any finite set of words on $X \cup Y$, $w_i(x_1, \ldots, x_s, y_1, \ldots, y_t)$, $i = 1, \ldots, n$, that are not 1 in G and any finite set of relations in R, $r_i(x_1, \ldots, x_s, y_1, \ldots, y_t)$, $i = 1, \ldots, m$ (where the $x_j \in X$ and the $y_j \in Y$) there exist $h_1, \ldots, h_s \in H$ such that $w_i(h_1, \ldots, h_s, y_1, \ldots, y_t) \neq 1$ for $i = 1, \ldots, m$ and $r_i(h_1, \ldots, h_s, y_1, \ldots, y_t) = 1$ for $i = 1, \ldots, m$.

A consequence of the next lemma (and its proof) is that these definitions do not depend on the presentations.

LEMMA 2.2. (1) *Let G and H be two groups; if H discriminates G, then G satisfies the universal theory of H.*

(2) *Let G be a group and H a subgroup of G; if H strongly discriminates G, then H is existentially closed in G.*

Proof. We could prove, using Tietze transformations, that Definition 2.1 does not depend on the presentations and then we could prove (1) and (2) using simple computations. To avoid any computation and to reconcile logicians with Definition 2.1, we are going to use the compactness theorem of first-order logic. Let G be a group with a presentation $(X \mid R)$ and H a group which discriminates G. To prove that G satisfies the universal theory of H, we just have to embed G in a group H_1 with the same theory as H. We consider the following set of formulas of the language of groups with a new constant $\bar{x}$ for each $x \in X$,

$$\Delta = \{w(\bar{x}_1, \ldots, \bar{x}_s) \neq 1 \mid w(x_1, \ldots, x_s) \neq 1 \text{ in } G\} \cup \{r(\bar{x}_1, \ldots, \bar{x}_t) = 1 \mid r \in R\}.$$

Since H discriminates G, by compactness there exists a group H_1 with the same theory as H and with for each $x \in X$ an element h_x such that H_1 satisfies Δ, if we substitute $\bar{x}$ by h_x for all $x \in X$. Now by definition of Δ there exists a homomorphism f of G into H_1 (put $f(x) = h_x$ and apply von Dyck's theorem) which is a monomorphism. Now suppose that H is a subgroup of G and that H strongly discriminates G (for a given presentation $(X \cup Y \mid R)$); then we can prove as above that there exists a group H_1 which contains G and such that H_1 and H satisfies the same sentences with parameters in H; this implies that H is existentially closed in G. ∎

Lemma 2.2 improves Theorem 1 of [9]. Indeed, let V be a variety of groups and let r be a cardinal. If $F_r(V)$ discriminates V, then for any $r_1, r_2 \geq r$, $F_{r_1}(V)$ discriminates $F_{r_2}(V)$ and, by Lemma 2.2, we see that for any $r_1, r_2 \geq r$, $F_{r_1}(V)$ and $F_{r_2}(V)$ have the same universal theory. If $F_r(V)$ strongly discriminates V, then for any $r_1, r_2 \geq r$ with $r_1 \leq r_2$, $F_{r_1}(V)$ strongly discriminates $F_{r_2}(V)$ under any canonical embedding (i.e., embedding which maps free generators to free generators); by Lemma 2.2, for any $r_1, r_2 \geq r$ with $r_1 \leq r_2$, $F_{r_1}(V)$ is existentially closed in $F_{r_2}(V)$ under any canonical embedding.

An immediate consequence of Theorem 2 of [11] is that $F_2(n)$ strongly discriminates the variety of solvable groups of class $\leq n$, for any $n \geq 2$; moreover, it is easy to see that $F_1(1)$ strongly discriminates the variety of abelian groups. We obtain a result which is contained in [9].

PROPOSITION 2.3. *For any integer $n \geq 1$ and for any cardinals $r_1, r_2 \geq 2$ ($r_1, r_2 \geq 1$ if $n = 1$) with $r_1 \leq r_2$, $F_{r_1}(n)$ is existentially closed in $F_{r_2}(n)$ under any canonical embedding; in particular, $F_{r_1}(n)$ and $F_{r_2}(n)$ have the same universal theory.*

We turn our attention to wreath products. It is announced in [25] that if A_1 and A_2 are two groups with the same universal theory and B_1 and B_2 are two groups with the same universal theory, then $A_1 \operatorname{wr} B_1$ and $A_2 \operatorname{wr} B_2$ have the same universal theory. Our next lemma is a particular case of this result; however, we include a proof of this lemma for the sake of completeness.

LEMMA 2.4. *If B_1 and B_2 are two groups with the same universal theory and if A_1 and A_2 are two nontrivial torsion-free abelian groups, then $A_1 \operatorname{wr} B_1$ and $A_2 \operatorname{wr} B_2$ have the same universal theory.*

Proof. Let A be a nontrivial torsion-free abelian group and B a group. Let $b_1 f_1, \dots, b_n f_n$ be a finite set of elements of $A \operatorname{wr} B$ where the $b_i \in B$ and where the $f_i \in A^{(B)}$; $\langle b_1 f_1, \dots, b_n f_n \rangle$ is contained in $\langle b_1, \dots, b_n, f_1, \dots, f_n \rangle$. Since there is a finite number of elements of A not equal to 1 in the image of each f_i, and since a finitely generated torsion-free abelian is isomorphic to a free abelian group, we see that there exists an integer k such that $\langle b_1 f_1, \dots, b_n f_n \rangle$ can be embedded in $\mathbb{Z}^{(k)} \operatorname{wr} B$. Hence any finitely generated subgroup of $A \operatorname{wr} B$ can be embedded in $\mathbb{Z}^{(\aleph_0)} \operatorname{wr} B$ and thus $A \operatorname{wr} B$ satisfies the universal theory of $\mathbb{Z}^{(\aleph_0)} \operatorname{wr} B$. Moreover, since A is torsion-free and nontrivial, $\mathbb{Z} \operatorname{wr} B$ can be embedded in $A \operatorname{wr} B$. We see that to prove the lemma, we just have to prove that if B is a group, then $\mathbb{Z} \operatorname{wr} B$ and $\mathbb{Z}^{(\aleph_0)} \operatorname{wr} B$ have the same universal theory and that if B_1 and B_2 are two groups with the same universal theory, then $\mathbb{Z} \operatorname{wr} B_1$ and $\mathbb{Z} \operatorname{wr} B_2$ have the same universal theory.

We first show that if B is a group, then $\mathbb{Z} \operatorname{wr} B$ and $\mathbb{Z}^{(\aleph_0)} \operatorname{wr} B$ have the same universal theory. By Lemma 2.2 and since $\mathbb{Z} \operatorname{wr} B$ can be embedded in $\mathbb{Z}^{(\aleph_0)} \operatorname{wr} B$, we just have to prove that $\mathbb{Z} \operatorname{wr} B$ discriminates $\mathbb{Z}^{(\aleph_0)} \operatorname{wr} B$. Let $(b_i,\ i \in \kappa \mid s_i(\bar{b}),\ i \in \theta)$ be a presentation of B, then $\mathbb{Z} \operatorname{wr} B$ has a presentation

$$\left(a, b_i, i \in \kappa \mid s_i(\bar{b}), i \in \theta; [a^x, a^y], x, y \in B\right)$$

and $\mathbb{Z}^{(\aleph_0)} \operatorname{wr} B$ has a presentation

$$\left(a_i, i \in \aleph_0; b_i, i \in \kappa \mid s_i(\bar{b}), i \in \theta; \left[a_i^x, a_j^y\right], x, y \in B, i, j \in r\right).$$

