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∀-FREE METABELIAN GROUPS

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§1. Introduction. In 1965, during the first All-Union Symposium on Group Theory, Kargapolov presented the following two problems: (a) describe the universal theory of free nilpotent groups of class m; (b) describe the universal theory of free groups (see [18, 1.28 and 1.27]). The first of these problems is still open and it is known [25] that a positive solution of this problem for an $m \geq 2$ should imply the decidability of the universal theory of the field of the rationals (this last problem is equivalent to Hilbert's tenth problem for the field of the rationals which is a difficult open problem; see [17] and [20] for discussions on this problem). Regarding the second problem, Makanin proved in 1985 that a free group has a decidable universal theory (see [15] for stronger results), however, the problem of deriving an explicit description of the universal theory of free groups is open. To try to solve this problem Remeslennikov gave different characterization of finitely generated groups with the same universal theory as a noncyclic free group (see [21] and [22] and also [11]). Recently, the author proved in [8] that a free metabelian group has a decidable universal theory, but the proof of [8] does not give an explicit description of the universal theory of free metabelian groups.

The aim of this paper is to describe the groups with the same universal theory as a noncyclic free metabelian group (we call these groups ∀-free metabelian) and to give an explicit description of the universal theory of noncyclic free metabelian groups (two noncyclic free metabelian groups have the same universal theory). This is done in Sections 3 and 4: we state our main theorem in Section 3 and we prove it in Section 4. We then obtain a new proof of the decidability of the universal theory of a noncyclic free metabelian group which is more natural for a logician than the "combinatorial" proof of [8] (this paper does not use the results of [8]). Moreover, we obtain an abstract version of the so called Magnus embedding. In Section 5, we give some applications of the main theorem of Section 3: we characterize ∀-free metabelian groups in terms of residual properties, we prove that noncyclic parafree metabelian groups are ∀-free metabelian and we study 2-generator subgroups of ∀-free metabelian groups. In the last section of this paper we make a few remarks. In Section 2, we recall some notations and elementary facts from group theory.

We proved in [7] that if the field of the rationals has an undecidable universal theory, then a noncyclic free solvable group of class ≥ 3 has also an undecidable universal theory. So, the problem of deriving an explicit description of the universal theory of free solvable groups of class m, for $m \geq 3$, is either impossible or a very difficult task. Moreover, we have recently studied the universal theory of free

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metabelian groups from the point of view of "model theoretic algebra" (see the last paragraph of the last section).

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§2. Preliminaries and notations. Our notation is consistent with [6] for logic and with [23] for group theory and we refer the reader to these books for any unexplained notion (see also [19] for group theory and [9] for logic). The knowledge of logic needed for this paper is elementary and does not exceed compactness and completeness theorems and basic properties of universal formulas. We work most often with the language of groups: $\mathcal{L}_{gp} = \{., ^{-1}, 1\}$. If $\phi(x)$ is a formula with one free variable and if G is a model then we denote by $\phi(G)$ the set $\{g \in G \mid G \models \phi(g)\}$. We say that a sentence is universal if its prenex form is a universal sentence; such a sentence in $\mathcal{L}_{gp}$ is equivalent modulo the theory of groups to a sentence of the form

$$\forall x_1 \cdots \forall x_r \Big(\bigwedge_{i=1}^{n} \big(\bigvee_{j=1}^{k_i} w_{i,j}(x_1, \ldots, x_r) = 1 \ \vee \bigvee_{j=k_i+1}^{l_i} w_{i,j}(x_1, \ldots, x_r) \neq 1 \big) \Big)$$

where the $w_{i,j}$ are elements of the free groups on $x_1, \ldots, x_r$. A group G has a decidable universal theory if there is an algorithm which decides whether or not a universal sentence is true in G. This is equivalent to decide whether a system of equations and inequations in $\mathcal{L}_{gp}$ (without parameters from G) has a solution in G.

Let G be a group. If $g_1, \ldots, g_n$ are elements of G, then we denote by $\langle g_1, \ldots, g_n \rangle$ the subgroup of G generated by $g_1, \ldots, g_n$. We say that G is an n-generator group if there exist $g_1, \ldots, g_n \in G$ such that $G = \langle g_1, \ldots, g_n \rangle$. We denote by $Fit(G)$ the Fitting subgroup of G. We recall that $Fit(G)$ is the subgroup of G generated by all the nilpotent normal subgroups of G (see [23, Chapter 5]). If H is a subgroup of G, H is said to be isolated in G if, for all $g \in G$, $g \in H$ whenever $g^n \in H$ for some integer $n \geq 1$. If H is normal in G, H is isolated in G iff G/H is torsion-free. For every set X we denote by $G^{(X)}$ the group of functions f from X into G such that $\{x \in X | f(x) \neq 1\}$ is finite. We denote by $\mathbb{Z}[G]$ the integral group ring of G. Let A and B be two groups, we denote by $AwrB$ the (restricted) wreath product of A by B. We recall that $AwrB$ is the semidirect product of $A^{(B)}$ by B, in which the automorphism of $A^{(B)}$ produced by an element $b \in B$ is given by: $f^b(x) = f(xb^{-1})$ for $f \in A^{(B)}$. If L is a right $\mathbb{Z}[A]$-module we put

$$M(A, L) = \left\{ \begin{pmatrix} a & 0 \\ v & 1 \end{pmatrix} \mid \text{for } a \in A \text{ and } v \in L \right\}.$$

$M(A, L)$ is a group under matrix multiplication, and this group is of course the usual semidirect product of L by A. We denote by (a, v) the element $\left(\begin{smallmatrix} a & 0 \\ v & 1 \end{smallmatrix}\right)$ of a group of the form $M(A, L)$. A few moments of thought will convince the reader that if $A \simeq \mathbb{Z}^{(r)}$ where r is a cardinal, then $AwrB$ is isomorphic to $M(A, L)$ where L is the free $\mathbb{Z}[A]$-module of rank r.

Let g and h be two elements of a group G, we put $[g, h] = g^{-1}h^{-1}gh$ and $g^h = h^{-1}gh$. The derived subgroup G' of G is the subgroup of G generated by all the $[g, h]$. G is abelian iff G satisfies the identity $\forall xy\ [x, y] = 1$. G is metabelian (i.e., solvable of class ≤ 2) iff G satisfies the identity $\forall xyzt\ [[x, y], [z, t]] = 1$. More concretely, G is metabelian iff G has an abelian normal subgroup H such that $\bar{G} = G/H$ is abelian. Then, H is a $\mathbb{Z}[\bar{G}]$-module where the action of $\mathbb{Z}[\bar{G}]$ on H is (well) defined as follows: if $g \in G$ and $v \in H$, $v.\bar{g} = g^{-1}vg = v^g$. So, adopting a multiplicative notation, if $\sum n_{\bar{g}}\bar{g} \in \mathbb{Z}[\bar{G}]$ and if $v \in H$ then

$$v.\sum n_{\bar{g}}\bar{g} = v^{\sum n_{\bar{g}} g} = \prod g^{-1} v^{n_{\bar{g}}} g\ .$$

As examples of metabelian groups, we have the groups of the form $M(A, L)$ where A is an abelian group and where L is a $\mathbb{Z}[A]$-module. Identifying L with the matrices of the form $(1, v)$, L is an abelian normal subgroup of $M(A, L)$ and $M(A, L)/L \simeq A$ (of course in this case we use an additive notation for L). Notice that there are only $\aleph_0$ metabelian groups on a finite number of generators (this follows from [23, 15.3.1]). But, if G is metabelian on a finite number of generators, then, in general, G contains subgroups which are not finitely generated, and G may contain $2^{\aleph_0}$ nonisomorphic subgroups (see [4]).

