ELSEVIER Journal of Pure and Applied Algebra 131 (1998) 13–24

JOURNAL OF PURE AND APPLIED ALGEBRA

On the theories of free solvable groups[1]

Olivier Chapuis *

Institut Girard Desargues - CNRS, Bâtiment des Mathématiques, Université Lyon I, 43, boulevard du 11 Novembre 1918, 69622 Villeurbanne Cedex, France

Communicated by M.-F. Roy; received 19 December 1994; received in revised form 6 March 1997

Abstract

We prove that the terms of the derived series of a free solvable group are definable by existential formulae. We use this result to prove some 'model theoretic' results about free solvable groups. For example, we prove that if Hilbert's 10th problem has a negative answer for the field of the rationals, then the universal theory of a noncyclic free solvable group of class ≥ 3 is undecidable. © 1998 Elsevier Science B.V. All rights reserved.

AMS Classification: 20F10; 20A15; 03C60; 20F16

1. Introduction and results

In 1960, Malcev [11] proved that a noncyclic free solvable group of class ≥ 2 has an undecidable theory. One of the crucial points of the proof of this result is that we can define by a (first-order) formula of the language of groups $\mathscr{L} = \{., ^{-1}, 1\}$ the derived subgroup of a free solvable group. More precisely, Malcev exhibits a sequence $(\mu_i(x))_{i\geq 0}$ of universal formulae of $\mathscr{L}$ with one free variable such that if F is a noncyclic free solvable group of class $n \geq 2$, then for $k = 1, \ldots, n$ the kth term of the derived series $\delta_k F$ is defined by $\mu_{n-k}(x)$ (i.e., $\delta_k F = \{g \in F \mid F \models \mu_{n-k}(g)\}$). In this paper, using results of [11, 2], we will prove the following result which allows us to prove some 'model theoretic' results about free solvable groups and which answers a question of G. Sabbagh.

Theorem A. *There exists (and we will effectively construct) a sequence $(\theta_i(x))_{i\geq 0}$ of existential formulae of $\mathscr{L}$ such that if F is a noncyclic free solvable group of class $n \geq 2$, then for $k = 1, \ldots, n$ the kth term of the derived series of F is defined by $\theta_{n-k}(x)$.*

* E-mail: chapuis@desargues.univ-lyon1.fr.

[1] This paper is a version of Chapter III of the author's doctoral thesis [4].

PII: S0022-4049(98)00024-3

Before stating some applications of this result let us note that a result of Rhemtulla [18, Corollary 3] implies that the derived subgroup of a finitely generated solvable group of class ≤ 3 is definable by an existential formula of the form

$$\exists y_1 \ldots y_{2p} \quad x = [y_1, y_2] \cdots [y_{2p-1}, y_{2p}],$$

where p is an integer which depends on the number of generators (the case of solvable groups of class ≥ 4 is open). However, it is not possible to define in this way the derived subgroup of a free solvable group of class ≥ 2 and infinite rank, neither the second term of the derived series of a free solvable group of class ≥ 3 and rank ≥ 3 (see [23]).

One of the outstanding questions in the model theory of absolutely free groups is whether two noncyclic free groups of finite rank have the same theory. It follows easily from [11, Lemma] that two free solvable groups of class $n \geq 1$ and different finite rank do not have the same $\forall\exists\forall$ positive theory (see [20]); and using the result of Rhemtulla quoted above it is easy to prove that two free solvable groups of class $1 \leq n \leq 3$ and different finite rank do not have the same $\forall\exists$ positive theory. Moreover, it is known that two noncyclic free solvable groups of the same class have the same universal theory (see [8]). Using Theorem A we will prove

Corollary B1. *Two free solvable groups of class $n \geq 1$ and different finite rank do not have the same $\forall\exists$ theory.*

We turn our attention to algorithmic problems. In [22], Roman'kov proved that there is no algorithm which decides if a formula of the form

$$\exists x_1 \ldots x_s \quad w(x_1, \ldots, x_s) = v(a, b)$$

is true (or not) in a noncyclic free metabelian group where a and b are two free generators. We will see that this result holds for free solvable groups of class 3 (using the result of Rhemtulla and the fact that a canonical embedding between two noncyclic free solvable groups of the same class is existentially closed). Whether this result holds for solvable groups of class ≥ 4 seems to be open. Using Theorem A and the result of Roman'kov we will prove

Corollary B2. *Let F be a noncyclic free solvable group of class $n \geq 2$ and let a and b be two free generators of F. There is no algorithm which decides if a system of $max(1, 2^{n-3})$ equations with parameters a and b has a solution in F. In particular, in the language of groups with two new constants for two free generators, a noncyclic free solvable group of class ≥ 2 has an undecidable universal (existential) theory.*

In [5], we prove that a free metabelian group has a decidable universal theory (see also [6]). We do not know if this result is false for noncyclic free solvable groups of class ≥ 3. Nevertheless, the following result gives a good information on this problem.