Let $w_i(b_0, \dots, b_s, a_0, \dots, a_t)$, $i = 1, \dots, n$, be a finite set of words not equal to 1 in $\mathbb{Z}^{(\aleph_0)} \operatorname{wr} B$; for $i = 1, \dots, n$ we have

$$w_i(b_0, \dots, b_s, a_0, \dots, a_t) = v_i(b_0, \dots, b_s) \prod_{k=0}^{t} a_k^{f_{k,i}},$$

where the $f_{k,i} \in \mathbb{Z}B$ and where the v_i are words on $b_0, \dots, b_s$. If we find integers $m_0, \dots, m_t$ such that $w_i(b_0, \dots, b_s, a^{m_0}, \dots, a^{m_t}) \neq 1$ in $\mathbb{Z} \operatorname{wr} B$ for all $i = 1, \dots, n$, then we can conclude that $\mathbb{Z} \operatorname{wr} B$ discriminates $\mathbb{Z}^{(\aleph_0)} \operatorname{wr} B$ because all the relations satisfied by $b_0, \dots, b_s, a_0, \dots, a_t$ are satisfied by $b_0, \dots, b_s, a^{m_0}, \dots, a^{m_t}$. To find these integers, it is clear that we may suppose that $v_i(b_0, \dots, b_s) = 1$ for all $i = 1, \dots, n$ and we will proceed by induction on n. But before that we note that

$$w_i(b_0, \dots, b_s, a^{m_0}, \dots, a^{m_t}) = a^{m_0 f_{0,i} + \cdots + m_t f_{t,i}}$$

in $\mathbb{Z} \operatorname{wr} B$ for all $i = 1, \dots, n$. Thus we have to show that there exist integers $m_0, \dots, m_t$ such that $m_0 f_{0,i} + \cdots + m_t f_{t,i} \neq 0$ for all $i = 1, \dots, n$.

If $n = 1$, since $w_1 \neq 1$ there exists $j \in \{0, \dots, t\}$ such that $f_{j,1} \neq 0$ and we take $m_j = 1$ and $m_k = 0$ for $k \neq j$. If $n \geq 2$, then by induction hypothesis there exist integers $m'_0, \dots, m'_t, m''_0, \dots, m''_t$ such that

$$m'_0 f_{0,i} + \cdots + m'_t f_{t,i} \neq 0 \qquad \text{for } i = 1, \dots, n-1$$

and

$$m''_0 f_{0,n} + \cdots + m''_t f_{t,n} \neq 0.$$

If $m'_0 f_{0,n} + \cdots + m'_t f_{t,n} \neq 0$ then it is not difficult to see that there exists an integer p such that $(pm'_0 + m''_0) f_{0,i} + \cdots + (pm'_t + m''_t) f_{t,i} \neq 0$ for $i = 1, \dots, n$. We have proved that $\mathbb{Z} \operatorname{wr} B$ discriminates $\mathbb{Z}^{(\aleph_0)} \operatorname{wr} B$.

Let B_1 and B_2 be two groups with the same universal theory; we have to prove that $\mathbb{Z} \operatorname{wr} B_1$ and $\mathbb{Z} \operatorname{wr} B_2$ have the same universal theory. By Lemma 2.2 and by symmetry, we just have to show that $\mathbb{Z} \operatorname{wr} B_1$ discriminates $\mathbb{Z} \operatorname{wr} B_2$. If B_2 has a presentation $(b_i,\ i \in \kappa \mid R)$, then $\mathbb{Z} \operatorname{wr} B_2$ has a presentation

$$(a, b_i, i \in \kappa \mid R, [a^x, a^y], x, y \in B_2).$$

Let $w_i(b_0, \dots, b_s, a)$, $i = 1, \dots, n$, be a finite set of words not equal to 1 in $\mathbb{Z} \operatorname{wr} B_2$ and $r_i(b_0, \dots, b_s)$, $i = 1, \dots, m$, be a finite set of relations in R. For $i = 1, \dots, n$, we have

$$w_i(b_0, \dots, b_s, a) = v_i(b_0, \dots, b_s) a^{f_i} \qquad \text{with } f_i = \sum_{k=1}^{l_i} e_{k,i} v_{k,i}(b_0, \dots, b_s)$$

where the $e_{k,i} = \pm 1$ and where the $v_{k,i}$ are words on $b_0, \dots, b_s$. We are going to construct an existential sentence of the language of groups. Let $i \in \{1, \dots, n\}$. If $v_i(b_0, \dots, b_s) \neq 1$, then we put $\varphi_i(x_0, \dots, x_s) = (v_i(x_0, \dots, x_s) \neq 1)$. If $v_i(b_0, \dots, b_s) = 1$, then $f_i \neq 0$. If f_i is not in the augmentation ideal of $\mathbb{Z}B$, then we put $\varphi_i = (x_0 = x_0)$; if f_i is in the augmentation ideal of $\mathbb{Z}B$, then there exists a formula $\varphi_i(x_0, \dots, x_s)$ of the form

$$\bigwedge \bigvee v_{k_1,i}(x_0, \dots, x_s) \neq v_{k_2,i}(x_0, \dots, x_s)$$

such that if G is a group

$$\mathbb{Z}G \vDash \exists x_0 \cdots x_s \in G \sum_{k=1}^{l_i} e_{k,i} v_{k,i}(x_0, \dots, x_s) \neq 0$$

if and only if

$$G \models \exists x_0 \cdots x_s\ \varphi_i(x_0, \ldots, x_s).$$

We denote by φ the formula

$$\exists x_0 \cdots x_s \left(\bigwedge_{i=1}^{m} r_i(x_0, \ldots, x_s) = 1 \wedge \bigwedge_{i=1}^{n} \varphi_i(x_0, \ldots, x_s) \right).$$

By construction of φ, $B_2 \models \varphi$ and, since B_2 satisfies the universal theory of B_1, $B_1 \models \varphi$. Thus if we put $\mathbb{Z} \operatorname{wr} B_1 = \langle a' \rangle \operatorname{wr} B_1$, then there exist $b'_0, \ldots, b'_s \in B_1$ such that

$$w_i(b'_0, \ldots, b'_s, a') \neq 1 \qquad \text{and} \qquad r_j(b'_0, \ldots, b'_s) = 1$$

for all $i = 1, \ldots, n$ and for all $j = 1, \ldots, m$. We see that $\mathbb{Z} \operatorname{wr} B_1$ discriminates $\mathbb{Z} \operatorname{wr} B_2$. ∎

The main result of this section is

THEOREM 2.5. (1) *If $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are two finite sequences of nontrivial torsion-free abelian groups, then $\operatorname{wr}_{i=1}^{n} A_i$ and $\operatorname{wr}_{i=1}^{n} B_i$ have the same universal theory.*

(2) *If r_1 and r_2 are cardinals ≥ 2 and if A and B are two nontrivial torsion-free abelian groups, then $F_{r_1}(2)$, $M_{r_2}(2)$, and $A \operatorname{wr} B$ have the same universal theory.*

(3) *If r is a cardinal ≥ 2 and if A_1, A_2, A_3 are three nontrivial torsion-free abelian groups, then $M_r(3)$ and $\operatorname{wr}_{i=1}^{3} A_i$ have the same universal theory.*

Proof. It is well known that two nontrivial torsion-free abelian groups have the same universal theory. Thus the first part of Theorem 2.5 is a direct consequence of Lemma 2.4. If r is a cardinal ≥ 2, then $M_r(2) \simeq \mathbb{Z}^{(r)} \operatorname{wr} \mathbb{Z}^{(r)}$; thus by Lemma 2.4, Proposition 2.3, and the first part of Theorem 2.5, to prove the second part of Theorem 2.5 we just have to prove that $F_2(2)$, $M_2(2)$, and $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ have the same universal theory. By Lemma 2.4, $M_2(2)$ and $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ have the same universal theory. Since $F_2(2)$ can be embedded in $M_2(2)$ every universal sentence true in $M_2(2)$ is true in $F_2(2)$. By [1, Corollary 3], $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ can be embedded in $F_2(2)$ and thus every universal sentence true in $F_2(2)$ is true in $\mathbb{Z} \operatorname{wr} \mathbb{Z}$. Since $M_2(2)$ and $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ have the same universal theory, the second part of Theorem 2.5 is proved. If r is a cardinal ≥ 2, then $M_r(3) \simeq \mathbb{Z}^{(r)} \operatorname{wr} F_r(2)$; thus by Lemma 2.4 and by Proposition 2.3, to prove the last part of Theorem 2.5 we just have to prove that $M_2(3)$ and $\mathbb{Z} \operatorname{wr}(\mathbb{Z} \operatorname{wr} \mathbb{Z})$ have the same universal theory. By the second part of Theorem 2.5, $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ and $F_2(2)$ have the same universal theory and we can apply Lemma 2.4. ∎

The second part of Theorem 2.5 led us to study in a systematic manner the groups with the same universal theory as $F_2(2)$. In [5] we give a description of the universal theory of $F_2(2)$ and we prove that if G is a nonabelian group, then G has the same universal theory as $F_2(2)$ iff for all $g_1, \dots, g_n \in G$ there exist $k, r \in \mathbb{N}$ such that $\langle g_1, \dots, g_n \rangle$ can be embedded in $\mathbb{Z}^{(k)} \operatorname{wr} \mathbb{Z}^{(r)}$.