A variety V of groups is the class of models of a theory in $\mathcal{L}_{gp}$ consisting of the axioms of groups and of sentences of the form $\forall \bar{x}\ w_j(\bar{x}) = 1$ where the w_j are terms of $\mathcal{L}_{gp}$ (i.e., elements of the free group on the variables x_i). The w_j are called the laws of V. Let r be a cardinal. We say that G is a free group of V of rank r if $G \in V$ and if G has a generating system $\{a_i\}_{i \in r}$ such that for all $H \in V$, any function from $\{a_i\}_{i \in r}$ into H can be extended to a homomorphism from G into H. Then, we say that the a_i are the free generators of G. Such a group exists and it is unique up to isomorphism. If W is the set of laws of V then the free group of rank r of V is isomorphic to $F_r/W(F_r)$ where F_r is the (absolutely) free group of rank r and where $W(F_r)$ is the normal subgroup $\langle w(\bar{g}) \mid w \in W \text{ and } g_i \in F_r\rangle$. For example, the free abelian group of rank r is $\mathbb{Z}^{(r)}$. We refer the reader to [19] for an introduction to the theory of the varieties of groups.

The class of metabelian groups is a variety of groups. We denote by $F_r(\mathcal{A}^2)$ the free metabelian group of rank r. A fundamental result on free solvable groups is the existence of the Magnus embedding that we now define in the context of free metabelian groups. Let r be cardinal number. We denote by a_i, $i \in r$, free generators of $F_r(\mathcal{A}^2)$. Then, the quotient of $F_r(\mathcal{A}^2)$ by its derived subgroup is a free abelian group on r generator, freely generated by the images $\bar{a}_i$ of the a_i. We denote this quotient by $F_r(\mathcal{A})$. Let L be the free $\mathbb{Z}[F_r(\mathcal{A})]$-module on $\{v_i\}_{i \in r}$. The function f from $\{a_i\}_{i \in r}$ into the group $M(F_r(\mathcal{A}), L) \simeq \mathbb{Z}^{(r)} wr \mathbb{Z}^{(r)}$ defined by

$$f(a_i) = \begin{pmatrix} \bar{a}_i & 0 \\ v_i & 1 \end{pmatrix}$$

can be extended to a monomorphism f from $F_r(\mathcal{A}^2)$ into $M(F_r(\mathcal{A}), L)$. f is called the Magnus embedding and the most important properties of free metabelian groups can be deduced from it. For example, with this result it is easy to prove that a free metabelian group has a decidable word problem and more work allows us to prove that if $r \geq 2$, then $Fit(F_r(\mathcal{A}^2)) = F_r(\mathcal{A}^2)'$. We refer the reader to [12, Chapters I and II] for a complete treatment of Magnus embedding.

Let G be a group, let $(e_1, \dots, e_n)$ be a sequence on $\{-1, 1\}$ and let $g_1, \dots, g_n \in G$. If $\sum_{i=1}^{n} e_i \neq 0$, then $e_1 g_1 + \cdots + e_n g_n \neq 0$ in $\mathbb{Z}[G]$. If $\sum_{i=1}^{n} e_i = 0$, then $e_1 g_1 + \cdots + e_n g_n = 0$ in $\mathbb{Z}[G]$ iff there exists a partition of $\{1, \dots, n\}$ into two elements set $\{i_1, j_1\}, \dots, \{i_{\frac{n}{2}}, j_{\frac{n}{2}}\}$ such that $g_{i_k} = g_{j_k}$ and $e_{i_k} = -e_{j_k}$ for $k = 1, \dots, \frac{n}{2}$. Thus, we can express in $\mathscr{L}_{gp}$ the equality $e_1 g_1 + \cdots + e_n g_n = 0$ in $\mathbb{Z}[G]$: if $\sum_{i=1}^{n} e_i \neq 0$ then we consider the formula $g_1 \neq g_1$ and if $\sum_{i=1}^{n} e_i = 0$ then we consider the formula

$$\bigvee_{I \in \mathscr{J}} \bigwedge_{\{i,j\} \in I} g_i g_j^{-1} = 1$$

where $\mathscr{J}$ is the set of partitions of $\{1, \dots, n\}$ into two elements set $\{i, j\}$ such that $e_i = -e_j$ (since $\sum_{i=1}^{n} e_i = 0$, $\mathscr{J}$ is not empty).

§3. Main theorem. It is known that two noncyclic free metabelian groups have the same universal theory (see [10] or [8, Section 2]). With this fact in mind, we say that a group is a $\forall$-free metabelian group if it has the same universal theory as a noncyclic free metabelian group. An important fact for our propose is that if A and B are two nontrivial torsion-free abelian groups, then $AwrB$ is a $\forall$-free metabelian group (see Lemma 4.1 below or [8, Section 2]).

We set $fit^{\forall}(x) \sim \forall t\ [x, x^t] = 1$ and $fit^{\exists}(x) \sim \exists t\ \ [x, x^t] = 1 \wedge [x, t] \neq 1$. These formulas will define the Fitting subgroup. We denote by $\mathfrak{A}$ the following set of universal sentences of $\mathscr{L}_{gp}$

$$\phi_1 \sim \big(\forall xyz\ (xy)z = x(yz)\big) \wedge \big(\forall x\ 1x = x\big) \wedge \big(\forall x\ x^{-1}x = 1\big)$$
$$\phi_2 \sim \forall xyzt\ \big[[x, y], [z, t]\big] = 1$$
$$\phi_3(n) \sim \forall x\quad (x^n = 1) \Rightarrow (x = 1)\quad \text{for all } n \geq 2$$
$$\phi_4 \sim \forall xyz\quad ([x, y] = 1 \wedge [y, z] = 1 \wedge y \neq 1) \Rightarrow ([x, z] = 1)$$
$$\phi_5(n) \sim \forall x\quad (fit^{\exists}(x^n)) \Rightarrow (fit^{\forall}(x))\quad \text{for all } n \geq 2$$
$$\phi_6(\bar{e}) \sim \forall x y_1 \cdots y_n\quad (fit^{\exists}(x) \wedge x^{e_1 y_1} \cdots x^{e_n y_n} = 1) \Rightarrow (e_1 y_1 + \cdots + e_n y_n = 0 \text{ in } \mathbb{Z}[\ /fit^{\forall}])$$

for all finite sequence $\bar{e}$ on $\{-1, 1\}$.

The right hand side of the sentences $\phi_6(\bar{e})$ is an abbreviation for a formula of the form $y_1 \neq y_1$ or of the form $\bigvee \bigwedge fit^{\forall}(y_i y_j^{-1})$ as explained in the end of Section 2. Note that ϕ_4 says that the relation "commute" is an equivalence relation on $G \backslash \{1\}$. This is a strong condition and it implies that a nonabelian subgroup of a model of $\mathfrak{A}$ has a trivial center. An important fact for understanding the signification of the sentences $\phi_5(n)$ and $\phi_6(\bar{e})$ is that if G is a model of $\mathfrak{A}$, then $Fit(G) = fit^{\forall}(G)$ and if G is a nonabelian model of $\mathfrak{A}$, then $Fit(G) \backslash \{1\} = fit^{\exists}(G)$. These follow from the Main Theorem and its proof (Lemmas 4.5 and 4.7). To stress the signification of the set of sentences $\mathfrak{A}$ we introduce the following terminology: we say that a group G is a ρ-group if (i) G is a torsion-free metabelian group; (ii) the Fitting of G, $Fit(G)$, is abelian and isolated in G; (iii) $Fit(G)$ is torsion-free as a $\mathbb{Z}[\bar{G}]$-module, where $\bar{G} = G/Fit(G)$ (i.e., if $v \in Fit(G)$ and $\alpha \in \mathbb{Z}[\bar{G}]$, then $v.\alpha = 1$ implies $\alpha = 0$).

We are now ready to state our main result.