Theorem C1. *If the universal theory of the field of the rationals is undecidable, then the universal theory of a noncyclic free solvable group of class* ≥ 3 *is undecidable.*

The problem of the decidability of the universal theory of the field of rationals is a well-known open problem and it is equivalent to Hilbert's 10th problem for the field of the rationals (it is also equivalent to the problem of the decidability of the universal theory of a noncyclic free nilpotent group of class ≥ 2 (see [21])). It seems sensible to conjecture a negative answer (notice that some mathematicians conjecture a positive answer; see [16, 14] for discussions on this problem).

Using Matiyasevich's theorem [12, 13] (i.e., the negative answer of Hilbert's 10th problem for the ring of the integers), we will prove

Theorem C2. *A noncyclic free solvable group of class* ≥ 2 *has an undecidable* $\forall\exists$ *theory.*

It is possible to prove that a noncyclic free solvable group of class 2 or class 3 and of finite rank has an undecidable $\forall\exists$ positive theory (see the last section, where it is also explained why we can not prove Theorem C1 for a free metabelian group).

Corollary B2, Theorems C1 and C2 generalize Malcev's undecidability results on free solvable groups. Let us make some comments. Let F be a noncyclic free solvable group of class $n \geq 2$. Malcev gives an interpretation of the theory of the ring of the integers using $\exists\forall$ formulae which involve two free generators ($\mathbb{Z}$ is realized as the cyclic subgroup generated by a free generator, an other free generators is need to define the multiplication). Then, Malcev invokes the essential undecidability of the ring of the integers to obtain the undecidability of the theory of F without constants in the language. Also, with Matiyasevich's theorem we obtain that F has an undecidable $\forall\exists$ theory in the language of groups with two new constants for two free generators. Using Theorem A we can prove that Malcev's interpretation can be realized with existential formulae. This gives the second part of Corollary B2, however, we use Roman'kov's result to obtain a more precise one. The main difference between Theorems C1 and C2 and Corollary B2 and Malcev's strategy is that we obtain (relative) undecidability results for the theory of F without constants in the language and with good bounds on the alternation of quantifiers. To obtain Theorem C1 (respectively, Theorem C2) we construct a universal (respectively, an existential) formula in two free variables so that if it is satisfied by two elements g and h of F, then g and h are not to far from being two free generators of F (respectively, of F'). In the case of Theorem C2 we obtain the rationals (and not the integers) because g and h are (essentially) uncountrollable powers of free generators of F'. Moreover, The proof of Theorem C1 (respectively, Theorem C2) do not gives an interpretation of the integers (respectively, of the rationals): we give an algorithm which given a polynomial equation $P(X_1,\dots,X_s)=0$ constructs a $\forall\exists$ (respectively, existential) sentence ψ of $\mathscr{L}$ such that $P=0$ has a solution in $\mathbb{Z}$ (respectively, in $\mathbb{Q}$) if and only if $F \models \phi$.

This paper is organized as follows. In the next section, we recall some results which are important for us. In Section 3, we prove the above results. Finally, in the last section, we make a few remarks. In particular, using Corollary B2 and a result of Hartley on commutators in finite solvable groups we show that a noncyclic free solvable group of class ≥ 2 is not 1-residually finite for the equations. For any unexplained notion or notation we refer the reader to [19, 15] for group theory and to [3, 7] for logic.

2. Preliminaries

A free solvable group of class n and rank r can be defined as the quotient of a free group of rank r by the nth term of the derived series. From this it follows that if F is a free solvable group of class $n \geq 2$ and rank $r \geq 2$ then (i) if $1 \leq m \leq n$, $F/\delta_m F$ is a free solvable group of class m and rank r and $\delta_m F$ is isolated in F; (ii) F' is a free solvable group of class $n-1$ and infinite rank. But, the most important properties of free solvable groups can be deduced from the Magnus–Remeslennikov–Sokolov embedding (see [9, Ch. I and II], see also [15], and see [1] for an other point of view). Here we recall five results which are essential for the proof of our results.

(a) A noncyclic free solvable group of class ≥ 2 has a trivial center (see [1] and see also [9] and [15] for a proof which uses Magnus' embedding).

(b) A free solvable group is an R-group, namely, if $g^s = h^s$ for an integer $s \neq 0$, then $g = h$. This is the corollary of Theorem 2 of [11] (see also [9]).

(c) Let F be a noncyclic free solvable group of class $n \geq 2$. If g and h are elements $\neq 1$ of F which commute, then either $g, h \in \delta_{n-1} F$ or there exist two non zero integers s and t such that $g^s = h^t$. This result is stated in [11] after the proof of Theorem 1 and it is a direct consequence of this theorem (see also [9, Theorem II.1.14] for a proof of this result which use the MRS embedding in the place of the results of [1]).