REMARK 2.6. A free noncyclic solvable group of class 3 and $\mathbb{Z} \operatorname{wr}(\mathbb{Z} \operatorname{wr} \mathbb{Z})$ do not have the same universal theory. Indeed, we consider the formula φ

$$\exists x y_1 y_2 z_1 z_2 z_3 \big[[x, x^{z_1}], [x, x^{z_1}]^{z_2}\big] \neq 1 \wedge \big[[y_1, y_2], [y_1, y_2]^{z_3}\big] \neq 1 \wedge \big[[y_1, y_2], [y_1, y_2]^{x}\big] = 1.$$

We put $F = F_2(3)$. Assume that F satisfies φ; then there exist $a, b_1, b_2 \in F$ such that if we put $b = [b_1, b_2]$, we have $[b, b^a] = 1$ and then we have, by the lemma of [14], $a \in F - F'$ and $b \in F' - \delta_2 F$. By [14, Theorem 1] there exist two nonzero integers n and m such that $b^n = b^{ma}$ and if we apply [1, Theorem 7] we obtain $n = ma$ in $\mathbb{Z}(F/F')$. This is absurd, thus F does not satisfy φ. Moreover, it is not very difficult to see that $\mathbb{Z} \operatorname{wr}(\mathbb{Z} \operatorname{wr} \mathbb{Z}) \models \varphi$.

3. BOUNDED AND RECURSIVELY BOUNDED ORE GROUP RINGS

Let G be a group; we say (by "abus de langage") that G is Ore if $\mathbb{Z}G$ is a domain and if for any $a, b \in \mathbb{Z}G^*$ there exist $c, d \in \mathbb{Z}G^*$ such that $ac = bd$. We define a norm $\| \ \|$ on $\mathbb{Z}G$: if $\Sigma n_g g \in \mathbb{Z}G$ then we put $\|\Sigma n_g g\| = \Sigma |n_g|$.

DEFINITION 3.1. Let G be a group. We say that G is a bounded Ore group if $\mathbb{Z}G$ is a domain and if there exists a function f of $\mathbb{N}^*$ into $\mathbb{N}$ such that for any $a, b \in \mathbb{Z}G^*$ there exist $c, d \in \mathbb{Z}G^*$ such that $ac = bd$ and such that $\|c\| \leq f(\max(\|a\|, \|b\|))$ and $\|d\| \leq f(\max(\|a\|, \|b\|))$. Further, if f can be taken recursive we say that G is a recursively bounded Ore group.

In our Definition 3.1 we impose the right Ore condition, but for group rings the left Ore condition and the right Ore condition are equivalent; moreover, this equivalence preserves our function f (this is because we have the antiautomorphism $(\Sigma n_g g)^* = \Sigma n_g g^{-1}$). Moreover, if G is torsion-free abelian, then $\mathbb{Z}G$ is a domain and it is then clear that G is a recursively bounded Ore group.

LEMMA 3.2. *Let G be a bounded Ore group with a function f. If a group satisfies the universal theory of G, then it is bounded Ore with f.*

Proof. Let G_1 be a bounded Ore group with a function f. If H is a subgroup of G_1, then $\mathbb{Z}G_1$ is a free right $\mathbb{Z}H$-module on a left transversal of H in G_1 (see [20, 1.1.3]). With this elementary fact (and its proof) it is easy to see that if H is a subgroup of G_1, then H is a bounded Ore group with the same function f. So, by [3, 5.2.2] or by [3, Sect. 3.2], to prove the lemma it suffices to prove that if G is a bounded Ore group with a function f, then if a group G_1 satisfies the theory of G it is bounded Ore with f. Let G be a bounded Ore group with a function f. If $(n, m) \in \mathbb{N}^* \times \mathbb{N}^*$ then we denote by $\varphi_{n,m}$ the sentence

$$\forall x_1 \cdots x_n \forall y_1 \cdots y_n$$

$$\bigwedge_{\mathbf{e}, \boldsymbol{\epsilon} \in I_n} \Bigg[(e_1 x_1 + \cdots + e_n x_n \neq 0 \wedge \epsilon_1 y_1 + \cdots + \epsilon_n y_n \neq 0)$$

$$\Rightarrow \Bigg(\exists z_1 \cdots z_m \exists t_1 \cdots t_m$$

$$\bigvee_{\mathbf{e}', \boldsymbol{\epsilon}' \in I_m} ((e_1 x_1 + \cdots + e_n x_n)(e'_1 z_1 + \cdots + e'_m z_m)$$

$$= (\epsilon_1 y_1 + \cdots + \epsilon_n y_n)(\epsilon'_1 t_1 + \cdots + \epsilon'_m t_m)$$

$$\wedge\, e'_1 z_1 + \cdots + e'_m z_m \neq 0 \wedge \epsilon'_1 t_1 + \cdots + \epsilon'_m z_m \neq 0) \Bigg) \Bigg],$$

where $I_k = \{(e_1, \ldots, e_k) \mid e_i \in \{-1, 0, 1\}, (e_1, \ldots, e_k) \neq (0, \ldots, 0)\}$. It is not difficult to see that this sentence is equivalent to a sentence in the language of groups. By hypothesis on G, for all $n \in \mathbb{N}^*$, $G \models \varphi_{n, f(n)}$. Moreover, it is clear that there is a set of (universal) sentences of the language of groups which say that $\mathbb{Z}G$ is a domain. Hence, if a group G_1 satisfies the theory of G, then $\mathbb{Z}G_1$ is a domain and since $G_1 \models \varphi_{n, f(n)}$ for all $n \in \mathbb{N}^*$, G_1 is bounded Ore with f. ∎

The following lemma is an application of the compactness theorem (of first-order logic); I thank Zoé Chatzidakis who suggested it.

LEMMA 3.3. *Let G be a group such that every group which satisfies the (universal) theory of G is Ore. Let $a(\mathbf{x})$ be an element of $\mathbb{Z}X$ where X is a free group on $x_1, \ldots, x_n$ and let $b(\mathbf{y})$ be an element of $\mathbb{Z}Y$ where Y is a free group on $y_1, \ldots, y_m$. Then there exists a finite set of couples $\{(c_i(\mathbf{x}, \mathbf{y}), d_i(\mathbf{x}, \mathbf{y}))\}_{i \in p}$ where the c_i and the d_i are elements of $\mathbb{Z}X * Y$ such that for all*

$g_1, \ldots, g_n, h_1, \ldots, h_m \in G$, *if* $a(\mathbf{g}) \neq 0$ *and* $b(\mathbf{h}) \neq 0$ *then* $a(\mathbf{g})c_i(\mathbf{g}, \mathbf{h}) = b(\mathbf{h})d_i(\mathbf{g}, \mathbf{h})$, $c_i(\mathbf{g}, \mathbf{h}) \neq 0$, *and* $d_i(\mathbf{g}, \mathbf{h}) \neq 0$ *for a* $i \in p$.