MAIN THEOREM. *Let G be a group. The following properties are equivalent:*
(1) *G satisfies the universal theory of a noncyclic free metabelian group;*
(2) *G is a subgroup of a group of the form $M(A, L)$ where A is a torsion-free abelian group and L is a torsion-free $\mathbb{Z}[A]$-module;*
(3) *G satisfies $\mathfrak{A}$;*
(4) *G is a ρ-group;*
(5) *for all $g_1, \dots, g_n \in G$ there exist $k, r \in \mathbb{N}$ such that the group $\langle g_1, \dots, g_n \rangle$ can be embedded in $\mathbb{Z}^{(k)} wr \mathbb{Z}^{(r)}$.*

Moreover, if G is a nonabelian group satisfying one of the properties above, then G is a ∀-free metabelian group.

We prove this result in the next section. For finitely generated groups we obtain

COROLLARY 3.1. *If G is a finitely generated nonabelian group, then G is ∀-free metabelian iff there exist $k, r \in \mathbb{N}$ such that G can be embedded in $\mathbb{Z}^{(k)} wr \mathbb{Z}^{(r)}$.* ⊣

One can see the result above as a metabelian analogue of the classification of torsion-free abelian groups. Moreover, we obtain a generalization of the Magnus embedding (see Lemma 4.8 and its proof).

We put $\mathfrak{A}^+ = \mathfrak{A} \cup \{\exists xy\ [x, y] \neq 1\}$. By the Main Theorem, if G is a group, we have: $G \models \mathfrak{A}^+$ iff G is a ∀-free metabelian group. Thus, by the completeness theorem, if ϕ is an existential or a universal sentence, then $\mathfrak{A}^+ \vdash \phi$ or $\mathfrak{A}^+ \vdash \neg\phi$. Since $\mathfrak{A}^+$ is a recursive set of sentences, we obtain a new proof of the decidability of the universal theory of a noncyclic free metabelian group.

COROLLARY 3.2. *$\mathfrak{A}^+$ is complete for the universal sentences. The universal theory of a noncyclic free metabelian group is decidable.* ⊣

We remark that in the language of groups with two constants for two free generators the universal theory of a noncyclic free metabelian is undecidable (this follows from [24]); and that the ∀∃ theory of a noncyclic free metabelian is undecidable (see [7]).

§4. Proof of the main theorem.

We decompose the proof of the Main Theorem in a sequence of lemmas. Through this section G will be a group and then we set $Fit = Fit(G)$ and $\bar{G} = G/Fit$.

The following lemma is contained in [8, Section 2].

LEMMA 4.1. *If A and B are two nontrivial torsion-free abelian groups, then $A wr B$ is a ∀-free metabelian group.*

PROOF. In [27] Timoshenko proved that if A_1 and A_2 are two groups with the same universal theory and if B_1 and B_2 are two groups with the same universal theory, then $A_1 wr B_1$ and $A_2 wr B_2$ have the same universal theory. Moreover, it is well known that two nontrivial torsion-free abelian groups have the same universal theory. Thus, to prove the lemma it suffices to prove that every universal sentence true in $F_2(\mathscr{A}^2)$ is true in $\mathbb{Z} wr \mathbb{Z}$ and that every universal sentence true in $\mathbb{Z}^{(2)} wr \mathbb{Z}^{(2)}$ is true in $F_2(\mathscr{A}^2)$. These follow form the fact that $\mathbb{Z} wr \mathbb{Z}$ can be embedded in $F_2(\mathscr{A}^2)$ (see [1, Corollary 3]), and from Magnus embedding which implies that $F_2(\mathscr{A}^2)$ can be embedded in $\mathbb{Z}^{(2)} wr \mathbb{Z}^{(2)}$. ⊣

LEMMA 4.2. *If G satisfies the universal theory of $\mathbb{Z}wr\mathbb{Z}$, then there exist a torsion-free abelian group A and a torsion-free $\mathbb{Z}[A]$-module L such that G can be embedded in $M(A, L)$.*

PROOF. The proof is a standard application of compactness theorem, hence we just give the idea of the proof (morever if we just want to prove that the properties (1), (3), (4) and (5) of the Main Theorem are equivalent we do not need Lemma 4.2). We consider a (two-sorted first-order) language: $\{(P_A, ., ^{-1}, 1), (P_L, +, -, 0), *\}$, where P_A and P_L are unary relation symbols and where $*$ is a function symbol for a function from $P_L \times P_A$ into P_L. In this language we have a theory Σ_0 which says that P_A is a torsion-free abelian group and that P_L is a torsion-free $\mathbb{Z}[P_A]$-module. Now for each $g \in G$ we introduce two new constants (symbol) a_g and v_g and we consider the following set of sentences

$$\begin{aligned}\Sigma_1 \;=\;& \{\, P_A(a_g) \wedge P_L(v_g) \mid g \in G \,\}\\ &\cup \{\, a_g a_h = a_{gh} \wedge *(v_g, a_h) + v_h = v_{gh} \mid g, h \in G \,\}\\ &\cup \{\, a_g a_h \neq a_f \vee *(v_g, a_h) + v_h \neq v_f \mid g, h, f \in G \text{ with } gh \neq f \,\}.\end{aligned}$$

We put $\Sigma = \Sigma_0 \cup \Sigma_1$ (note that Σ implies that $v_g + v_h = v_{gh}$). Since G satisfies the universal theory of $\mathbb{Z}wr\mathbb{Z}$ and $\mathbb{Z}wr\mathbb{Z} \simeq M(\langle X\rangle, \mathbb{Z}[X, X^{-1}])$, by compactness, Σ has a model (A, L). Then, by definition of Σ_1, the function which maps $g \in G$ to $(a_g, v_g) \in M(A, L)$ is a monomorphism of groups. ⊣

LEMMA 4.3. *If A is a torsion-free abelian group and if L is a torsion-free right $\mathbb{Z}[A]$-module, then $M(A, L) \models \mathfrak{A}$.*

PROOF. Let $\bar{e} = (e_1, \ldots, e_n)$ be a sequence on $\{-1, 1\}$, let $x = (a, v)$, $y = (b, w)$, $y_1 = (a_1, v_1), \ldots, y_n = (a_n, v_n)$ be elements of $M(A, L)$, and let n be an integer ≥ 2. Since A is abelian we have

$$\begin{aligned}&[x, y] = \big(1, v(b-1) - w(a-1)\big)\\ &x^n = \big(a^n, v(a^{n-1} + \cdots + a + 1)\big)\\ &[x, x^y] = \big(1, (v(b-1) - w(a-1))(a-1)\big)\\ &[x^n, x^{ny}] = \big(1, (v(b-1) - w(a-1))(a^n - 1)(a^{n-1} + \cdots + a + 1)\big)\\ &\text{if } a = 1,\; x^{e_1 y_1} \cdots x^{e_n y_n} = \big(1, v(e_1 a_1 + \cdots + e_n a_n)\big).\end{aligned}$$

Now, since A is torsion-free, since $\mathbb{Z}[A]$ is an integral domain (see [23, 15.3.10]), and since L is a torsion-free right $\mathbb{Z}[A]$-module, simple computations show that $M(A, L)$ satisfies $\mathfrak{A}$. ⊣

LEMMA 4.4. *Assume that G satisfies ϕ_4. Then, all the nonabelian subgroups of G have a trivial center and all the nilpotent subgroups of G are abelian.*

PROOF. Since ϕ_4 is a universal sentence and since a nilpotent group has a nontrivial center, it suffices to prove that a nonabelian group which satisfies ϕ_4 has a trivial center. Let G be a nonabelian group with $G \models \phi_4$. There exist $g, h \in G$ such that $[g, h] \neq 1$. Let f be an element of $Z(G)$, we have $[g, f] = 1$ and $[f, h] = 1$. Since $G \models \phi_4$, if $f \neq 1$, then $[g, h] = 1$, thus $f = 1$. ⊣

LEMMA 4.5. *If $G \models \mathfrak{A}$, then G is a ρ-group.*

PROOF. We suppose that $G \models \mathfrak{A}$. It is clear that G is a torsion-free metabelian group (sentences ϕ_1, ϕ_2 and $\phi_3(n)$). Obviously we may suppose that G is nonabelian.