(d) We define by induction on i a sequence $(\mu_i(x))_{i \geq 1}$ of universal formulae of $\mathscr{L}$ by

$$\mu_0(x) \sim x = 1,$$
$$\mu_{i+1}(x) \sim \forall y_{i+1} \;\; \mu_i([x, x^{y_{i+1}}]).$$

By [11, Lemma], if F is a noncyclic free solvable group of class $n \geq 2$, then for $k = 1, \ldots, n$ the kth term of the derived series of F is defined by $\mu_{n-k}(x)$. This result is a consequence of (c).

(e) Let F be a noncyclic free solvable group of class $n \geq 2$. If $g \in \delta_{n-1} F$ and if $g \neq 1$, then the normal closure of g in F is a free $\mathbb{Z}(F/\delta_{n-1} F)$-module. This result is a direct consequence of [2, Theorem 7].

3. Proof of the results

3.1. *Proof of Theorem A*

We define by induction on i a sequence $(\theta_i(x))_{i \geq 0}$ of existential formulae of $\mathscr{L}$ by

$$\theta_0(x) \sim x = 1,$$
$$\theta_{i+1}(x) \sim \exists z_{i+1} \;\; ((\theta_i([x, x^{z_{i+1}}]) \wedge \neg \mu_i([x, z_{i+1}])) \vee \theta_i(x)).$$

It is easy to see that the formulae θ_i are (logically equivalent to) existential formulae. Let F be a noncyclic free solvable group of class $n \geq 2$. We have to prove that if $k \in \{1,\dots,n\}$, then $g \in \delta_k F$ iff $F \models \theta_{n-k}(g)$. Or, equivalently that if $k \in \{0,\dots,n-1\}$, then $g \in \delta_{n-k}F$ iff $F \models \theta_k(g)$ (note that for $k=0$ this is obvious).

First, we show by induction on k that if $g \in \delta_{n-k}F$, then $F \models \theta_k(g)$. The case $k=0$ is obvious. We assume that $1 \leq k \leq n-1$ and that $g \in \delta_{n-k}F$. If $g \in \delta_{n-(k-1)}F$, then, by induction, $F \models \theta_{k-1}(g)$ and by definition of θ_k we have that $F \models \theta_k(g)$. Thus, we assume that $g \notin \delta_{n-(k-1)}F$ and we consider the quotient $\bar{F} = F/\delta_{n-(k-1)}F$ which is a noncyclic free solvable group of class $n-(k-1) \geq 2$ (we denote by $\bar{x}$ the image in $\bar{F}$ of an element x of F). Then, $\bar{g} \neq 1$ and $\bar{g} \in \delta_{n-k}\bar{F}$. Since $\bar{F}$ has a trivial center we have a $h \in F$ such that $[\bar{g},\bar{h}] \neq 1$. Thus, $[g,h] \notin \delta_{n-(k-1)}F$ and since $\delta_{n-(k-1)}F = \mu_{k-1}(F)$, we have that $F \models \neg\mu_{k-1}([g,h])$. Moreover, since $\delta_{n-k}\bar{F}$ is abelian and normal in $\bar{F}$, we have that $[\bar{g},\bar{g}^{\bar{h}}]=1$. Thus, $[g,g^h] \in \delta_{n-(k-1)}F$ and by the induction hypothesis $F \models \theta_{k-1}([g,g^h])$. We have found a $h \in F$ such that $F \models \theta_{k-1}([g,g^h]) \wedge \neg\mu_{k-1}([g,h])$. This completes the induction.

Now, we show that if $F \models \theta_1(g)$, then $g \in \delta_{n-1}F$. Obviously, we may assume that there exists a $h \in F$ such that $[g,g^h]=1$ and $[g,h] \neq 1$. By (c) of Section 2, we have two nonzero integers s and t such that $g^s = (g^h)^t$. We suppose that $g \notin \delta_{n-1}F$. Thus, $g \in \delta_m F \setminus \delta_{m+1}F$ for an integer m such that $0 \leq m \leq n-2$. We set $\bar{F} = F/\delta_{m+1}F$ which is a noncyclic free solvable group of class $m+1$ (note that $\bar{g} \neq 1$). If $m=0$, then $\bar{F}$ is a free abelian group and the equality $g^s = (g^h)^t$ implies that $s=t$; thus $g^s = (g^h)^s$ and since F is an R-group we obtain $g = g^h$; this implies that $[g,h]=1$, a contradiction. Thus, we may assume that $m+1 \geq 2$. In this case we may apply (e) of Section 2 with $\bar{F}$ and $\bar{g}$ and the equality $g^s = (g^h)^t$ implies that $s - h.t = 0$ in the group-ring $\mathbb{Z}(\bar{F}/\delta_m\bar{F})$. It follows that $s=t$ and we obtain $g^s = (g^h)^s$. As in the case $m=0$ we obtain a contradiction.