Proof. Let G be a group such that every group which satisfies the (universal) theory of G is Ore. Let $a(\mathbf{x})$ be an element of $\mathbb{Z}X$ where X is a free group on $x_1, \ldots, x_n$ and let $b(\mathbf{y})$ be an element of $\mathbb{Z}Y$ where Y is a free group on $y_1, \ldots, y_m$. We consider the infinite sentence

$$\forall \mathbf{x}\, \forall \mathbf{y}\big[(a(\mathbf{x}) \neq 0 \wedge b(\mathbf{y}) \neq 0) \\ \Rightarrow (\vee (a(\mathbf{x})c(\mathbf{x},\mathbf{y}) = b(\mathbf{y})d(\mathbf{x},\mathbf{y}) \wedge c(\mathbf{x},\mathbf{y}) \neq 0 \wedge d(\mathbf{x},\mathbf{y}) \neq 0))\big]$$

where the disjunction is taken over the set $\mathbb{Z}X * Y \times \mathbb{Z}X * Y$. It is easy to see that this infinite sentence can be written in the language of groups. We denote by φ this sentence. Let H be a group which satisfies the (universal) theory of G. Then by hypothesis H is Ore. Let $g_1, \ldots, g_n, h_1, \ldots, h_m \in H$ such that $a(\mathbf{g}) \neq 0$ and $b(\mathbf{h}) \neq 0$; by Lemma 3.2, $\langle g_1, \ldots, g_n, h_1, \ldots, h_m \rangle$ is an Ore group, so there exists $(c(\mathbf{x}, \mathbf{y}), d(\mathbf{x}, \mathbf{y})) \in \mathbb{Z}X * Y \times \mathbb{Z}X * Y$ such that $a(\mathbf{g})c(\mathbf{g}, \mathbf{h}) = b(\mathbf{h})d(\mathbf{g}, \mathbf{h})$, $c(\mathbf{g}, \mathbf{h}) \neq 0$, and $d(\mathbf{g}, \mathbf{h}) \neq 0$. Thus $H \models \varphi$. We have proved that any group which satisfies the (universal) theory of G satisfies φ. Hence, by compactness, there exists a finite subset $\{(c_i(\mathbf{x}, \mathbf{y}), d_i(\mathbf{x}, \mathbf{y}))\}_{i \in p}$ of $\mathbb{Z}X * Y \times \mathbb{Z}X * Y$ such that G satisfies the sentences

$$\forall \mathbf{x}\, \forall \mathbf{y}\Bigg[(a(\mathbf{x}) \neq 0 \wedge b(\mathbf{y}) \neq 0) \\ \Rightarrow \Bigg(\bigvee_{i \in p} (a(\mathbf{x})c_i(\mathbf{x},\mathbf{y}) = b(\mathbf{y})d_i(\mathbf{x},\mathbf{y}) \wedge c_i(\mathbf{x},\mathbf{y}) \neq 0 \wedge d_i(\mathbf{x},\mathbf{y}) \neq 0)\Bigg)\Bigg]. \quad \blacksquare$$

With these two lemmas we can prove the next theorem which will allow us to give "nontrivial" examples of (recursively) bounded Ore groups.

THEOREM 3.4. *Let G be a group.*

(1) *G is bounded Ore iff every group which satisfies the (universal) theory of G is Ore.*

(2) *If G can be embedded in a bounded Ore group with a decidable universal theory, then G is recursively bounded Ore (we can compute the* c_i *and the* d_i *of Lemma* 3.3).

Proof. The first part of Theorem 3.4 is an immediate consequence of Lemmas 3.2 and 3.3. By Lemma 3.2, to prove the second part of Theorem 3.4, it suffices to prove that a bounded Ore group with a decidable universal theory is recursively bounded Ore. Let G be a bounded Ore group with a decidable universal theory. Let n be an integer ≥ 1. We denote by X_n the free groups on $x_1, \dots, x_n$ and by Y_n the free groups on $y_1, \dots, y_n$. We denote by E_n an effective enumeration of $\mathbb{Z}X_n * Y_n \times \mathbb{Z}X_n * Y_n$, and if m is an integer ≥ 1, then we denote by $E_n(m)$ the m first elements of E_n. We define a function f_1 of $\mathbb{N}^*$ into $\mathbb{N}$ by

$$f_1(m) = \max\{\max\{\|c\|, \|d\|\} \mid (c, d) \in E_n(m)\}.$$

If m is an integer ≥ 1, we denote by $\varphi_{n,m}$ the following sentence which is equivalent to a universal sentence in the language of groups:

$$\forall x_1 \cdots x_n \forall y_1 \cdots y_n \bigwedge_{\mathbf{e}, \boldsymbol{\epsilon} \in I_n} \Bigg[(e_1 x_1 + \cdots + e_n x_n \neq 0 \wedge \epsilon_1 y_1 + \cdots + \epsilon_n y_n \neq 0) \Rightarrow \Bigg(\bigvee_{c, d \in E_n(m)} ((e_1 x_1 + \cdots + e_n x_n) c(\mathbf{x}, \mathbf{y}) = (\epsilon_1 y_1 + \cdots + \epsilon_n y_n) d(\mathbf{x}, \mathbf{y}) \wedge c(\mathbf{x}, \mathbf{y}) \neq 0 \wedge d(\mathbf{x}, \mathbf{y}) \neq 0) \Bigg) \Bigg]$$

where $I_k = \{(e_1, \dots, e_k) \mid e_i \in \{-1, 0, 1\}, (e_1, \dots, e_k) \neq (0, \dots, 0)\}$. Now we define a function f of $\mathbb{N}^*$ into $\mathbb{N}$ by

$$f(n) = f_1 (\text{smallest } m \in \mathbb{N}^* \text{ such that } G \models \varphi_{n,m}).$$

Since G is bounded Ore, by the first part of Theorem 3.4 and by Lemma 3.3, f is well defined. Since G has a decidable universal theory, since the E_i are effective enumerations and since the $\varphi_{i,j}$ are (effective) universal sentences, f is a recursive function. By construction of f, G is a bounded Ore group with f. ▮

Using Theorem 3.4 and classical results we obtain

COROLLARY 3.5. (1) *A right orderable solvable group is a bounded Ore group.* (2) *Any subgroup of the wreath product of two torsion-free abelian groups is a recursively bounded Ore group.*

Proof. By [20, 13.1 and 13.3.6], if G is a right orderable solvable group, then KG is an Ore domain for any commutative field K. Since being a right orderable group is a first-order property in the language of groups (use [2, 7.1.1]), we can apply Theorem 3.4 to prove the first part of Corollary 3.5.

By Theorem 2.5 and by Lemma 3.2, to prove the second part of Corollary 3.5 we just have to prove that $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ is a recursively bounded Ore group. $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ can be embedded in the group G of triangular 2×2 matrices $(\alpha_{i,j})$ where $\alpha_{1,1} \in \mathbb{R}^{*+}$, $\alpha_{1,2} = 0$, $\alpha_{2,1} \in \mathbb{R}$, and $\alpha_{2,2} = 1$. Clearly G is a torsion-free solvable group; moreover, G is an orderable group (take $\{(\alpha_{i,j}) \in G \mid \alpha_{1,1} > 1 \text{ or } \alpha_{1,1} = 1 \text{ and } \alpha_{2,1} > 0\}$ for the set of positive elements of G). Thus G is a bounded Ore group, and since $\mathbb{R}$ is a decidable field (see [8, Corollary 2.4.2]) it is not difficult to see that G is a decidable group. We can apply Theorem 3.4. ∎

Free solvable groups and iterated unrestricted wreath products of torsion-free abelian groups are examples of right orderable groups (see [2, 2.4.7 and 7.3.2]). Anyway, it has been proved recently that if G is a torsion-free elementary amenable group (see [6] for this notion) and if K is a division ring, then KG is an Ore ring (see [12]); using this deep result and Theorem 3.4 we see that if G is a torsion-free group such that G_1 is elementary amenable for any group G_1 which satisfies the theory of G, then G is bounded Ore. For example, we can prove that a *torsion-free solvable-by-finite group is bounded Ore*. Moreover, very recent results of Delon and Simonetta imply that a left-iterated wreath product of torsion-free abelian groups can be embedded in a decidable orderable solvable group (see [24]). With the above results this implies that a left-iterated wreath product of torsion-free abelian groups is recursively bounded Ore. We will prove this last statement, in the next section, without using the results of [24].