By Fitting's Theorem (see [23, 5.2.8]), *Fit* is locally nilpotent (i.e., all the finitely generated subgroups of *Fit* are nilpotent). Since $G \models \phi_4$, Lemma 4.4 shows that *Fit* is abelian. Then, $fit^{\forall}(G) = Fit$. Indeed, if $g \in fit^{\forall}(G)$, then the normal closure of g is abelian thus $g \in Fit$. If $g \in Fit$, then the normal closure of g is contained in *Fit* and so is abelian. It follows that $g \in fit^{\forall}(G)$.

Let $g \in G$ such that $g^n \in Fit$ for $n \geq 2$. If $g^n = 1$, then since G is torsion-free $g \in Fit$. If $g^n \neq 1$, then, since G is nonabelian, by Lemma 4.4, there exists $h \in G$ such that $[g^n, h] \neq 1$, moreover, since $fit^{\forall}(G) = Fit$ we have $[g^n, g^{nh}] = 1$. Thus, $G \models fit^{\exists}(g^n)$ and the sentence $\phi_5(n)$ implies that $g \in Fit$.

Let $g \in Fit$ with $g \neq 1$. As above $G \models fit^{\exists}(g)$. Since G satisfies the sentences $\phi_6(\bar{e})$ (and since $fit^{\forall}(G) = Fit$), if we have $v \in \mathbb{Z}[\bar{G}]$ such that $g^v = 1$ then $v = 0$. ⊣

The following lemma is well known.

LEMMA 4.6. *If G is a metabelian, then for all $g_1, g_2, g_3 \in G$ we have*

$$[g_1, g_2]^{g_3 - 1}[g_1, g_3]^{-(g_2 - 1)}[g_2, g_3]^{g_1 - 1} = 1.$$

PROOF. It is easy to see that the identity of the lemma holds in $\mathbb{Z}wr\mathbb{Z}$. Lemma 4.1 implies that the identity of the lemma holds in any metabelian group. ⊣

LEMMA 4.7. *If G is a ρ-group, then $G \models \mathfrak{A}$.*

PROOF. Let G be a group a ρ-group. Clearly G satisfies ϕ_1, ϕ_2 and $\phi_3(n)$ for all $n \geq 2$.

Let $g, h \in G$ with $[g, h] \neq 1$ and $[g, g^h] = 1$. We have $1 = [g, g^h] = [g, h]^{-g+1}$. Since *Fit* is a torsion-free $\mathbb{Z}[\bar{G}]$-module, $g \in Fit$. Thus, $fit^{\exists}(G) \subset Fit \backslash \{1\}$. Also, since *Fit* is abelian we have $Fit = fit^{\forall}(G)$ (see the proof of Lemma 4.5). It is then easy to prove that G satisfies the sentences ϕ_5 and $\phi_6(\bar{e})$.

It remains to prove that $G \models \phi_4$. First, note that since G is metabelian, $G' \leq Fit$. Let $g, h, f \in G$ such that $[g, h] = 1$, $[h, f] = 1$ and $h \neq 1$. We suppose that $[g, f] \neq 1$. By Lemma 4.6, we have

$$[g, h]^{f-1}[g, f]^{-h+1}[h, f]^{g-1} = 1,$$

thus $[g, f]^{-h+1} = 1$ and since *Fit* is a torsion-free $\mathbb{Z}[\bar{G}]$-module we have $h \in Fit$. Since $1 = [g, h] = h^{-g+1}$ and since $h \neq 1$ we see that $g \in Fit$. In the same way we see that $f \in Fit$. Since *Fit* is abelian, we obtain $[g, f] = 1$. This is absurd and we have proved that $G \models \phi_4$. ⊣

The following lemma is one of the key results of the proof of the Main Theorem.

LEMMA 4.8. *If G is a finitely generated ρ-group, then there exist two positive integers k and r such that G can be embedded in $\mathbb{Z}^{(k)}wr\mathbb{Z}^{(r)}$.*

PROOF. Let G be a finitely generated ρ-group. We set $R = \mathbb{Z}[\bar{G}]$. By hypothesis G is metabelian, thus $G' \leq Fit$ and $\bar{G}$ is abelian. By hypothesis *Fit* is isolated in G and G is finitely generated, thus $\bar{G}$ is a torsion-free finitely generated abelian group. Thus, there exists $r \in \mathbb{N}$ such that $\bar{G} \simeq \mathbb{Z}^{(r)}$. If $r = 0$, the lemma is obvious, thus we suppose that $r \geq 1$. Let $a_1, \ldots, a_r \in G$ be such that $\bar{G} =$

$\langle \bar{a}_1, \dots, \bar{a}_r \rangle$. Since G is metabelian, G satisfies max-n (see [23, 15.3.1]), thus *Fit* is the normal closure of a finite number of elements of *Fit*. This implies that *Fit* is a finitely generated R-module. Thus, there exist $v_1, \dots, v_s \in Fit$ such that $Fit = \langle v_1, \dots, v_s \rangle_R$. Since $\bar{G} \simeq \mathbb{Z}^{(r)}$, R is isomorphic to the ring of Laurent polynomials $\mathbb{Z}[X_1, \dots, X_r, X_1^{-1}, \dots, X_r^{-1}]$ (see [13, 2.2.6]), which is a commutative domain. We put $S = \mathbb{Z}[X_1, \dots, X_r, X_1^{-1}, \dots, X_r^{-1}]$.

Let us give a presentation of G. We have $G = \langle a_1, \dots, a_r, v_1, \dots, v_s \rangle$ and we want a presentation on $a_1, \dots, a_r, v_1, \dots, v_s$. Let $i, j \in \{1, \dots, r\}$ with $i < j$, then $[a_i, a_j] \in Fit$ and thus there exist $P_{i,j,1}, \dots, P_{i,j,s} \in S$ such that

$$[a_i, a_j] = v_1^{P_{i,j,1}(\bar{a})} \cdots v_s^{P_{i,j,s}(\bar{a})} \stackrel{\text{def}}{=} u_{i,j}.$$

And we set

$$T_1 = \{ [a_i, a_j] = u_{i,j} \mid 1 \le i < j \le r \}.$$

Note that if $r = 1$, then T_1 is empty. Since G is metabelian we set

$$T_2 = \{ [v_i^{P(\bar{a})}, v_j^{Q(\bar{a})}] = 1 \mid 1 \le i \le j \le s \text{ and } P, Q \in S \}.$$

Fit is an R-module generated by $v_1, \dots, v_s$, thus *Fit* has a presentation of the form

$$\left(v_1, \dots, v_s \mid \prod_{i=1}^{s} v_i^{Q_{i,j}(\bar{a})} = 1 \,,\, j \in J \right)$$

where J is a set of positive integers and where the $Q_{i,j} \in S$ (since R is noetherian we can take J finite, but this is of no importance here). We put

$$T_3 = \left\{ \prod_{i=1}^{s} v_i^{Q_{i,j}(\bar{a})} = 1 \mid j \in J \right\}.$$

Now, it is not difficult to prove that G has the presentation

$$(a_1, \dots, a_r, v_1, \dots, v_s \mid T_1, T_2, T_3).$$

Since *Fit* is a finitely generated torsion-free module over a commutative integral domain, *Fit* can be embedded in a free R-module of finite dimension k (we may suppose that $k \ge 1$). Since $R \simeq S$, we may suppose that $Fit \le \bigoplus_{i=1}^{k} S \stackrel{\text{def}}{=} L$. We are going to prove that G can be embedded in $\mathbb{Z}^{(k)} wr \mathbb{Z}^{(r)}$. We denote by A the free abelian group on $X_1, \dots, X_r$. We have $\mathbb{Z}^{(k)} wr \mathbb{Z}^{(r)} \simeq M(A, L)$ and so we identify $\mathbb{Z}^{(k)} wr \mathbb{Z}^{(r)}$ and $M(A, L)$.