To complete the proof of the theorem we show by induction on $k \geq 1$ that if $F \models \theta_k(g)$, then $g \in \delta_{n-k}F$. The case $k=1$ has already been settled and thus we assume that $2 \leq k \leq n-1$. Let $g \in F$ such that $F \models \theta_k(g)$. If $F \models \theta_{k-1}(g)$, then, by the induction hypothesis, $g \in \delta_{n-(k-1)}F \subseteq \delta_{n-k}F$. Thus, we may assume that there exits a $h \in F$ such that $F \models \theta_{k-1}([g,g^h]) \wedge \neg\mu_{k-1}([g,h])$. By (d) of Section 2 and the induction hypothesis, this says that $[g,g^h] \in \delta_{n-(k-1)}F$ and that $[g,h] \notin \delta_{n-(k-1)}F$. Moreover, $\bar{F} = F/\delta_{n-(k-1)}F$ is a noncyclic free solvable group of class $n-(k-1) \geq 2$. Then, (d) of Section 2 and the first induction in the proof show that $\bar{F} \models \theta_1(\bar{g})$. The paragraph above implies that $\bar{g} \in \delta_{n-k}\bar{F}$. Thus $g = g_1 g_2$ with $g_1 \in \delta_{n-k}\bar{F}$ and $g_2 \in \delta_{n-(k-1)}F$ and we obtain that $g \in \delta_{n-k}F$. This completes the proof of Theorem A. □

3.2. *Proof of Corollary B1*

Let F_1 and F_2 be two free solvable groups of class $n \geq 1$ and of different finite rank r_1 and r_2 respectively. We may suppose that $2 \leq r_1 < r_2$. F_1/F_1' and F_2/F_2' are free abelian group of rank r_1 and r_2, respectively, thus we know [25] that there exists a sentence φ of $\mathscr{L}$ of the form

$$\forall u_1 \ldots u_s \exists v_1 \ldots v_t \bigvee_{i=1}^{l} w_i(\vec{u}, \vec{v}) = 1,$$

where the w_i are terms of $\mathscr{L}$ such that $F_1/F_1' \models \varphi$ and $F_2/F_2' \models \neg\varphi$ (we can take for the sentence φ the sentence $\forall u_1 \ldots u_{2^{r_1}+1} \exists v \ \bigvee_{i \neq j} u_i u_j v^2 = 1$). Thus, we may suppose that $n \geq 2$ (if $n = 1$ then $F_1' = F_2' = 1$). By Theorem A, we have an existential formula $\theta(x)$ which define F_i' for $i = 1, 2$. We consider the following sentence:

$$\psi \sim \forall u_1 \ldots u_s \exists v_1 \ldots v_t \bigvee_{i=1}^{l} \theta(w_i(\vec{u}, \vec{v})).$$

Since θ is existential, ψ is (logically equivalent to) a $\forall\exists$ sentence. Then we have $F_1 \models \psi$ and $F_2 \models \neg\psi$. This completes the proof of Corollary B1. □

3.3. *Proof of Corollary B2*

If $n = 2$ the result follows from the result of [22] (see Section 1). Now, we proceed by induction on n. Let F be a free solvable group of class 3 and let a and b be two free generators of F. We are going to prove that we have no algorithm for solving equations of the form $w(x_1, \ldots, x_s) = v(a, b)$. Since $\langle a, b \rangle$ is existentially closed in F we may suppose that F is free on a and b (see [8] or [5, Section 2]). Then, by [18, Corollary 3], we have an integer p (which we can compute) such that $g \in F'$ iff there exist $g_1, \ldots, g_{2p} \in F$ with $g = [g_1, g_2] \cdots [g_{2p-1}, g_{2p}]$. Moreover, F' is a free metabelian group of infinite rank and $[a, b]$ and $[a, b]^a$ are free generators of F'. Let F_1 be a free metabelian group on a_1 and b_1. Using the fact that the homomorphism of F_1 into F' which maps a_1 to $[a, b]$ and b_1 to $[a, b]^a$ is existentially closed, we see that

$$F_1 \models \exists x_1 \ldots x_s \ w(x_1, \ldots, x_s) = v(a_1, b_1),$$

if and only if

$$F \models \exists x_{1,1} \ldots x_{1,2p} \ldots x_{s,1} \ldots x_{s,2p} \ w(y_1, \ldots, y_s) = v([a, b], [a, b]^a),$$

where $y_i = [x_{i,1}, x_{i,2}] \ldots [x_{i,2p-1}, x_{i,2p}]$. Thus an algorithm for solving equations of the form $w(x_1, \ldots, x_s) = v(a, b)$ in F yields an algorithm for solving equations of the form $w(x_1, \ldots, x_s) = v(a_1, b_1)$ in F_1. We can use the result of [22].