As we will see, it is not very difficult to construct an Ore group which is not a bounded Ore group, but we have not been able to construct a bounded Ore group which is not a recursively bounded Ore group.

EXAMPLE 3.6. *There exists an Ore group which is not a bounded Ore group (the group of the example is locally nilpotent and orderable).*

Proof. Let F be a free group on a and b. If n is an integer ≥ 1 then we put $G_n = F/\gamma_{n+1}F$ and we denote by a_n and b_n the image of a and b in G_n; G_n is a free nilpotent group of class n on a_n and b_n. We denote by G the direct product of the G_n and we identify G_n and its image in G. Clearly G is locally nilpotent and torsion-free, thus G is orderable (see [2, 1.3.2(c) and p. 37]). By [20, 13.1 and 13.3.6] G is an Ore group. Let u be a nonprincipal ultrafilter on $\mathbb{N}^*$; we put $H = G^u$. By the fundamental theorem of ultraproducts, H and G have the same theory; moreover, it is well known that F is not an Ore group (see [20, p. 598]). Thus, by Lemma 3.2, to prove that G is not a bounded Ore group it suffices to prove that H contains a group isomorphic to F. Let x and y be the elements of H defined by the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$. Let $w(x,y)$ be a word on x and y. If $w(x,y)=1$, then clearly $\{n\geq 1 \mid w(a,b)\in \gamma_{n+1}F\}\in u$. Since u is not principal we have $w(a,b)\in \bigcap_{n\geq 1}\gamma_{n+1}F$. But a free group is residually nilpotent (i.e., $\bigcap_{n\geq 1}\gamma_{n+1}F=1$; see [21, 6.1.10]), thus $w(a,b)=1$. We see that x and y freely generate a free group. ∎

Let R be an Ore domain. We define an equivalence relation $\sim$ on $R\times R^*$ by $(x,y)\sim(z,t)$ iff there exist $u,v\in R^*$ such that $xu=zv$ and $yu=tv$. With this equivalence relation we obtain the right ring of quotients of R: $R\times R^*/\sim$, which is a division ring (in a similar way we obtain the left ring of quotients of R). Using this equivalence relation, we can see that systems of equations in an Ore domain are really tractable (see [19]).

The relevance of Definition 3.1 for systems of equations and inequations is contained in

THEOREM 3.7. *Let G be a bounded Ore group. Then there exists a function $\bar{f}$ of $\mathbb{N}^*\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}^*$ into $\mathbb{N}$ such that if*

$$\begin{cases} \alpha_{i,1}X_1+\cdots+\alpha_{i,r}X_r=0, & i=1,\dots,t\\ \beta_{i,1}X_1+\cdots+\beta_{i,r}X_r\neq 0, & i=1,\dots,s,\end{cases} \qquad (*)$$

is a nontrivial left-hand system of equations and inequations in $\mathbb{Z}G$ with a solution, then $(*)$ has a solution $(x_1,\dots,x_r)$ in $\mathbb{Z}G$ such that

$$\|x_i\|\leq \bar{f}(r,t,s,l) \qquad \text{for } i=1,\dots,r$$

where $l=\max\{1,\|\alpha_{i,j}\| \mid \text{for } i=1,\dots,t \text{ and } j=1,\dots,r\}$. Furthermore, if G is a recursively bounded Ore group then $\bar{f}$ can be taken recursive.

Proof. Let G be a bounded Ore group. We have a function f of $\mathbb{N}^*$ into $\mathbb{N}$ such that for any $a,b\in\mathbb{Z}G^*$ there exist $c,d\in\mathbb{Z}G^*$ with $ac=bd$ and $\|c\|,\|d\|\leq f(\max(\|a\|,\|b\|))$. We define $\bar{f}$ by

$$\bar{f}(r,0,s,l)=1 \qquad \text{for any } r,l\geq 1 \text{ and any } s\geq 0$$

$$\bar{f}(r,t,0,l)=0 \qquad \text{for any } r,t,l\geq 1$$

$$\bar{f}(r,1,1,l)=2rf(l) \qquad \text{for any } r,l\geq 1$$

$$\bar{f}(r,t+1,1,l)=\bar{f}(r,t,1,2lf(l))f(rl\bar{f}(r,t,1,2lf(l))) \qquad \text{for any } r,t,l\geq 1$$

$$\bar{f}(r,t,s+1,l)=(s+1)\bar{f}(r,t,s,l)+\bar{f}(r,t,1,l) \qquad \text{for any } r,t,s,l\geq 1.$$

We see that if f is recursive, then $\bar{f}$ is also recursive. Let $(*)$ be a system of equations and inequations as in Theorem 3.7. We suppose that $(*)$ has a solution. If $s = 0$ or if $t = 0$ there is nothing to prove. We begin by supposing that $s = 1$ and that $t \geq 1$ and we are going to prove Theorem 3.7, in this case, by induction on t.

If $t = 1$, then we can write $(*)$ in the form

$$\begin{cases} \alpha_1 X_1 + \cdots + \alpha_p X_p = 0 \\ \beta_1 X_1 + \cdots + \beta_p X_p + \beta_{p+1} X_{p+1} + \cdots + \beta_r X_r \neq 0 \end{cases} \qquad (*)$$

with $\alpha_1 \neq 0, \ldots, \alpha_p \neq 0$ and $1 \leq p \leq r$. If $p = 1$, since $\mathbb{Z}G$ is a domain there is no problem. Thus we suppose that $p \geq 2$. Since $\alpha_i \neq 0$ for $i = 1, \ldots, p$ there exist $a_2, b_2, \ldots, a_p, b_p \in \mathbb{Z}G$ such that $\alpha_1 a_i = \alpha_i b_i$ and $\|a_i\|, \|b_i\| \leq f(l)$ for $i = 2, \ldots, p$. We put

$$\mathbf{x}_1 = (-a_2 - \cdots - a_p, b_2, \ldots, b_p, 0, \ldots, 0)$$

$$\mathbf{x}_i = (-a_2 - \cdots - 2a_i - \cdots - a_p, b_2, \ldots, 2b_i, \ldots, b_p, 0, \ldots, 0)$$

for $i = 2, \ldots, p$. It is clear that $\mathbf{x}_1, \ldots, \mathbf{x}_p$ are solutions of the equation of $(*)$. If one of the $\mathbf{x}_i$ is a solution of the inequation of $(*)$, then we have a solution $(c_1, \ldots, c_r)$ of $(*)$ in $\mathbb{Z}G$ such that $\|c_i\| \leq 2rf(l) = \bar{f}(r, 1, 1, l)$ for $i = 1, \ldots, r$. We thus suppose that none of the $\mathbf{x}_i$ is a solution of the inequation of $(*)$ and we are going to prove that if $(c_1, \ldots, c_p)$ is a solution in $\mathbb{Z}G$ of the equation of $(*)$ then $\beta_1 c_1 + \cdots + \beta_p c_p = 0$; this implies that $(*)$ has a solution of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ and the case $s = t = 1$ will be settled. None of the $\mathbf{x}_i$ is a solution of the inequation of $(*)$, thus

$$(\beta_2 b_2 - \beta_1 a_2) + \cdots + (\beta_j b_j - \beta_1 a_j) + \cdots + (\beta_p b_p - \beta_1 a_p) = 0$$

$$(\beta_2 b_2 - \beta_1 a_2) + \cdots + 2(\beta_i b_i - \beta_1 a_i) + \cdots + (\beta_p b_p - \beta_1 a_p) = 0$$