If $y_1, \dots, y_r \in L$ and if $z \in L$ with $z = (z_1, \dots, z_k)$ and $z_1 \ne 0, \dots, z_k \ne 0$, then we define a function $f_{\bar{y},z}$ from $\{a_1, \dots, a_r, v_1, \dots, v_s\}$ into $M(A, L)$ by

$$f_{\bar{y},z}(a_i) = (X_i, y_i) \text{ for } 1 \le i \le r \text{ and } f_{\bar{y},z}(v_i) = (1, v_i.z) \text{ for } 1 \le i \le s$$

where $v_i.z = (v_{i,1}, \dots, v_{i,k}).(z_1, \dots, z_k) = (v_{i,1}z_1, \dots, v_{i,k}z_k)$. It is clear that $f_{\bar{y},z}$ can be extended to a monomorphism from *Fit* into L. Thus, by von Dyck's theorem, we see that $f_{\bar{y},z}$ can be extended to a homomorphism from G into $M(A, L)$ iff

$$y_i(X_j - 1) - y_j(X_i - 1) = u_{i,j}.z \text{ for } 1 \le i < j \le r.$$

Hence, we can construct a homomorphism of the form $f_{\bar{y},z}$, if we can solve in S the following system of equations and inequalities

$$(*)\begin{cases} z_n \neq 0 \quad \text{for } n = 1, \dots, k \\ y_{i,n}(X_j - 1) - y_{j,n}(X_i - 1) - u_{i,j,n}z_n = 0 \quad \text{for } 1 \leq i < j \leq r \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } 1 \leq n \leq k \end{cases}$$

where we have put $u_{i,j} = (u_{i,j,1}, \dots, u_{i,j,k})$ and where the unknowns are the z_n and the $y_{i,n}$. In general it is not possible to solve $(*)$, but here we have some relations between the $u_{i,j,n}$: by Lemma 4.6, if $r \geq 3$ and if $1 \leq i < j < l \leq r$, then

$$u_{i,j,n}(X_l - 1) - u_{i,l,n}(X_j - 1) + u_{j,l,n}(X_i - 1) = 0 \ \text{ for } n = 1, \dots, k.$$

For solving $(*)$ it suffices to solve the k systems $(*_1), \dots, (*_k)$ where

$$(*_n)\begin{cases} z_n \neq 0 \\ y_{i,n}(X_j - 1) - y_{j,n}(X_i - 1) - u_{i,j,n}z_n = 0 \ \text{ for } 1 \leq i < j \leq r. \end{cases}$$

Let $n \in \{1, \dots, k\}$, we set $z = z_n$, $x_i = y_{i,n}$ and $\alpha_{i,j} = u_{i,j,n}$. Let us solve $(*_n)$. If $r = 1$, then we have to solve $z \neq 0$ and we can take $z = 1$. If $r = 2$, then we have to solve the system

$$\begin{cases} z \neq 0 \\ x_1(X_2 - 1) - x_2(X_1 - 1) - \alpha_{1,2}z = 0. \end{cases}$$

If $\alpha_{1,2} = 0$, then we can take $x_1 = x_2 = 0$ and $z = 1$. If $\alpha_{1,2} \neq 0$, then we can take $x_1 = x_2 = \alpha_{1,2}$ and $z = X_2 - X_1$. Now we suppose that $r \geq 3$. Let $i, j \in \{2, \dots, r\}$ with $i < j$. We denote by $(*_{i,j})$ the following subsystem of $(*_n)$

$$\begin{cases} x_1(X_i - 1) - x_i(X_1 - 1) - \alpha_{1,i}z = 0 \\ x_1(X_j - 1) - x_j(X_1 - 1) - \alpha_{1,j}z = 0 \\ x_i(X_j - 1) - x_j(X_i - 1) - \alpha_{i,j}z = 0. \end{cases}$$

Clearly $(*_{i,j})$ is equivalent to the system

$$\begin{cases} x_1(X_i - 1)(X_j - 1) - x_i(X_1 - 1)(X_j - 1) - \alpha_{1,i}(X_j - 1)z = 0 & (1) \\ x_1(X_j - 1)(X_i - 1) - x_j(X_1 - 1)(X_i - 1) - \alpha_{1,j}(X_i - 1)z = 0 & (2) \\ x_i(X_j - 1)(X_1 - 1) - x_j(X_i - 1)(X_1 - 1) - \alpha_{i,j}(X_1 - 1)z = 0 & (3) \end{cases}$$

and we have

$$(1) - (2) + (3) \ \Leftrightarrow \ (-\alpha_{1,i}(X_j - 1) + \alpha_{1,j}(X_i - 1) - \alpha_{i,j}(X_1 - 1))z = 0 \ \ (3').$$

By Lemma 4.6, the left-hand side of $(3')$ is equal to zero, thus

$$(*_{i,j}) \Leftrightarrow \begin{cases} x_1(X_i - 1) - x_i(X_1 - 1) - \alpha_{1,i}z = 0 \\ x_1(X_j - 1) - x_j(X_1 - 1) - \alpha_{1,j}z = 0, \end{cases}$$

it follows that

$$(*_n) \Leftrightarrow \begin{cases} z \neq 0 \\ x_1(X_i - 1) - x_i(X_1 - 1) - \alpha_{1,i}z = 0 \quad 2 \leq i \leq r. \end{cases}$$

If all the $\alpha_{1,i}$ are equal to zero, to solve $(*_n)$ we can take $x_1 = \cdots = x_r = 0$ and $z = 1$. Thus we suppose that there exists $i_0 \in \{2, \dots, r\}$ such that $\alpha_{1,i_0} \neq 0$ and we put $I = \{2, \dots, r\} - \{i_0\}$. We have

$$(*_n) \Leftrightarrow \begin{cases} z \neq 0 \\ \alpha_{1,i_0} z = x_1(X_{i_0} - 1) - x_{i_0}(X_1 - 1) \\ x_1 \alpha_{1,i_0}(X_i - 1) - x_i \alpha_{1,i_0}(X_1 - 1) - \alpha_{1,i}\alpha_{1,i_0} z = 0 \quad (i) \ i \in I. \end{cases}$$

If $i \in I$, then we have

$$(i) \Leftrightarrow x_1(\alpha_{1,i_0}(X_i - 1) - \alpha_{1,i}(X_{i_0} - 1)) + x_{i_0}\alpha_{1,i}(X_1 - 1) - x_i \alpha_{1,i_0}(X_1 - 1) = 0.$$

If $i_0 < i$, by Lemma 4.6, we have

$$\alpha_{1,i_0}(X_i - 1) - \alpha_{1,i}(X_{i_0} - 1) = -\alpha_{i_0,i}(X_1 - 1)$$

and we obtain

$$\begin{aligned} (i) \quad &\Leftrightarrow -x_1\alpha_{i_0,i}(X_1 - 1) + x_{i_0}\alpha_{1,i}(X_1 - 1) - x_i\alpha_{1,i_0}(X_1 - 1) = 0 \\ &\Leftrightarrow x_1\alpha_{i_0,i} - x_{i_0}\alpha_{1,i} + x_i\alpha_{1,i_0} = 0. \end{aligned}$$

If $i_0 > i$, as above we obtain

$$(i) \Leftrightarrow x_1\alpha_{i_0,i} + x_{i_0}\alpha_{1,i} - x_i\alpha_{1,i_0} = 0.$$

It follows that

$$(*_n) \Leftrightarrow \begin{cases} z \neq 0 \\ \alpha_{1,i_0} z = x_1(X_{i_0} - 1) - x_{i_0}(X_1 - 1) \\ x_1\alpha_{i_0,i} - x_{i_0}\alpha_{1,i} + x_i\alpha_{1,i_0} = 0 \ \text{ for } i_0 < i \leq r \\ x_1\alpha_{i_0,i} + x_{i_0}\alpha_{1,i} - x_i\alpha_{1,i_0} = 0 \ \text{ for } 2 \leq i < i_0. \end{cases}$$

It is then easy to solve $(*_n)$: we can take $x_1 = x_{i_0} = \alpha_{1,i_0}$, $x_i = \alpha_{1,i} - \alpha_{i_0,i}$ for $i_0 < i \leq r$, $x_i = \alpha_{1,i} + \alpha_{i_0,i}$ for $2 \leq i < i_0$ and $z = X_{i_0} - X_1$.