Now, we suppose that $n \geq 4$. Let F be a free solvable group of class n and let a and b be two free generators of F. By the induction hypothesis we have no algorithm which decides if a sentence of the form

$$\exists x_1 \ldots x_s \bigwedge_{i=1}^{t} w_i(x_1, \ldots, x_s, \bar{a}, \bar{b}) = 1, \qquad (*)$$

where $t = 2^{n-4}$, is true in $F/\delta_{n-1}F$. Theorem A and its proof show that $g \in \delta_{n-1}F$ iff $g = 1$ or there exists $h \in F$ such that $[g, g^h] = 1$ and $[g, h] \neq 1$. This implies that $g \in \delta_{m-1}F$ iff $[g, g^a] = 1$ and $[g, g^b] = 1$. Thus, a sentence of the form $(*)$ is true in $F/\delta_{n-1}F$ iff the sentence

$$\exists x_1, \ldots, x_s \bigwedge_{i=1}^{t} \left([w_i(\vec{x}, a, b), w_i(\vec{x}, a, b)^a] = 1 \wedge [w_i(\vec{x}, a, b), w_i(\vec{x}, a, b)^h] = 1 \right)$$

is true in F. This completes the proof of Corollary B2. □

3.4. *Proof of Theorem C1*

We need the following lemma.

Lemma 1. *Let F be a noncyclic free solvable group of class $n \geq 3$. There exists (and we will exhibit) an existential formula $\phi(x, y)$ such that the following properties are equivalent* (i) $F \models \phi(g, h)$; (ii) $g \in F' \backslash F''$ *and* $h \notin F'$; (iii) g *and* g^h *freely generate a free abelian group modulo* F'', $g \in F' \backslash F''$ *and* $h \notin F'$.

Proof. Let F be a noncyclic free solvable group of class $n \geq 3$. By Theorem A, we have an existential formula $\theta'(x)$ which defines F'. By (d) of Section 2, we have an universal formula $\mu'(x)$ which defines F' and we have an universal formula $\mu''(x)$ which define F''. We take for $\phi(x, y)$ the formula

$$\theta'(x) \wedge \neg\mu''(x) \wedge \neg\mu'(y).$$

$\phi(x, y)$ is (logically equivalent to) an existential formula. It is clear that (i) and (ii) are equivalent and that (iii) implies (ii). Let us prove that (ii) implies (iii). Let $g, h \in F$ such that $g \in F' \backslash F''$ and $h \in F \backslash F'$. Since g and g^h are in F', g and g^h generate an abelian group modulo F''. Suppose that there exist $s, t \in \mathbb{Z}$ with $(s, t) \neq (0, 0)$ such that $g^s = (g^h)^t \bmod(F'')$. Since $g \in F' \backslash F''$, we can use (e) of Section 2 to prove that $s = th$ in $\mathbb{Z}(F/F')$. Since h is not in F' we obtain a contradiction. Hence g and g^h are linearly independent modulo F''. □

Let F be a noncyclic free solvable group of class $n \geq 3$. By Theorem A, we have an existential formula $\theta(x)$ which defines F''. Moreover, we have an existential formula $\phi(x, y)$ with the property described in Lemma 1.

It is well-known that $(\mathbb{Q}, +, ., 0, 1)$ has a decidable universal theory iff there is an algorithm which decides if a homogeneous polynomial in several variables and with its coefficients in $\mathbb{Z}$ has a nontrivial zero in $\mathbb{Z}$ (see, for example, [16, Section 9] for a proof). Thus, to prove Theorem C1, it suffices to associate (in an effective way) to each homogeneous polynomial $P(\vec{X})$ in several variables and with its coefficients in $\mathbb{Z}$, an existential sentence ψ of $\mathscr{L}$ such that $P(\vec{X})$ has a nontrivial zero in $\mathbb{Z}$ iff $F \models \psi$.

Now following the proof of Theorem 3 of [11], we set

$$+(x,y,z) \sim \theta(xyz^{-1}),$$

$$\times(x,y,z,\alpha,\beta) \sim \exists uvw\ (ux\alpha^{\beta} = x\alpha^{\beta}u \wedge v\alpha^{\beta} = \alpha^{\beta}v \wedge w\alpha\alpha^{\beta} = \alpha\alpha^{\beta}w \wedge \theta(yvw^{-1}) \wedge \theta(uv^{-1}z^{-1})),$$

$$1(x,\alpha) \sim x = \alpha,$$

$$0(x) \sim x = 1.$$

These formulae are (logically equivalent to) existential formulae. Let $g,h \in F$ such that $F \models \phi(g,h)$ and let $x_1,x_2,x_3 \in F$ such that $x_i g = g x_i$ and $x_i \neq 1$ for $i = 1,2,3$. We know that g is not in $\delta_{n-1}F$, thus by (c) of Section 2, there exist $s_1,s_2,s_3 \in \mathbb{Z}^*$ and $t_1,t_2,t_3 \in \mathbb{Z}^*$ such that $x^{s_i} = g^{t_i}$ for $i = 1,2,3$.