for $i = 2, \ldots, p$. Since $\mathbb{Z}G$ is a domain, these equations give $\beta_i b_i = \beta_1 a_i$ for $i = 2, \ldots, p$. If one of the β_i, for $i = 1, \ldots, p$, is equal to zero, then since the a_i and the b_i are not equal to zero all of the β_i, for $i = 1, \ldots, p$, are equal to zero and we obtain what we want. We thus suppose that $\beta_i \neq 0$ for $i = 1, \ldots, p$. We have $\alpha_i b_i = \alpha_1 a_i$ for $i = 2, \ldots, p$. Thus $(\beta_i, \alpha_i) \sim (\beta_1, \alpha_1)$ for $i = 2, \ldots, p$ where $\sim$ is the equivalence relation on $\mathbb{Z}G \times \mathbb{Z}G^*$ with which we define the right ring of quotients of an Ore ring. Since $\sim$ is an equivalence relation, we have $(\beta_i, \alpha_i) \sim (\beta_j, \alpha_j)$ for all $i, j \in \{1, \ldots, p\}$. We are going to prove, by induction on $k \in \{1, \ldots, p\}$, that there exist $\theta_{1,k}, \theta_{2,k} \in \mathbb{Z}G^*$ such that $\theta_{1,k}\alpha_i = \theta_{2,k}\beta_i$ for $i = 1, \ldots, k$. If $k = 1$, then since G is an Ore group and since $\beta_1 \neq 0$ and $\alpha_1 \neq 0$ there exist $\theta_{1,1}, \theta_{2,1} \in \mathbb{Z}G^*$ such that $\theta_{1,1}\alpha_1 = \theta_{2,1}\beta_1$. If $k \geq 2$, then by the induction hypothesis we have $\theta_1 = \theta_{1,k-1}$ and $\theta_2 = \theta_{2,k-1}$ such that $\theta_1\alpha_i = \theta_2\beta_i$ for $i = 1,\dots,k-1$. We consider the system

$$\begin{cases} \theta_1\alpha_{k-1}X_{k-1} + \theta_1\alpha_k X_k = 0 \text{ (a)} \\ \theta_2\beta_{k-1}X_{k-1} + \theta_2\beta_k X_k = 0 \text{ (b)}. \end{cases} \qquad (*')$$

Let $c_1, c_2 \in \mathbb{Z}G^*$ such that $\theta_1\alpha_{k-1}c_1 = \theta_1\alpha_k c_2$. It is clear that

$$(\theta_2\beta_{k-1}c_1, \theta_1\alpha_{k-1}c_1) \sim (\theta_2\beta_{k-1}, \theta_1\alpha_{k-1})$$
$$\text{and } (\theta_2\beta_k c_2, \theta_1\alpha_k c_2) \sim (\theta_2\beta_k, \theta_1\alpha_k).$$

Since $(\beta_{k-1}, \alpha_{k-1}) \sim (\beta_k, \alpha_k)$, we have $(\theta_2\beta_{k-1}, \theta_1\alpha_{k-1}) \sim (\theta_2\beta_k, \theta_1\alpha_k)$ and thus we obtain

$$(\theta_2\beta_{k-1}c_1, \theta_1\alpha_{k-1}c_1) \sim (\theta_2\beta_k c_2, \theta_1\alpha_k c_2).$$

Thus we have $u_1, u_2 \in \mathbb{Z}G^*$ such that

$$\theta_2\beta_{k-1}c_1u_1 = \theta_2\beta_k c_2u_2 \qquad \text{and} \qquad \theta_1\alpha_{k-1}c_1u_1 = \theta_2\alpha_k c_2u_2.$$

Since $\beta_1\alpha_{k-1}c_1 = \theta_1\alpha_k c_2$, we have $u_1 = u_2$ and then clearly $\theta_2\beta_{k-1}c_1 = \theta_2\beta_k c_2$. We have proved that any nontrivial solution of (a) is a solution of (b) and moreover that the coefficients of $(*')$ are not equal to zero; thus, by [19, pp. 471–473], the left-hand Ore determinant of $(*')$ is not equal to zero and thus there exist $v_1, v_2 \in \mathbb{Z}G^*$ such that $v_1\theta_1\alpha_{k-1} = v_2\theta_2\beta_{k-1}$ and $v_1\theta_1\alpha_k = v_2\theta_2\beta_k$. We just have to put $\theta_{1,k} = v_1\theta_1$ and $\theta_{2,k} = v_2\theta_2$. Now, for $k = p$, we have $\theta_{1,p}, \theta_{2,p} \in \mathbb{Z}G^*$ such that $\theta_{1,p}\alpha_i = \theta_{2,p}\beta_i$ for $i = 1,\dots,p$. We see that if $(c_1,\dots,c_p)$ is a solution in $\mathbb{Z}G$ of the equation of $(*)$ then $\beta_1c_1 + \cdots + \beta_pc_p = 0$. We have proved Theorem 3.7 in the case where $s = t = 1$.

Now we suppose that $t \geq 2$ (and that $s = 1$). We can suppose that $(*)$ is of the form

$$\begin{cases} \alpha_{i,1}X_1 + \cdots + \alpha_{i,r}X_r = 0, \qquad i = 1,\dots,t \\ \beta_1X_1 + \cdots + \beta_rX_r \neq 0 \end{cases}$$

with $\alpha_{1,1} \neq 0$. Let $i \in \{2,\dots,t\}$; if $\alpha_{i,1} \neq 0$, then there exist $a_i, b_i \in \mathbb{Z}G^*$ such that $a_i\alpha_{i,1} = b_i\alpha_{1,1}$ and such that $\|a_i\|, \|b_i\| \leq f(l)$, and we denote by (i) the equation

$$(a_i\alpha_{i,1} - b_i\alpha_{1,2})X_2 + \cdots + (a_i\alpha_{i,r} - b_i\alpha_{1,r})X_r = 0;$$

if $\alpha_{i,1} = 0$, then we denote by (i) the equation $\alpha_{i,2}X_2 + \cdots + \alpha_{i,r}X_r = 0$. If $\beta_1 \neq 0$, then there exist $c, d \in \mathbb{Z}G^*$ such that $c\beta_1 = d\alpha_{1,1}$, and we denote by $(t+1)$ the inequation

$$(c\beta_2 - d\alpha_{1,2})X_2 + \cdots + (c\beta_r - d\alpha_{1,r})X_r \neq 0;$$

if $\beta_{1,1} = 0$, then we denote by $(t+1)$ the inequation $\beta_2 X_2 + \cdots + \beta_r X_r \neq 0$. We have

$$(*) \Leftrightarrow \begin{cases} \alpha_{1,1}X_1 = -\alpha_{1,2}X_2 - \cdots - \alpha_{1,r}X_r \\ (i) \qquad i = 2,\dots,t \\ (t+1), \end{cases}$$

and we denote by $(*')$ the system

$$\begin{cases} (i) \qquad i = 2,\dots,t \\ (t+1). \end{cases}$$

Since $(*)$ has a solution in $\mathbb{Z}G$, $(*')$ has a solution in $\mathbb{Z}G$; thus by the induction hypothesis we have a solution $(x_2,\dots,x_r)$ of $(*')$ in $\mathbb{Z}G$ such that

$$\|x_i\| \le \bar{f}(r, t-1, 1, 2lf(l)).$$

It is then clear that $\mathbf{x} = (-\alpha_{1,1}^{-1}(\alpha_{1,2}x_2 + \cdots + \alpha_{1,r}x_r), x_2,\dots,x_r)$ is a solution of $(*)$ in the left ring of quotients of $\mathbb{Z}G$. If $\alpha_{1,2}x_2 + \cdots + \alpha_{1,r}x_r = 0$, then $\mathbf{x}$ is a solution of $(*)$ in $\mathbb{Z}G$ and clearly this solution satisfies the required increase. If $\alpha_{1,2}x_2 + \cdots + \alpha_{1,r}, x_r \neq 0$, then there exist $\mu, \lambda \in \mathbb{Z}G^*$ such that

$$(\alpha_{1,2}x_2 + \cdots + \alpha_{1,r}x_r)\mu = \alpha_{1,1}\lambda$$

and such that $\|\mu\|, \|\lambda\| \le f(rl\bar{f}(r, t-1, 1, 2lf(l)))$. Then

$$(x'_1,\dots,x'_r) = (-\mu, x_2\lambda,\dots,x_r\lambda)$$

is a solution of $(*)$ in $\mathbb{Z}G$ such that

$$\|x'_i\| \le \bar{f}(r, t-1, 1, 2lf(l)) f\big(rl\bar{f}(r, t-1, 1, 2lf(l))\big) = \bar{f}(r, t, 1, l)$$

for all $i \in \{1,\dots,r\}$.