We have solved $(*_n)$ for each $n \in \{1, \dots, k\}$ and thus we can solve $(*)$ in S. It follows that there exist $y_1, \dots, y_r \in L$ and $z \in L$ such that $f_{\bar{y},z}$ can be extended to a homomorphism f from G into $M(A, L)$. It remains to prove that f is a monomorphism. It is clear that the restriction of f on *Fit* is a monomorphism. Hence $\ker(f) \cap Fit = 1$, and since G is solvable, by [23, 5.4.4], $\ker(f) = 1$. ⊣

LEMMA 4.9. *Let G be a ρ-group. If g belongs to $G \backslash Fit$ and if h belongs to $Fit \backslash \{1\}$, then $\langle g, h \rangle \simeq \mathbb{Z}wr\mathbb{Z}$.*

PROOF. For all $n \in \mathbb{Z}$ we set $H_n = \langle g^{-n}hg^n \rangle$. Since G is torsion-free, H_n is infinite cyclic. We set $H = \langle H_n \mid n \in \mathbb{Z} \rangle$. Since *Fit* is abelian, H is abelian. Moreover, since *Fit* is a torsion-free $\mathbb{Z}[\bar{G}]$-module, it is clear that $H_n \cap H_m = 1$ for all $n, m \in \mathbb{Z}$ with $n \neq m$. Thus H is the direct sum of the H_n. By definition, H is normal in $\langle g, h \rangle$. Since *Fit* is isolated in G, it is clear that $\langle g \rangle \cap H = 1$. Thus $\langle g, h \rangle$ is the semidirect product of H by $\langle g \rangle$ and it follows that $\langle g, h \rangle \simeq \mathbb{Z}wr\mathbb{Z}$. ⊣

PROOF OF THE MAIN THEOREM. Let G be a group. If G satisfies (1), then, by Lemma 4.1, G satisfies the universal theory of $\mathbb{Z}wr\mathbb{Z}$, and by Lemma 4.2, G satisfies (2). Since $\mathfrak{A}$ is a set of universal sentences, by Lemma 4.3, the implication (2) ⇒ (3) holds. The equivalence of the properties (3) and (4) follows immediately from

Lemmas 4.5 and 4.7. Let us prove that (4) implies (5). Suppose that G is a ρ-group. By Lemma 4.7, $G \models \mathfrak{A}$ and since $\mathfrak{A}$ is a set of universal sentences, every subgroup of G satisfies $\mathfrak{A}$. Thus, by Lemma 4.5, every finitely generated subgroup of G is a ρ-group. By Lemma 4.8, G satisfies (5). Let us prove that (5) implies (1). Suppose that G satisfies (5). Then, every finitely generated subgroups of G satisfies the universal theory of a noncyclic free metabelian group (Lemma 4.1). It is then easy to prove that G satisfies the universal theory of a noncyclic free metabelian group.

We have proved that the five properties of the Main Theorem are equivalent. It remains to prove (for example) that a nonabelian ρ-group is $\forall$-free metabelian. Let G be a nonabelian ρ-group. By the first part of the Main Theorem, G satisfies the universal theory of a noncyclic free metabelian group. Thus, to prove that G is $\forall$-free metabelian, it suffices to prove that G contains a group isomorphic to $\mathbb{Z}wr\mathbb{Z}$ (Lemma 4.1). Since G is metabelian there exists $h \in Fit$ with $h \neq 1$. Since Fit is a torsion-free module, $Z(G) = 1$. Thus there exists $g \in G$ such that $[g, h] \neq 1$. Since Fit is abelian, $g \in G \backslash Fit$. By Lemma 4.9, $\langle g, h\rangle \simeq \mathbb{Z}wr\mathbb{Z}$. This completes the proof of the Main Theorem. ⊣

§5. **Some applications of the main theorem.** Let $\mathscr{X}$ be a class of groups and n an integer ≥ 1. We say that a group G is n-approximable by (respectively, n-residually a) $\mathscr{X}$-group if for all $h_1, \dots, h_n \in G\backslash\{1\}$ there exists a homomorphism f from G into (respectively, onto) a group $H \in \mathscr{X}$ such that $f(h_1) \neq 1, \dots, f(h_n) \neq 1$. We say that a group G is $\aleph_0$-approximable by (respectively, $\aleph_0$-residually a) $\mathscr{X}$-group if G is n-approximable by (respectively, n-residually a) $\mathscr{X}$-group for all $n \geq 1$. If $\mathscr{X}$ is a class of groups closed with respect to forming subgroup (for example if $\mathscr{X}$ is the class of free groups), then the previous two notions coincide. Remeslennikov proved in [21] that a finitely generated nonabelian group has the same universal theory as a noncyclic free group iff it is $\aleph_0$-approximable by free groups (respectively, by noncyclic free groups). The Main Theorem implies that if a nonabelian group satisfies the universal theory of a noncyclic free metabelian group, then it is $\forall$-free metabelian. We use this fact to prove

COROLLARY 5.1. *Let $\mathscr{X}$ be a nonempty class of $\forall$-free metabelian groups. A nonabelian finitely generated group is $\forall$-free metabelian iff it is $\aleph_0$-approximable by $\mathscr{X}$-group.*

In particular, a nonabelian finitely generated group is $\forall$-free metabelian iff it is $\aleph_0$-approximable by a free metabelian group of rank 2 (respectively, $\aleph_0$-residually a finitely generated subgroup of a free metabelian group of rank 2).

PROOF. Let $\mathscr{X}$ be a nonempty class of $\forall$-free metabelian groups and G a nonabelian finitely generated group.

We suppose that G is $\forall$-free metabelian. Since finitely generated metabelian groups satisfy max-n (see [23, 15.3.1]), G has a finite presentation $(g_1, \dots, g_s \mid r_1(\bar{g}), \dots, r_k(\bar{g}))$ in the variety of metabelian groups. Let us prove that G is $\aleph_0$-approximable by $\mathscr{X}$-group. Let n be an integer ≥ 1 and let $h_1, \dots, h_n$ be elements of G not equal to 1. We have $h_1 = w_1(\bar{g}), \dots, h_n = w_n(\bar{g})$ where the w_i are words of the free group on $x_1, \dots, x_s$. Clearly G satisfies the following existential sentence in the language of groups

$$\phi \sim \exists x_1, \dots, x_s \Big(\bigwedge_{i=1}^{n} w_i(\bar{x}) \neq 1 \wedge \bigwedge_{i=1}^{k} r_i(\bar{x}) = 1 \Big).$$

Let $H \in \mathscr{X}$. By hypothesis $H \models \phi$, and there are $a_1, \dots, a_s \in H$ such that $w_i(\bar{a}) \neq 1$ for $i = 1, \dots, n$ and $r_i(\bar{a}) = 1$ for $i = 1, \dots, k$. By von Dick's theorem, we define a homomorphism f from G into H, by setting $f(g_i) = a_i$ for $i = 1, \dots, s$. Clearly, we have $f(h_i) \neq 1$ for $i = 1, \dots, n$.

Conversely, suppose that G is $\aleph_0$-approximable by $\mathscr{X}$-group. Since G is nonabelian, by the Main Theorem, to prove that G is $\forall$-free metabelian it suffices to prove that a noncyclic free metabelian group satisfies the existential theory of G. Let F be a noncyclic free metabelian group. Let ϕ be an existential sentence true in G. We may suppose that ϕ is in the form

$$\exists x_1 \dots x_s \Big(\bigwedge_{i=1}^{n} w_i(\bar{x}) \neq 1 \wedge \bigwedge_{i=1}^{k} v_i(\bar{x}) = 1 \Big).$$

By hypothesis on G there exists $H \in \mathscr{X}$ such that $H \models \phi$. Since H is $\forall$-free metabelian we have $F \models \phi$. ⊣

The Main Theorem implies that if A is a torsion-free metabelian group and if L is a torsion-free $\mathbb{Z}[A]$-module, then any nonabelian subgroup of $M(A, L)$ is $\forall$-free metabelian. We can use this fact to prove that certain groups which are close from being free metabelian groups are $\forall$-free metabelian.