We claim that (i) $F \models +(x_1,x_2,x_3)$ iff $t_1/s_1 + t_2/s_2 = t_3/s_3$; (ii) if $F \models \times(\vec{x},g,h)$ then $t_1t_2/s_1s_2 = t_3/s_3$; and (iii) if $t_1t_2/s_1s_2 = t_3/s_3$ and $s_i \mid t_i$ for $i = 1,2,3$ then $F \models \times(x_1,x_2,x_3, g,h)$. We just prove (ii). Suppose that $F \models \times(x_1,x_2,x_3,g,h)$, then we have $u,v,w \in F$ such that $uxg^h = xg^h u$, $vg^h = g^h v$, $wgg^h = gg^h w$, $yv = w \bmod(F'')$ and $u = vz \bmod(F'')$. Using (c) of Section 2, we see that there exist $l_1,l_2,l_3 \in \mathbb{Z}^*$ and $k_1,k_2,k_3 \in \mathbb{Z}$ such that $u^{l_1} = (xg^h)^{k_1}$, $v^{l_2} = (g^h)^{k_2}$ and $w^{l_3} = (gg^h)^{k_3}$. Since F'' is isolated in F we have

$$(yv)^{s_2l_2l_3} = w^{s_2l_2l_3} \bmod(F'') \quad \text{and} \quad u^{s_1s_3l_1l_3} = (vz)^{s_1s_3l_1l_3} \bmod(F'').$$

Since all our elements are in F' we obtain

$$g^{t_2l_2l_3}(g^h)^{k_2s_2l_3} = (gg^h)^{k_3l_2s_2} \bmod(F''),$$

$$g^{t_1k_1s_3l_2}(g^h)^{k_1s_1s_3l_2} = (g^h)^{k_2s_1s_3l_1} g^{t_3s_1l_1l_2} \bmod(F'').$$

Since g and g^h are linearly independent modulo F'', it is then easy to obtain $t_1t_2/s_1s_2 = t_3/s_3$.

Then, if $P(X_1,\dots,X_r)$ is a homogeneous polynomial with its coefficients in $\mathbb{Z}$, we can construct, by induction on the complexity of $P(X_1,\dots,X_r)$ and using the formulae $+(x,y,z)$, $\times(x,y,z,\alpha,\beta)$, $1(x,\alpha)$ and $0(x)$, an existential formula $\psi'(\alpha,\beta,x_1,\dots,x_r)$ such that if we consider the existential sentence ψ

$$\exists\alpha\beta\exists x_1\dots x_r\ \left(\phi(\alpha,\beta) \wedge \bigwedge_{i=1}^{r}(x_i\alpha = \alpha x_i \wedge \neg 0(x_i)) \wedge \psi'(\alpha,\beta,x_1,\dots,x_r)\right),$$

then $F \models \psi$ iff $P(\vec{X})$ has a nontrivial zero in $\mathbb{Z}$ (by construction of ψ if P has a nontrivial zero in $\mathbb{Z}$ then $F \models \psi$; if $F \models \psi$, then P has a nontrivial zero in $\mathbb{Q}$ and since P is homogeneous, P has a nontrivial zero in $\mathbb{Z}$). We have proved Theorem C1. □

3.5. *Proof of Theorem C2*

First, we show that we may restrict ourselves to noncyclic free metabelian groups. Let F be a noncyclic free solvable group of class $n \geq 3$ and φ a $\forall\exists$ sentence. By

(d) of Section 2 and Theorem A, we have an universal sentence $\mu(x)$ and an existential sentence $\theta(x)$ which define F'' in F. To φ, we associate the sentence φ' which is obtained from φ by replacing each atomic subformula of φ of the form $w_1(\vec{x}) = w_2(\vec{x})$ by $\theta(w_1(\vec{x})w_2(\vec{x})^{-1})$ and each negation of atomic subformula of φ of the form $w_1(\vec{x}) \neq w_2(\vec{x})$ by $\neg\mu(w_1(\vec{x})w_2(\vec{x})^{-1})$. Then φ' is logically equivalent to a $\forall\exists$ sentence and we have $F \models \varphi'$ iff $F/F'' \models \varphi$. Since F/F'' is a free metabelian group of the same rank as F, we see that if F has a decidable $\forall\exists$ theory then the free metabelian group of the same rank as F has a decidable $\forall\exists$ theory (φ' can be effectively constructed from φ). Note that a similar argument (with the fact that two noncyclic free solvable groups of the same class have the same universal theory) shows that if a noncyclic free solvable group of class ≥ 3 has a decidable universal theory, then every free solvable group of class $\leq n$ has a decidable universal theory.

To prove Theorem C2 for free metabelian groups we need the following lemma.