We have now to prove Theorem 3.7 for $t \ge 1$ and $s \ge 1$. We proceed by induction on s. The case $s = 1$ has already been settled; thus we suppose that $s \ge 2$. By the induction hypothesis, we have a solution $(x_1,\dots,x_r)$ in $\mathbb{Z}G$ of

$$\begin{cases} \alpha_{i,1}X_1 + \cdots + \alpha_{i,r}X_r = 0 & i = 1,\dots,t \\ \beta_{i,1}X_1 + \cdots + \beta_{i,r}X_r \neq 0 & i = 1,\dots,s-1 \end{cases}$$

such that $\|x_i\| \le \bar{f}(r, t, s-1, l)$ for $i = 1,\dots,r$; and we have a solution

$(y_1, \ldots, y_r)$ in $\mathbb{Z}G$ of

$$\begin{cases} \alpha_{i,1}X_1 + \cdots + \alpha_{i,r}X_r = 0 & i = 1, \ldots, t \\ \beta_{s,1}X_1 + \cdots + \beta_{s,r}X_r \neq 0 \end{cases}$$

such that $\|y_i\| \leq \bar{f}(r, t, 1, l)$ for $i = 1, \ldots, r$. Now it is not very difficult to see that if $(x_1, \ldots, x_r)$ is not a solution of $(*)$, then there exists an integer $n \in \{0, \ldots, s\}$ such that $(nx_1 + y_1, \ldots, nx_r + y_r)$ is a solution of $(*)$. We see that we have a solution $(z_1, \ldots, z_r)$ of $(*)$ such that

$$\|z_i\| \leq s\bar{f}(r, t, s-1, l) + \bar{f}(r, t, 1, l)$$

for $i = 1, \ldots, r$. ∎

4. SOME DECIDABLE UNIVERSAL THEORIES

Malcev proved that a noncyclic free solvable group has an undecidable theory (see [14]). More recently, Noskov in [18] proved that a finitely generated solvable-by-finite group is decidable iff it is abelian-by-finite. This implies that the wreath product of a nontrivial finitely generated solvable group by an infinite finitely generated solvable group has an undecidable theory.

Matiyasevich's theorem (see [16]) implies that there is no algorithm which decides if a diophantine equation has a solution or not (i.e., Hilbert's tenth problem for the ring of the integers has a negative answer): using this theorem it is possible to obtain undecidability results for the universal theories of certain algebraic structures. Roman'kov proved in [22] that there exists a finitely generated metabelian nilpotent group with an undecidable universal theory and that the decidability of the universal theory of a noncyclic free nilpotent group of class 2 is equivalent to the decidability of the universal theory of the field of the rationals (this last assertion is equivalent to Hilbert's tenth problem for the field of the rationals which is open, and it is possible to prove that the same result holds for any class ≥ 2). As has been remarked by Ershov in [7], it is not difficult to deduce from [15] that the universal theory of a noncyclic free nilpotent group of class ≥ 2 with two constants for two free generators is undecidable. Roman'kov proved in [23] that there is no algorithm which decides if a formula of the form $\exists x_1 \ldots, x_s \; w(x_1, \ldots, x_s) = v(a, b)$ is true or not in a noncyclic free metabelian group where a and b are two free generators. In [4], we prove some undecidability results for free solvable groups: (i) the universal theory of a noncyclic free solvable group of class

≥ 2 with two constants for two free generators is undecidable; (ii) a noncyclic free solvable group of class ≥ 2 has an undecidable $\forall\exists$-theory; (iii) if the universal theory of the field of the rationals is undecidable, then the universal theory of a noncyclic free solvable group of class ≥ 3 is undecidable.

A free solvable group and an iterated wreath product of free abelian groups have a solvable word problem (for free solvable groups see [10, Corollary 2.2] for example). This implies (with Proposition 2.3) that there exists an algorithm which decides if a formula φ of the form, $\forall x_1 \cdots x_s\, w(x_1, \ldots, x_s, a_{i_1}, \ldots, a_{i_t}) = 1$ is true or not in a given free solvable group $F_r(n)$ where the a_i, $i \in r$, are free generators. Indeed, by Proposition 2.3, we have $F_r(n) \models \varphi$ iff $F_{\aleph_0}(n) \models \varphi$ (we may suppose that $r \geq 2$); there exist free generators of $F_{\aleph_0}(n)$, $a_{j_1}, \ldots, a_{j_s}$, such that $\{a_{i_1}, \ldots, a_{i_t}\} \cap \{a_{j_1}, \ldots, a_{j_s}\} = \varnothing$. Since $F_{\aleph_0}(n)$ is a relatively free group, we have $F_{\aleph_0}(n) \models \varphi$ iff $F_{\aleph_0}(n) \models w(a_{j_1}, \ldots, a_{j_s}, a_{i_1}, \ldots, a_{i_t}) = 1$, and since $F_{\aleph_0}(n)$ has a solvable word problem we can conclude.

All our decidability results are based on the next theorem which can be seen as a corollary of Theorem 3.7.

THEOREM 4.1. *If A is a torsion-free abelian group and if B is a bounded Ore group with a decidable universal theory, then $A \operatorname{wr} B$ has a decidable universal theory.*

Proof. Let B be a bounded Ore group with a decidable universal theory. By Theorem 3.4, B is recursively bounded Ore. By Lemma 2.4, to prove the theorem we just have to prove that $\mathbb{Z} \operatorname{wr} B$ has a decidable universal theory. By Theorem 3.7, since B is a recursively bounded Ore group, we have a recursive function $\bar{f}$ with the property described in Theorem 3.7. Since B has a decidable universal theory, to prove Theorem 4.1 we just have to give a uniform procedure which associates to any existential sentence φ of the language of groups an existential sentence θ of the language of groups such that $\mathbb{Z} \operatorname{wr} B \models \varphi$ iff $B \models \theta$. Let φ be an existential sentence of the language of groups; we can suppose that φ is of the form

$$\exists x_1 \cdots x_n \left(\bigwedge_{i=1}^{m} w_i(x_1, \ldots, x_n) = 1 \wedge \bigwedge_{i=m+1}^{l} w_i(x_1, \ldots, x_n) \neq 1 \right).$$

If we use the matrix representation of $\mathbb{Z} \operatorname{wr} B$, we can write any elements of $\mathbb{Z} \operatorname{wr} B$ in the form (b, h) where $b \in B$ and $h \in \mathbb{Z}B$ and we have

$$1 = (1, 0) \qquad \text{and} \qquad (b_1, h_1)(b_2, h_2) = (b_1 b_2, h_1 b_2 + h_2).$$

Thus $\mathbb{Z} \operatorname{wr} B \models \varphi$ iff $\exists b_1, \ldots, b_n \in B\ \exists h_1, \ldots, h_n \in \mathbb{Z}B$,

$$\bigwedge_{i=1}^{m} w_i((b_1, h_1), \ldots, (b_n, h_n)) = (1,0) \wedge \bigwedge_{i=m+1}^{l} w_i((b_1, h_1), \ldots, (b_n, h_n)) \neq (1,0).$$