We recall that a group G is termed parafree metabelian if G is residually nilpotent and if G has the same lower central sequence as some free metabelian group (see [2] for the existence of nonfree parafree metabelian groups). Baumslag shows in [3] that many properties of free metabelian groups persist in parafree metabelian groups. Here we obtain

Corollary 5.2. *A noncyclic parafree metabelian group is $\forall$-free metabelian.*

Proof. The proof of [3, Theorem 3.2 and its corollary] shows that a parafree metabelian group can be embedded in a group of the form $M(A, L)$ where L is the ring of power-series (over $\mathbb{Z}$) in the variables $a_{i,\lambda}$ ($i = 1, 2$ and $\lambda \in \Lambda$), where A is the multiplicative subgroup of L generated by the elements $1 + a_{2,\lambda}$ ($\lambda \in \Lambda$), and where Λ is a well-ordered set. Now it is easy, using simple computations, to prove that the natural morphism between $\mathbb{Z}[A]$ and L is an embedding. Since L is an integral domain, it follows that L is a torsion-free $\mathbb{Z}[A]$-module. Moreover, A is torsion-free and a noncyclic parafree metabelian group is nonabelian. Thus Corollary 5.2 follows from the Main Theorem. ⊣

We remark that Gaglione and Spellman proved in [11] that there exists a (absolutely) parafree group which does not statisfy the universal theory of (absolutely) free groups.

Let π be a set of prime numbers. We say that a group G is a D_π-group if for all $p \in \pi$ and all $g \in G$ there exists a unique $h \in G$ such that $h^p = g$. A group G is D_π-free metabelian if it is free in the class of metabelian D_π-groups (see [14]).

Ledlie proved in [14] that a D_π-free metabelian group can be embedded in a group $M(A, L)$ where A is a divisible torsion-free abelian group (written multiplicatively) and where L is a vector space over the quotient field of $\mathbb{Z}[A]$. Hence, a D_π-free metabelian group of rank ≥ 2 is ∀-free metabelian.

It is known that a nonabelian 2-generator subgroup of a (para)free metabelian group is either $F_2(\mathscr{A}^2)$ or $\mathbb{Z}wr\mathbb{Z}$ (see [3, Theorem 4.3 and pp. 525]). We can prove this for ∀-free metabelian groups.

COROLLARY 5.3. *A nonabelian 2-generator subgroup of a ∀-free metabelian groups is either $F_2(\mathscr{A}^2)$ or $\mathbb{Z}wr\mathbb{Z}$. In particular, a 2-generator ∀-free metabelian group is either $F_2(\mathscr{A}^2)$ or $\mathbb{Z}wr\mathbb{Z}$.*

It is probably "well known" that a nonabelian 2-generator subgroup of a wreath product of the form $\mathbb{Z}^{(k)}wr\mathbb{Z}^{(r)}$ is either $F_2(\mathscr{A}^2)$ or $\mathbb{Z}wr\mathbb{Z}$. With this fact, Corollary 5.3 is an immediate consequence of the Main Theorem. That may be, since we do not find any complete proof in the literature, we will prove Corollary 5.3.

PROOF. Let G be a ∀-free metabelian group and let $H = \langle g, h\rangle$ be a nonabelian subgroup of G. By the Main Theorem, H is a ρ-group. We put $Fit(H) = Fit$ and $\bar{H} = H/Fit$.

We begin by supposing that g and h are linearly independent modulo Fit. Let F be a free metabelian group on a and b. Since H is metabelian, we define an epimorphism f from F onto H, by $f(a) = g$ and $f(b) = h$. Let us prove that f is an isomorphism. Let x be an element of F such that $f(x) = 1$. We can write

$$x = a^n b^m \prod_{i=1}^{l} [a, b]^{n_i a^{m_{1,i}} b^{m_{2,i}}}$$

where $n, m, n_i, m_{k,i} \in \mathbb{Z}$ and where $(m_{1,i}, m_{2,i}) \neq (m_{1,j}, m_{2,j})$ for $i \neq j$ (clearly $x = a^n b^m \bmod(F')$ and use [23, 5.1.5] to prove that F' is generated by $[a, b]$ as an F/F'-module). We have

$$f(x) = g^n h^m \prod_{i=1}^{l} [g, h]^{n_i g^{m_{1,i}} h^{m_{2,i}}}.$$

Hence, since $H' \leq Fit$, we have $g^n h^m = 1 \bmod(Fit)$. By hypothesis on g and h, it follows that $n = m = 0$. Clearly, $[g, h] \neq 1$. Since Fit is a torsion-free $\mathbb{Z}[\bar{H}]$-module we have

$$\sum_{i=1}^{l} n_i g^{m_{1,i}} h^{m_{2,i}} = 0 \text{ in } \mathbb{Z}[\bar{H}].$$

But, by hypothesis on the $m_{k,i}$ and on g and h we have $g^{m_{1,i}} h^{m_{2,i}} \neq g^{m_{1,j}} h^{m_{2,j}} \bmod(Fit)$, for $i \neq j$. Hence, we must have $n_1 = \cdots = n_l = 0$. Thus $x = 1$ and f is an isomorphism.

Now we suppose that g and h are not linearly independent modulo Fit and we are going to prove that $\langle g, h\rangle \simeq \mathbb{Z}wr\mathbb{Z}$. By Lemma 4.9, it suffices to prove that there exist $u, v \in H$ such that $H = \langle u, v\rangle$, $u \in H \backslash Fit$ and $v \in Fit$. For this, we need the following well-known lemma.

LEMMA 5.4. *If H is a 2-generator group and if A is a normal subgroup of H such that H/A is infinite cyclic, then there exist $u, v \in H$ such that $H = \langle u, v \rangle$ and $v \in A$.*

PROOF. The following proof has been communicated to me by Oger. By hypothesis there exists $u \in H$ such that $H = \langle u, A \rangle$ and there exist $x_1, x_2 \in H$ such that $H = \langle x_1, x_2 \rangle$. We have $x_1 = u^{n_1} x_1'$ and $x_2 = u^{n_2} x_2'$ where $x_1', x_2' \in A$ and $n_1, n_2 \in \mathbb{Z}$. We can choose x_1, x_2 so as $\max\{|n_1|, |n_2|\}$ is minimal, and then we may suppose that $n_1 \geq 0$ and $n_2 \geq 0$. If n_1 or n_2 is equal to zero, then we obtain what we want. So, we may suppose that $1 \leq n_1 \leq n_2$. We put $y_1 = x_1$ and $y_2 = x_2 x_1^{-1}$. Clearly $H = \langle y_1, y_2 \rangle$, and we have $y_1 = u^{m_1} y_1'$ and $y_2 = u^{m_2} y_2'$ where $y_1', y_2' \in A$ and $m_1, m_2 \in \mathbb{Z}$. Since H/A is infinite cyclic, $m_1 = n_1$ and $m_2 = n_2 - n_1$. Then $\max\{|m_1|, |m_2|\} < \max\{|n_1|, |n_2|\}$. This contradicts the choice of x_1 and x_2. ⊣

We come back to the proof of Corollary 5.3. By Lemma 5.4, it remains to prove that H/Fit is infinite cyclic. By hypothesis on g and h, there exist $n, m \in \mathbb{Z}$ with $(n, m) \neq (0, 0)$ such that $g^n h^m = 1 \bmod(Fit)$. Since Fit is isolated in H, we may suppose that $n \neq 0$, $m \neq 0$ and that g.c.d(n, m)=1. Thus there exist $t, s \in \mathbb{Z}$ such that $nt + ms = 1$. Then, it is easy to see that $H/Fit = \langle g^{-s} h^t Fit \rangle$. So, H/Fit is infinite cyclic. ⊣

In the class of $\forall$-free metabelian group we can distinguish the groups whose 2-generator nonabelian subgroups are isomorphic to $\mathbb{Z} wr \mathbb{Z}$. We can prove that if G is finitely generated the following properties are equivalent: (1) G is $\forall$-free metabelian and the 2-generator nonabelian subgroups of G are isomorphic to $\mathbb{Z} wr \mathbb{Z}$; (2) G is a ρ-group and $G/Fit(G)$ is infinite cyclic; (3) $G \simeq M(\langle X \rangle, L)$ where L is a nontrivial torsion-free $\mathbb{Z}[X, X^{-1}]$-module. Moreover, using [4], we can prove that (1) is equivalent to (4) G is $\forall$-free metabelian and contains at most $\aleph_0$ nonisomorphic subgroups. Moreover, since $\mathbb{Z}[X, X^{-1}]$ contains $\aleph_0$ ideals on 2 generators which are not isomorphic as $\mathbb{Z}[X, X^{-1}]$-modules, we have $\aleph_0$ nonisomorphic 3-generator $\forall$-free metabelian groups.