Lemma 2. *There exists (and we will exhibit) an universal formula $\phi'(x, y)$ such that if F is a noncyclic free metabelian group and if $g, h \in F$, then $F \models \phi'(g, h)$ iff g and h are linearly independent modulo F'.*

Proof. We have an existential formula $\theta'(x)$ which defines the derived subgroup of any noncyclic free metabelian group (Theorem A). We take for $\phi'(x, y)$ the formula

$$\begin{aligned} \neg\theta'(x) \wedge \neg\theta'(y) \wedge (\forall c x_1 y_1 (([x, x_1] &= 1 \wedge [y, y_1] \\ &= 1 \wedge x_1 \neq 1 \wedge y_1 \neq 1 \wedge \theta'(c) \wedge c \neq 1) \\ &\Rightarrow ([x_1, c] \neq [y_1, c]))). \end{aligned}$$

It is clear that $\phi'(x, y)$ is logically equivalent to an universal formula. Let F be a free metabelian group and $g, h \in F$.

Suppose that $F \models \phi'(g, h)$. We assume that there exist $s, t \in \mathbb{Z}$ such that $g^s = h^t c_1$ with $c_1 \in F'$ and $(s, t) \neq (0, 0)$. Since g and h are not in F' and since F' is isolated in F we may suppose that s and t are not equal to 0. Let $c \in F'$ with $c \neq 1$; we put $x_1 = g^s$ and $y_1 = h^t$. Then $[g, x_1] = 1$, $[h, y_1] = 1$, $x_1 \neq 1$ and $y_1 \neq 1$; moreover,

$$[x_1, c] = [h^t c_1, c] = [h^t, c][[h^t, c], c_1][c_1, c] = [h^t, c].$$

Hence, we obtain $[x_1, c] = [y_1, c]$ and this is absurd.

Conversely, we suppose that g and h are linearly independent modulo F'. It is obvious that g and h are not in F'. Let $c \in F' \setminus \{1\}$ and $x_1, y_1 \in F$ such that $[g, x_1] = 1$, $[h, y_1] = 1$, $x_1 \neq 1$ and $y_1 \neq 1$. Then, by (c) of Section 2, there exist $s_1, s_2, t_1, t_2 \in \mathbb{Z}^*$ such that $x_1^{s_1} = g^{t_1}$ and $y_1^{s_2} = h^{t_2}$. We suppose that $[x_1, c] = [y_1, c]$. Then $c^{-x_1+1} = c^{-y_1-1}$; since $c \in F' \setminus \{1\}$, it follows from (e) of Section 2 that $-x_1 + 1 = -y_1 + 1$ in $\mathbb{Z}(F/F')$; thus $x_1 = y_1 \bmod(F')$ and this implies that $g^{t_1 s_2} = h^{t_2 s_1} \bmod(F')$. This is absurd so $F \models \phi'(g, h)$. $\square$

Let us now prove Theorem C2 for free metabelian groups. Let F be a noncyclic free metabelian group. By Theorem A, we have an existential formula $\theta'(x)$ which define F'; moreover, we have a formula $\phi'(x,y)$ with the property of Lemma 2. By Matiyasevich's theorem, to prove that F has an undecidable $\forall\exists$ theory, it suffices to associate (in an effective way) to each polynomial $P(\vec{X})$ in several variables and with its coefficients in $\mathbb{Z}$ a $\forall\exists$ sentence ψ of $\mathscr{L}$ such that $P(\vec{X})$ has a zero in $\mathbb{Z}$ iff $F \models \psi$. Let $P(X_1,\dots,X_r)$ be such a polynomial. As in the proof of Theorem C1, we can construct, by induction on the complexity of $P(X_1,\dots,X_r)$, an existential formula $\psi'(\alpha,\beta,x_1,\dots,x_r)$ corresponding to P, where we put $\theta'(x)$ in the place of $\theta(x)$ and β in the place of α^β in the definition of $+(\vec{x})$ and $\times(\vec{x},\alpha,\beta)$. Then we consider the sentence ψ

$$\forall\alpha\beta\left(\phi'(\alpha,\beta) \Rightarrow \left(\exists x_1\dots x_r \bigwedge_{i=1}^{r} x_i\alpha = \alpha x_i \wedge \psi'(\alpha,\beta,x_1,\dots,x_r)\right)\right).$$

Since ϕ' is universal ψ is a $\forall\exists$ sentence. Using the proof of Theorem C1, it is easy to prove that if P has a zero in $\mathbb{Z}$, then $F \models \psi$. Suppose that $F \models \psi$. Let g and h be two free generators of F. Then, g has no roots in F and g and h are linearly independent modulo F'. By hypothesis and Lemma 2 we have

$$F \models \exists x_1\dots x_r\left(\bigwedge_{i=1}^{r} x_i g = g x_i \wedge \psi'(g,h,x_1,\dots,x_r)\right).$$