We can effectively compute (formally) some words $u_{i,j}$ of $\mathbb{Z}X$ where X is the free group on $x_1, \ldots, x_n$ such that $\mathbb{Z} \operatorname{wr} B \models \varphi$ iff $\exists b_1, \ldots, b_n \in B$ $\exists h_1, \ldots, h_n \in \mathbb{Z}B$

$$\bigwedge_{i=1}^{m} (w_i(b_1, \ldots, b_n), h_1 u_{i,1}(b_1, \ldots, b_n) + \cdots + h_n u_{i,n}(b_1, \ldots, b_n)) = (1,0) \wedge \bigwedge_{i=m+1}^{l} (w_i(b_1, \ldots, b_n), h_1 u_{i,1}(b_1, \ldots, b_n) + \cdots + h_n u_{i,n}(b_1, \ldots, b_n)) \neq (1,0);$$

and thus $\mathbb{Z} \operatorname{wr} B \models \varphi$ iff $\exists b_1, \ldots, b_n \in B$, $\exists h_1, \ldots, h_n \in \mathbb{Z}B$,

$$\bigwedge_{i=1}^{m} (w_i(b_1, \ldots, b_n) = 1 \wedge h_1 u_{i,1}(b_1, \ldots, b_n) + \cdots + h_n u_{i,n}(b_1, \ldots, b_n) = 0) \wedge \bigwedge_{i=m+1}^{l} (w_i(b_1, \ldots, b_n) \neq 1 \vee h_1 u_{i,1}(b_1, \ldots, b_n) + \cdots + h_u u_{i,n}(b_1, \ldots, b_n) \neq 0).$$

We see that the truth of φ in $\mathbb{Z} \operatorname{wr} B$ depends only on the truth of a finite number of sentences of the form

$$\exists b_1, \ldots, b_n \in B\ \exists h_1, \ldots, h_n \in \mathbb{Z}B \left[\left(\bigwedge w_i(\mathbf{b}) = 1 \right) \wedge \left(\bigwedge w_i(\mathbf{b}) \neq 1 \right) \wedge \left(\bigwedge_{i=1}^{t} h_1 \alpha_{i,1}(\mathbf{b}) + \cdots + h_n \alpha_{i,n}(\mathbf{b}) = 0 \right) \wedge \left(\bigwedge_{i=t+1}^{t+s} h_1 \beta_{i,1}(\mathbf{b}) + \cdots + h_n \beta_{i,n}(\mathbf{b}) \neq 0 \right) \right]$$

where the $\alpha_{i,j}$ and the $\beta_{i,j}$ are words of $\mathbb{Z}X$; we denote this sentence by φ'. If we put $k = \max\{1, \|\alpha_{i,j}\|\}$ and $p = \bar{f}(n, t, s, k)$; then, by Theorem

3.7, φ' is true iff the following formula is true:

$$\exists b_1, \ldots, b_n \in B \ \exists c_{1,1}, \ldots, c_{p,1}, \ldots, c_{1,n}, \ldots, c_{p,n} \in B$$

$$\Bigg[\Big(\big(\bigwedge w_i(\mathbf{b}) = 1\big) \wedge \big(\bigwedge w_i(\mathbf{b}) \neq 1\big)\Big)$$

$$\wedge \left(\bigvee \left(\left(\bigwedge_{i=1}^{t} \left(\sum_{j=1}^{p} e_{j,1} c_{j,1} \right) \alpha_{i,1}(\mathbf{b}) + \cdots + \left(\sum_{j=1}^{p} e_{j,n} c_{j,n} \right) \alpha_{i,n}(\mathbf{b}) = 0 \right) \right.\right.$$

$$\left.\left. \wedge \left(\bigwedge_{i=t+1}^{t+s} \left(\sum_{j=1}^{p} e_{j,1} c_{j,1} \right) \beta_{i,1}(\mathbf{b}) + \cdots + \left(\sum_{j=1}^{p} e_{j,n} c_{j,n} \right) \beta_{i,n}(\mathbf{b}) \neq 0 \right) \right) \right) \Bigg]$$

where the disjunction is taken over the set

$$\{(e_{1,1}, \ldots, e_{p,1}, \ldots, e_{1,n}, \ldots, e_{p,n}) \mid e_{i,j} \in \{-1, 0, 1\}\}$$

(here we have a right-hand system, but since we have the antiautomorphism $*$ of $\mathbb{Z}B$: $(\Sigma n_g g)^* = \Sigma n_g g^{-1}$, we can apply Theorem 3.7). Now it is not difficult to see that the previous sentence is equivalent to an existential sentence in the language of groups, and thus we have an existential sentence φ'' such that φ' is true iff $B \models \varphi''$. Since the function $\bar{f}$ is recursive, the formula φ'' can be effectively constructed. ∎

A trivial group has a decidable universal theory, and a left-iterated wreath product of the torsion-free abelian group is orderable (see [2, 2.1.1]). Hence, using Theorem 3.4, Corollary 3.5 (1), Theorem 4.1, and an induction, we obtain

THEOREM 4.2. *If $A_1, \ldots, A_n$ is a finite sequence of torsion-free abelian groups then the left-iterated wreath product $\mathrm{wr}_{i=1}^{n} A_i$ has a decidable universal theory (and any subgroup of $\mathrm{wr}_{i=1}^{n} A_i$ is recursively bounded Ore).*

Using Theorem 2.5, we solve our original problem (of course we can extract an "elementary" proof of this result).

THEOREM 4.3. *A free metabelian group has a decidable universal theory (and a Magnus group of class 3 has a decidable universal theory).*

Of side interest to these results is the fact that if G is a subgroup of a left-iterated wreath product of torsion-free abeian groups (for example, if G is a free solvable group) then we can compute the c_i and the d_i of Lemma 3.2.

Let us have a slight disgression. We put $F = F_2(2)$. We have an existential formula $\theta(x)$ such that $a \in \gamma_3 F$ iff $F \models \theta(a)$ (see [23] for example); thus using Theorem 4.3, if we can find a universal formula $\phi(x)$ such that $a \in \gamma_3 F$ iff $F \models \phi(a)$, then we can easily prove the decidability of the universal theory of a free nilpotent group of class 2 and rank 2 (by the proposition of [22] this last assertion implies the decidability of the universal theory of the field of the rational numbers). Moreover, if there existed an existential formula $\varphi(x, y)$ consistent with F such that if $F \models \varphi(a, b)$ then a and b are linearly independent modulo F', then, using Theorem 4.3, we could prove (with a nontrivial but classical argument) the decidability of the universal theory of the field of the rational numbers. To reassure the reader, we will prove

PROPOSITION 4.4. *Let r be a cardinal* ≥ 2.

(1) *There is no universal formula* $\phi(x)$ *such that* $a \in \gamma_3 F_r(2)$ *iff* $F_r(2) \models \phi(a)$.

(2) *There is no existential formula* $\varphi(x, y)$ *consistent with* $F_r(2)$ *such that if* $F_r(2) \models \varphi(a, b)$, *then a and b are linearly independent modulo* $F_r(2)'$.

Proof. We put $F = F_r(2)$. Suppose that there exists a formula $\phi(x)$ such that $a \in \gamma_3 F$ iff $F \models \phi(a)$. We denote by $\varphi(x, y)$ the formula $\neg \phi([x, y])$. Clearly $\varphi(x, y)$ is consistent with F. If we have $a, b \in F$ such that $F \models \varphi(a, b)$, then $[a, b] \neq 1$ in $F/\gamma_3 F$ and it is then easy to see that a and b are linearly independent modulo F'. If ϕ is universal then φ is existential, thus the second part of Proposition 4.4 implies the first part.

We suppose that we have an existential formula $\varphi(x, y)$ consistent with F such that if $F \models \varphi(a, b)$, then a and b are linearly independent modulo F'. Since $\varphi(x, y)$ is consistent with F, we have $F \models \exists x \exists y\, \varphi(x, y)$. Let a_1 and a_2 be two free generators of F. By [1, Corollary 3], the subgroup $G = \langle a_1, [a_1, a_2] \rangle$ of F is isomorphic to $\mathbb{Z} \operatorname{wr} \mathbb{Z}$. By Theorem 2.5, $G \models \exists x \exists y\, \varphi(x, y)$ and thus we have $a, b \in G$ such that $G \models \varphi(a, b)$. Since $\varphi(x, y)$ is existential, $F \models \varphi(a, b)$ and a and b are linearly independent modulo F'. But this is absurd because $a = a_1^n c_1$ and $b = a_1^m c_2$ for some integers n and m and for some $c_1, c_2 \in F'$. ∎

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