§6. **Remarks.** We can deduce from [5] that there is an algorithm which decides whether or not a given finitely generated metabelian group is a ρ-group (we thank the authors of this paper for their helpful comments on [5]). Hence, by the Main Theorem, we have an algorithm which decides whether or not a given finitely generated metabelian group is a $\forall$-free metabelian group (respectively a subgroup of a group of the form $\mathbb{Z}^{(k)} wr \mathbb{Z}^{(r)}$).

If $A_1, \ldots, A_m$ is a finite sequence of groups then we denote by $wr_{i=1}^m A_i$ the left-iterated wreath product $A_1 wr(A_2 wr(\cdots (A_{m-1} wr A_m) \cdots))$. In [8] we proved that if the A_i are torsion-free abelian groups then the universal theory of $wr_{i=1}^m A_i$ is decidable (if the A_i are nontrivial and if $B_1, \ldots, B_m$ are nontrivial torsion-free abelian groups, then $wr_{i=1}^m A_i$ and $wr_{i=1}^m B_i$ have the same universal theory); but if $m \geq 3$ and if the A_i are nontrivial we do not have an explicit description of the universal theory of $wr_{i=1}^m A_i$. It will be interesting to obtain such a description. Notice that if $m \geq 3$ then $wr_{i=1}^m \mathbb{Z}$ does not have the same universal theory as a noncyclic free solvable group of class m (see [8, Section 2]).

We have recently studied the theory $\mathfrak{A}$ from the point of view of "model theoretic algebra." We have described explicitly the existentially complete models of $\mathfrak{A}$ and we have found a natural complete theory $\mathfrak{T}$ which is decidable and classifiable (w-stable and bidimensional) such that every existentially complete model of $\mathfrak{A}$ is a model of $\mathfrak{T}$ (in particular all the existentially complete models of $\mathfrak{A}$ are elementarily equivalent). Unfortunately the theory $\mathfrak{A}$ has no model companion, but the theory $\mathfrak{T}$ is a natural extension of the theory $\mathfrak{A}$: $\mathfrak{T}$ and $\mathfrak{A}$ are cotheory and any formula of $\mathscr{L}_{gp}$ is equivalent modulo $\mathfrak{T}$ to a $\forall\exists$ formula. Notice that this gives another point of view on a recent work of Delon and Simonetta (see [26, Chapitre 4]) which is also connected with the question of the above paragraph. We consider these and other questions in a future paper.[1]

REFERENCES

[1] G. BAUMSLAG, *Wreath products and extensions*, ***Mathematische Zeitschrift***, vol. 81 (1963), pp. 286–299.

[2] ———, *Groups with the same lower central sequence as relatively free groups I*, ***Transaction of the American Mathematical Society***, vol. 129 (1967), pp. 308–321.

[3] ———, *Groups with the same lower central sequence as relatively free groups II*, ***Transaction of the American Mathematical Society***, vol. 142 (1969), pp. 507–538.

[4] ———, *Some reflexions on finitely generated metabelian groups*, ***Contemporary Mathematics***, vol. 109 (1990), pp. 1–9.

[5] G. BAUMSLAG, F. B. CANNONITO, and D. J. S. ROBINSON, *The algorithmic theory of finitely generated metabelian groups*, ***Transaction of the American Mathematical Society***, vol. 344 (1994), pp. 629–648.

[6] C. C. CHANG and H. J. KEISLER, ***Model theory***, North-Holland, Amsterdam, 1973.

[7] O. CHAPUIS, *On the theories of free solvable groups*, submitted for publication.

[8] ———, *Universal theory of certain solvable groups and bounded Ore group-rings*, ***Journal of Algebra***, vol. 176 (1995), pp. 368–391.

[9] YU. L. ERSHOV, I. A. LAVROV, A. D. TAIMANOV, and M. A. TAITSLIN, *Elementary theory*, ***Russian Mathematical Survey***, vol. 20 (1965), pp. 35–105.

[10] A. M. GAGLIONE and D. SPELLMAN, *The persistence of universal formulae in free algebras*, ***Bulletin of the Australian Mathematical Society***, vol. 36 (1987), pp. 11–17.

[11] ———, *More model theory of free groups*, ***Houston Journal of Mathematics***, vol. 21 (1995), pp. 225–245.

[12] N. GUPTA, ***Free group-rings***, American Mathematical Society, Providence, 1987.

[13] G. KARPILOVSKY, ***Commutative group algebras***, Dekker, New York, 1983.

[14] J. F. LEDLIE, *Representation of free metabelian D_π-groups*, ***Transaction of the American Mathematical Society***, vol. 153 (1971), pp. 307–346.

[15] A. G. MAKANIN, *Decidability of the universal and positive theories of a free group*, ***Mathematical USSR Izvestia***, vol. 25 (1985), pp. 75–88.

[16] A. I. MALCEV, *On free solvable groups*, ***Soviet Mathematical Doklady***, vol. 1 (1960), pp. 65–68.

[17] B. MAZUR, *Questions of decidability and undecidability in number theory*, ***Journal of Symbolic logic***, vol. 59 (1994), pp. 353–371.

[18] V. D. Mazurov, Yu. I. Merzlyakov, and V. A. Tchurkin (editors), ***The Kourovka notebook (unsolved problems in group theory, seventh augmented edition)***, vol. 121, 1983, American Mathematical Society translation.

[19] H. NEUMAN, ***Varieties of groups***, Springer-Verlag, Berlin, 1967.

[20] T. PHEIDAS, *Extentions of Hilbert's tenth problem*, this JOURNAL, vol. 59 (1994), pp. 372–397.

[21] V. N. REMESLENNIKOV, *∃-free groups*, ***Siberian Mathematical Journal***, vol. 30 (1989), pp. 998–1001.

[22] ———, *∃-free groups as a group with a length function*, ***Ukranian Mathematical Journal***, vol. 44 (1992), pp. 733–738.

[23] D. J. S. ROBINSON, ***A course in the theory of groups***, Springer-Verlag, New York, 1982.

[1]Added in proof. See, O. CHAPUIS, *From "Metabelian Q-vector spaces" to new ω-stable groups*, ***Bulletin of Symbolic Logic***, vol. 2 (1996), pp. 84–93.

[24] V. A. ROMAN'KOV, *Equation in free metabelian groups*, ***Siberian Mathematical Journal***, vol. 20 (1979), pp. 469–471.

[25] ———, *Universal theory of nilpotent groups*, ***Mathematical Notes***, vol. 25 (1979), pp. 252–258.

[26] P. SIMONETTA, *Décidabilité et interpretabilité dans les corps et les groupes non commutatifs*, ***Thèse de doctorat***, Université Paris VII, 1994.

[27] E. I. TIMOSHENKO, *Preservation of elementary and universal equivalence under the wreath product*, ***Algebra and Logic***, vol. 7 (1968), pp. 273–276.

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