Since g has no roots in F, if $xg = gx$ then, by (c) of Section 2, we have $s \in \mathbb{Z}$ such that $x = g^s$. Then, it follows from the proof of Theorem C1 that P has a zero in $\mathbb{Z}$. This completes the proof of Theorem C2. □

4. Remarks

4.1. Let F be a free solvable group of class ≤ 3 and of finite rank r. The result of [18] implies that there exists a p_1 (which depends on r) such that F' is definable by the formula

$$\exists y_1\dots y_{2p_1}\ x = [y_1,y_2]\ \dots[y_{2p_1-1},y_{2p_1}].$$

Moreover, it is possible to prove (using the ideas of [18]) that there exists a p_2 which depends on r, such that $\gamma_3 F$, the 3th term of the lower central series, is definable by the formula

$$\exists y_1\dots y_{3p_2}\ x = [[y_1,y_2],y_3]\dots[[y_{3p_2-2},y_{3p_2-1}],y_{3p_2}].$$

(see also [4, Ch. I] for some generalizations). Since it is not difficult to prove that if $g,h \in F$ and $[g,h] \neq 1 \operatorname{mod}(\gamma_3 F)$ then g and h are linearly independent modulo F', we can prove exactly as in the proof of Theorem C2 (without using the reduction to free metabelian groups), but using the previous formulae, that F has an undecidable

$\forall\exists$ theory. In this case, since the above formulae are positive, it is easy to see that we obtain the undecidability of the $\forall\exists$ positive theory of F.

4.2. It follows from [15, Theorem 42.55] that the property (iii) of Lemma 1 is equivalent to (iii') g and g^h freely generate a free solvable subgroup of F of class $n-1$. The same result implies that in Lemma 2, $F \models \phi'(g,h)$ iff g and h freely generate a free metabelian subgroup of F. Moreover, in [5] we prove that if F is a noncyclic free metabelian group there is no existential formula $\varphi(x,y)$ consistent with F such that if $F \models \varphi(g,h)$, then g and h are linearly independent modulo F'. This explains why our proof of Theorem C1 does not work for free metabelian groups.

4.3. In [6], we show that the universal theory of free metabelian groups and the groups with the same universal theory as a noncyclic free metabelian group have a lot of remarkable properties. Theorem C1 seems to prevent full generalization for free solvable groups of class ≥ 3; however, we can hope that some of these properties can be extended to free solvable groups of class ≥ 3. For example, it is possible to describe the groups on two generators with the same universal theory as a noncyclic free solvable group of class $n \geq 3$; these groups are the free solvable group of class n and rank 2 and the $(n-1)$-solvable verbal wreath product of two infinite cyclic groups (see for example [24] for the notion of verbal wreath product, and see [6] for $n=2$). In an other direction we ask the following question: let F be a free solvable group of class ≥ 3; for each infinite set of words on a fix number of variables $\{w_i(x_1,\dots,x_n)\}_{i\in I}$ does there exist a finite subset J of I such that for all $g_1,\dots,g_n \in F$, $w_i(\vec{g})=1$ for all $i \in I$ iff $w_i(\vec{g})=1$ for all $i \in J$? This question arose during discussions with Myasnikov and it was motivated by the will to prove analogues of the main result of [17] for free solvable groups of class ≥ 3 (see [6] for free metabelian groups).

4.4. Let n be an integer ≥ 1. We say that a finitely generated group G is n-residually finite for the equations if for every system $(*)$ of n equations with parameters from G, $(*)$ has a solution in G iff for every finite image H of G (the image of) $(*)$ has a solution in H. It is easy to see that G is n-residually finite for the equations iff for every system $(*)$ of n equations with parameters from G, if $(*)$ has a solution in the profinite completion of G then $(*)$ has a solution in G. If G is a finitely presented group in a variety defined by a law which is n-residually finite for the equations, then a classical argument shows that there exists an algorithm which decides if a system of n equations with parameters from G has a solution in G. Thus, Corollary B2 implies that a noncyclic finitely generated free solvable group of class m is not $\max(1,2^{m-3})$-residually finite for the equations. In fact using [10, Theorem 3], we can prove that a noncyclic finitely generated free solvable group is not 1-residually finite for the equations. Indeed, let F be a free solvable group of class $m \geq 4$ and finite rank $r \geq 2$. By [10, Theorem 3], we have an integer p such that for every finite image H of F, every element of H' can be written as product of p commutators. If every element of F' can be written as product of p commutators, then we can prove using the proof of Corollary

B2 that there is no algorithm which decides if an equation of the form $w(\vec{x}) = v(a, b)$ has a solution in F. If not, then we have $g_1, \ldots, g_{2(p+1)} \in F$ such that the equation

$$[x_1, x_2] \ldots [x_{2p-1}, x_{2p}] = [g_1, g_2] \ldots [g_{2p+1}, g_{2(p+1)}]$$

has no solution in F; and this equation has a solution in every finite image of F.

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